Lecture 21: (Chap. 7, cont.)

We will now see that the solutions of stochastic differential equations can be used to provide probabilistic representation formulas for the solutions of certain second order partial differential equations.

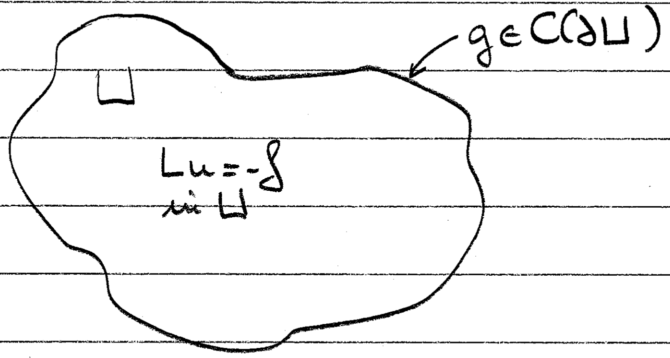
We consider the following Dirichlet-Poisson problem:

$U \neq \emptyset$ is a bounded open subset of $\mathbb{R}^d$,

$f \in C_b(U)$, $g \in C(\partial U)$,

and we look for $u \in C^2(U) \cap C(\overline{U})$ such that, see (7.25) for the notation:

$$(7.29)\quad \begin{cases} Lu(x) = -f(x), & \text{for } x \in U, \\ u(x) = g(x), & \text{for } x \in \partial U. \end{cases}$$



The Dirichlet problem corresponds to $f = 0$, and the Poisson equation to $g = 0$ in (7.29).

In addition to the local boundedness and measurability of $b(\cdot)$, $\sigma(\cdot)$, we assume the following ellipticity condition:

(7.30) There is $c > 0$, so that ${}^t\xi\, a(x)\, \xi \ge c|\xi|^2$, for $\xi \in \mathbb{R}^d$, $x \in \overline{U}$.

It is known that when $\sigma(\cdot)$, $b(\cdot)$ in addition satisfy (7.3), (in fact a Hölder condition is good enough), when f is bounded Hölder continuous in U, and U satisfies an exterior sphere condition:
$\forall z \in \partial U$, there is an open ball B, with $B \cap \overline{U} = \{z\}$,
the problem (7.29) has a solution, cf. Gilbarg-Trudinger, p.106, "Elliptic

partial differential equations of second order".

<u>Theorem</u>: ($b(\cdot)$, $\sigma(\cdot)$, measurable locally bounded, (7.30))
If u is a solution of (7.29), and X_t, $t \geq 0$, satisfies (7.23), for some $x \in U$, then the exit time of $X_\cdot$ from U

(7.31) $T_U = \inf\{s \geq 0;\ X_s \notin U\}$ is P-integrable,

and

(7.32) $u(x) = E\big[g(X_{T_U}) + \int_0^{T_U} f(X_s)\,ds\big]$.

<u>Proof</u>:

(7.31):

Pick $\varphi(y) = C(e^{\alpha R} - e^{\alpha y_1})$, where $y = (y_1, \dots, y_d) \in \overline{U}$, then

$L\varphi(y) = -C\, e^{\alpha y_1}(\alpha^2 a_{1,1}(y) + \alpha\, b_1(y))$
$\overset{(7.30)}{\leq} -C e^{\alpha y_1}(\alpha^2 c - \alpha M)$, with $M = \sup_{\overline{U}} |b_1(\cdot)|$.

Choosing α, R large and then C large enough, we can make sure that

(7.33): $L\varphi \leq -1$, on $\overline{U}$,

(7.34): $\varphi > 0$, on $\overline{U}$.

By (7.24) we find that under P,
$\varphi(X_{t\wedge T_U}) - \varphi(x) - \int_0^{t\wedge T_U} L\varphi(X_u)\,du$, $t \geq 0$,
is a local martingale, which is bounded. Hence it is a martingale.
Taking expectation, we find that
$E[\varphi(X_{t\wedge T_U})] - \varphi(x) - E[\int_0^{t\wedge T_U} L\varphi(X_u)\,du] = 0$,
and keeping in mind (7.33), (7.34), we thus find that

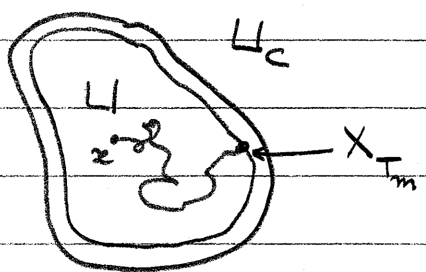
(7.35) $\sup_{\overline{U}} \varphi \geq E[\varphi(X_{t\wedge T_U})] - E[\int_0^{t\wedge T_U} L\varphi(X_u)\,du] \geq E[t \wedge T_U]$.

Letting $t \to \infty$, we obtain (7.31) and in fact the more precise:

(7.36) $E[T_U] \leq \sup_{\overline{U}} \varphi$.

- (7.32):

For $m \geq 1$ large enough so that $\frac{1}{m} < d(x, U^c)$, we define
$T_m = \inf\{s \geq 0;\ d(X_s, U^c) \leq \frac{1}{m}\}$



and construct $u_m \in C_c^2(\mathbb{R}^d, \mathbb{R})$ such that
(7.37) $u = u_m$ on $\{z \in U;\ d(z, U^c) \geq \frac{1}{m}\}$.

By (7.24), we see that P-a.s.,

(7.38) $u_m(X_{t \wedge T_m}) - u_m(x) - \int_0^{t \wedge T_m} L u_m(X_s)\,ds$
$= u(X_{t \wedge T_m}) - u(x) + \int_0^{t \wedge T_m} f(X_s)\,ds$

is a bounded continuous local martingale, and hence a martingale. Taking expectation we conclude that

$$E[u(X_{t \wedge T_m})] - u(x) + E\Big[\int_0^{t \wedge T_m} f(X_s)\,ds\Big] = 0.$$

Since $T_m \uparrow T_U < \infty$, and T_U is integrable, we can let $t \to \infty$, and then $m \to \infty$, and conclude that

(7.39) $u(x) = E[u(X_{T_U})] + E\Big[\int_0^{T_U} f(X_s)\,ds\Big]$
$\overset{(7.29)}{=} E\Big[g(X_{T_U}) + \int_0^{T_U} f(X_s)\,ds\Big]$,

whence our claim (7.32). $\square$

We will now discuss some features of the <u>martingale problem</u> (7.2), and its link with SDE's.

<u>Assumptions and notation:</u>

$b(\cdot): \mathbb{R}^d \to \mathbb{R}^d$, $\sigma(\cdot): \mathbb{R}^d \to M_{d \times n}$ are measurable, locally bounded, $a(\cdot) = (\sigma\, {}^t\sigma)(\cdot)$, and for $f \in C^2$

(7.40) $Lf(y) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(y)\, \partial^2_{ij} f(y) + \sum_{i=1}^d b_i(y)\, \partial_i f(y)$, $y \in \mathbb{R}^d$.

Theorem:

If on some $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$ endowed with an n-dimensional $(\mathcal{G}_t)$-Brownian motion $B_t, t\geq 0$, a continuous adapted process $(Y_t)_{t\geq 0}$ satisfies P-a.s., for $t \geq 0$:

(7.41) $Y_t = x + \int_0^t b(Y_s)ds + \int_0^t \sigma(Y_s).dB_s$,

then

(7.42) the law P_x of $(Y_t)_{t\geq 0}$, on $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F})$ is a solution of the martingale problem (7.2).

Conversely, if P_x is a solution of the martingale problem (7.2), then there exists an $(\Omega, (\mathcal{G}_t), \mathcal{G}, P)$ endowed with an n-dimensional Brownian motion $(\beta_t)_{t\geq 0}$, and a continuous adapted process $Z_t, t\geq 0$, such that P-a.s.:

(7.43) $Z_t = x + \int_0^t b(Z_s)ds + \int_0^t \sigma(Z_s)d\beta_s$, for $t \geq 0$,

and the law of Z is P_x.

Remark:

One should not expect $Z_.$ (or $Y_.$) to be strong solutions of the SDE, as in (7.22). An example of this feature comes for instance when considering $d = 1 = n$,

(7.44) $\sigma(x) = \text{sign}(x) = 1$, when $x \geq 0$

$= -1$ when $x < 0$,

and $Y_. = X_.$ the canonical Brownian motion on $(C(\mathbb{R}_+, \mathbb{R}), \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, W_0)$. Then in this case

(7.45) $B_t = \int_0^t \text{sign}(X_s)dX_s$, $t \geq 0$,

is thanks to Levy's characterization (6.44), (6.45) a Brownian motion. Moreover one has the identity: P-a.s. for all $t\geq 0$,

$$X_t = \int_0^t \text{sign}(X_s)^2 dX_s \overset{(5.92)}{=} \int_0^t \text{sign}(X_s)dB_s = \int_0^t \sigma(X_s)dB_s,$$

In other words $Y_. = X_.$ solves (7.41), but one can prove, see

Karatzas-Shreve, p.302, that, in the notation of (7.22), that for all $t>0$,
$F_t^{B.} = F_t^{|X.|}$ & $F_t^X = F_t$.
As a matter of fact one can show that whenever $Y.$ satisfies (7.41) with σ as in (7.44), then F_t^B & F_t^Y, for $t>0$. □

Proof:

– (7.42):

We know from (7.24) that for $f \in C_c^2(\mathbb{R}^d, \mathbb{R})$, under P

(7.46) $f(Y_t) - f(Y_0) - \int_0^t Lf(Y_s)\,ds$ is a $(\mathcal{G}_t)$-martingale.

Hence for $0 \le s_0 < \dots < s_m \le s < t$, $g_0, \dots, g_m \in bB(\mathbb{R}^d)$, denoting with P_x the law on $C(\mathbb{R}_+, \mathbb{R}^d)$ of Y under P, we see that:

canonical process

$$E^{P_x}\Big[\Big(f(X_t) - f(X_s) - \int_s^t (Lf)(X_u)\,du\Big)\, g_0(X_{s_0}) \dots g_m(X_{s_m})\Big] \overset{P_x = \text{law of } Y.}{=}$$
$$E\Big[\Big(f(Y_t) - f(Y_s) - \int_s^t (Lf)(Y_u)\,du\Big)\, g_0(Y_{s_0}) \dots g_m(Y_{s_m})\Big] \overset{(7.46)}{=} 0.$$

Using Dynkin's lemma, it follows that under P_x,
$M_t^f \overset{\text{def}}{=} f(X_t) - f(X_0) - \int_0^t Lf(X_u)\,du$ is an $(\mathcal{F}_t)$-martingale for any $f \in C_c^2(\mathbb{R}^d, \mathbb{R})$. Moreover since $Y_0 = x$, P-a.s., we see that $P_x[X_0 = x] = 1$. Hence P_x is a solution of the martingale problem (7.2).

– (7.43):

We will only prove (7.43) in a special case, namely when

(7.47) $n = d$, and $a(x)$ is locally elliptic, (i.e. for $U \neq \phi$ a bounded open subset of $\mathbb{R}^d$, $\exists\, c(U) > 0$, such that ${}^t\xi\, a(y)\, \xi \ge c(U)\, |\xi|^2$, for all ξ in $\mathbb{R}^d$, and y in U)

For a proof in the general case we refer to the book of D. Stroock: "Lectures on Stochastic Analysis: Diffusion Theory", p. 91.

Note that due to (7.47) and $a(\cdot) = \sigma(\cdot)\,{}^t\sigma(\cdot)$, $\sigma(\cdot)$ is invertible and for $y \in U$, $\xi \in \mathbb{R}^d$ one has

(7.48) $|\xi|^2 = {}^t\xi\,\sigma^{-1}(y)\,a(y)\,{}^t\sigma^{-1}(y).\xi \geq c(U)\,|{}^t\sigma^{-1}(y)\xi|^2$,

so that (using the explicit formula for $\sigma^{-1}(\cdot)$):

(7.49) $\sigma^{-1}(\cdot)$ is locally bounded measurable.

On $(C(\mathbb{R}_+,\mathbb{R}^d), \mathcal{F}, P_x)$, we introduce for $t \geq 0$, the σ-algebra $\mathcal{H}_t$ generated by $\mathcal{F}_t$ and the negligible sets of P_x, and $\mathcal{G}_t = \mathcal{H}_t^+$, $t \geq 0$, (satisfying the usual conditions). We define

(7.50) $M_t^i = X_t^i - X_0^i - \int_0^t b_i(X_s)\,ds$, $i = 1,\dots,d$.

If we apply (7.2) and stopping we see, since for $f(y) = y^i$, $Lf(y) = b_i(y)$, that the M_t^i are continuous $(\mathcal{G}_t)$-local martingales, (see exercise below). Analogously choosing $f(y) = y^i y^j$, so that $Lf(y) = a_{ij}(y) + y^i b_j(y) + y^j b_i(y)$, we see that

(7.51) $X_t^i X_t^j - X_0^i X_0^j - \int_0^t a_{ij}(X_s) + X_s^j b_i(X_s) + X_s^i b_j(X_s)\,ds$ is a continuous $(\mathcal{G}_t)$-local martingale under P_x.

From Ito's formula we know that P_x-a.s.,

(7.52) $X_t^i X_t^j = X_0^i X_0^j + \int_0^t X_s^i\,dX_s^j + \int_0^t X_s^j\,dX_s^i + \langle X^i, X^j\rangle_t$
$= X_0^i X_0^j + \int_0^t X_s^i b_j(X_s) + X_s^j b_i(X_s)\,ds + \langle M^i, M^j\rangle_t$
$+$ continuous local martingale.

Comparing (7.51) and (7.52), we conclude that P-a.s.

(7.53) $\langle M^i, M^j\rangle_t = \int_0^t a_{ij}(X_s)\,ds$, for $t \geq 0$.

We now define

(7.54) $\beta_t = \int_0^t \sigma^{-1}(X_s).\,dM_s$, $t \geq 0$,
(that is $\beta_t^i = \sum_{j=1}^d \int_0^t \sigma_{ij}^{-1}(X_s)\,dM_s^j$),

so that

(7.55) β_t, $t \geq 0$, is an $\mathbb{R}^d$-valued continuous $(\mathcal{G}_t)$-local martingale,

and

(7.56) $\langle \beta^i, \beta^j\rangle_t = \langle \sum_{k=1}^d \int_0^\cdot \sigma_{ik}^{-1}(X_s)\,dM_s^k, \sum_{\ell=1}^d \int_0^\cdot \sigma_{j\ell}^{-1}(X_s)\,dM_s^\ell \rangle_t =$

$$\sum_{k,\ell=1}^{d} \int_0^t \sigma^{-1}_{i,k}(X_s)\, \sigma^{-1}_{j,\ell}(X_s)\, d\langle M^k, M^\ell\rangle_s \overset{(7.53)}{=}$$

$$\sum_{k,\ell=1}^{d} \int_0^t \sigma^{-1}_{i,k}(X_s)\, a_{k,\ell}(X_s)\, \sigma_{j,\ell}(X_s)\, ds = \int_0^t (\sigma^{-1}(X_s)\, a(X_s)\, {}^t\sigma^{-1}(X_s))_{ij}\, ds$$
$$= \delta_{ij}\, t.$$

It thus follows from Paul Levy's characterization, cf. (6.44), (6.45), that

(7.57) $(\beta_t)_{t\geq 0}$ is a d-dimensional $(\mathcal{G}_t)$-Brownian motion under P_x.

Now from (7.54) we deduce that P_x-a.s., for $t \geq 0$,

(7.58) $\int_0^t \sigma(X_s)\, d\beta_s \overset{(5.92)}{=} \int_0^t \sigma(X_s) . \sigma^{-1}(X_s)\, dM_s = M_t$,

and therefore P_x-a.s, for $t \geq 0$,

(7.59) $X_t = x + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s) . d\beta_s$.

This yields the representation (7.43). □

<u>Exercise:</u>

Show that when P_x is a solution of the martingale problem (7.2), on $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F})$, then for any $f \in C^2(\mathbb{R}^d)$

$f(X_t) - f(X_0) - \int_0^t Lf(X_s)\, ds$, $t \geq 0$,

is a $(\mathcal{G}_t)$-local martingale, where $(\mathcal{G}_t)_{t\geq 0}$ is the filtration $\mathcal{G}_t = \mathcal{H}_t^+ (= \bigcap_{\varepsilon > 0} \mathcal{H}_{t+\varepsilon})$, where $\mathcal{H}_t = \sigma(\mathcal{F}_t, \mathcal{N})$, where $\mathcal{N}$ is the collection of P_x-negligible sets of $\mathcal{F}$, (so that $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F}, (\mathcal{G}_t)_{t\geq 0}, P_x)$ satisfies the usual conditions). □