Lecture 22: (Chap 7, cont.)

We further discuss the martingale problem (7.2) and its link with the SDE (7.1). As an application of the theorems p.123 and 132 we have the following

Corollary:

Assume that $b(\cdot): \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma(\cdot): \mathbb{R}^d \to M_{d\times n}$ satisfy the Lipschitz condition (7.3). Then for any $x \in \mathbb{R}^d$,

(7.60) there is a unique solution of the martingale problem (7.2) attached to $L = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(\cdot)\,\partial^2_{ij} + \sum_{i=1}^{d} b_i(\cdot)\,\partial_i$, with $a(\cdot) = \sigma(\cdot)\,{}^t\sigma(\cdot)$, (one says that the martingale problem attached to L is well-posed).

Proof:

– existence:

We consider some filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$ endowed with an n-dimensional $(\mathcal{G}_t)$-Brownian motion $B_t, t\geq 0$. One can for instance pick the canonical space $(C(\mathbb{R}_+, \mathbb{R}^n), \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, W_0)$, and $B_\cdot = X_\cdot$, the canonical process). By (7.4) we know that we can construct a "solution", i.e. continuous adapted $(Y_t)_{t\geq 0}$, such that P-a.s., for $t\geq 0$,

(7.61) $Y_t = x + \int_0^t b(Y_s)\,ds + \int_0^t \sigma(Y_s).\,dB_s$.

It then follows from (7.42) that

(7.62) the law of $(Y_t)_{t\geq 0}$ on $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F})$ is a solution of the martingale problem attached to L and x.

– uniqueness:

Assume that P_x is a solution of the martingale problem (7.2) attached to L and x. By (7.43) we know that we can find some $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$ and $(\beta_t)_{t\geq 0}$, which is an n-dimensional Brownian motion, and a continuous $(\mathcal{G}_t)$-adapted $\mathbb{R}^d$-valued process $Z_t, t\geq 0$, such that P-a.s., for $t\geq 0$:

(7.63) $Z_t = x + \int_0^t b(Z_s)ds + \int_0^t \sigma(Z_s).d\beta_s$, and
P_x = law of $(Z_t)_{t\geq 0}$ on $(C(\mathbb{R}_+,\mathbb{R}^d), \mathcal{F})$.

From (7.11), (7.21), we see that P-a.s., for all $t \geq 0$,

(7.64) $Z_t = X_t^\infty$,

where X_t^∞ is defined as the P-a.s. uniform limit on compact intervals of X_t^m, $t \geq 0$, where

(7.65) $X_t^0 \equiv x$, $X_t^1 = x + \int_0^t b(X_s^0)ds + \int_0^t \sigma(X_s^0).d\beta_s$, and for $m \geq 1$,
$X_t^{m+1} = x + \int_0^t b(X_s^m)ds + \int_0^t \sigma(X_s^m).d\beta_s$, for $m \geq 1$.

By inspection of (7.65) we see that the law of $(X^m_t)_{t\geq 0}$ or of $(X^\infty_t)_{t\geq 0}$ are unchanged if instead of $\beta.$ one uses the canonical n-dimensional Brownian motion. Combining this observation with (7.64), we see that the law of $Z.$ (i.e. P_x) is uniquely determined.

□

Remark:

A not very satisfactory feature of the above theorem has to do with the fact that the assumptions are made on the coefficients $\sigma(.)$ and $b(.)$, that appear in (7.3), but the conclusion concerns the martingale problem where only $a(.)$ and $b(.)$ are involved.

It is clear that it does not suffice to assume $a(.)$ Lipschitz continuous in order to find $\sigma(.)$ Lipschitz continuous such that $\sigma\, {}^t\sigma = a$, (for instance when $d=1$, $a(x) = |x|$ yields such an example).

However one can show that when $a(.): \mathbb{R}^d \to M_{d\times d}$ satisfies a global ellipticity condition: for some $\varepsilon > 0$,

(7.66) ${}^t\xi\, a(x)\, \xi \geq \varepsilon |\xi|^2$, for all x, ξ in $\mathbb{R}^d$,

and a Lipschitz condition

(7.67) $|a(y) - a(z)| \leq K|y-z|$, for $y, z \in \mathbb{R}^d$, then $a^{1/2}(\cdot)$ satisfies a Lipschitz condition as well, cf. Stroock "Lectures on Stochastic Analysis: Diffusion Theory", p. 97. Of course $a^{1/2}(\cdot)$ can then play the role of $\sigma(\cdot)$ in the corollary p. 136.

A similar Lipschitz property of $a^{1/2}(\cdot)$ can be proved when instead of (7.66), (7.67) one assumes that

(7.68) $\sup_{x\in\mathbb{R}^d} |a(x)| \le C < \infty$, and

(7.69) $x \in \mathbb{R}^d \to a(x) \in M_{d\times d}$ is a C^2-function and

$\sup_{1\le i,j\le d} \sup_{x\in\mathbb{R}^d} |\partial^2_{i,j} a(x)| \le C < \infty$.

In this last case, $a(\cdot)$ need not be uniformly elliptic. As direct application of the corollary p. 136, one now has:

(7.70) if $b(\cdot)$ satisfies a global Lipschitz condition, and $a(\cdot)$ either satisfies (7.66), (7.67), or (7.68), (7.69), then the martingale problem attached to L is well-posed.

$\square$