Lecture 23: (chap 7, cont.)

Girsanov transformations and applications to martingale problems.

We will now bring into play certain exponential martingales, and use them as a tool to solve various martingales problems. An important role is played by a theorem due in various forms to Cameron-Martin (1944), Girsanov (1960), Maruyama (1954, 1955).

Setting:

$(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$ is a filtered probability space satisfying the usual conditions, cf. (4.5), (4.6).

$M_t, t \geq 0$, is a continuous local martingale such that

(7.71) $Z_t = \exp\{M_t - \frac{1}{2}\langle M\rangle_t\}, t \geq 0$, is a martingale.

As we know from Novikov's criterion, cf. (6.72), (6.74), this is for instance the case when $E[\exp\{\frac{1}{2}\langle M\rangle_t\}] < \infty$, for each $t \geq 0$.

We pick a fixed $T > 0$, and since $E[Z_T] = 1 \; (= E[Z_0])$, we introduce the new probability Q on $(\Omega, \mathcal{G})$ defined by

(7.72) $Q \overset{\text{def}}{=} Z_T\, P$.

Of course when $A \in \mathcal{G}_t$, $t \leq T$, one has

$Q(A) = E[1_A Z_T] = E[1_A Z_t]$, so that

(7.73) $\left.\dfrac{dQ}{dP}\right|_{\mathcal{G}_t} = Z_t, \; 0 \leq t \leq T$,

in other words Z_t, for $t \leq T$, represents the density of the restriction of Q to $\mathcal{G}_t$ with respect to the restriction of P to $\mathcal{G}_t$. Note that P and Q have the same null sets.

<u>Theorem:</u> (Cameron-Martin, Girsanov, Maruyama)

If $(N_t)_{0\le t\le T}$ is a continuous local martingale under P, the "Girsanov transform of N":

(7.74) $\tilde{N}_t = N_t - \langle N, M\rangle_t$, $0\le t\le T$, is a continuous local martingale under Q.

Moreover if N^1, N^2 are continuous local martingales under P, then P-a.s., (equivalently Q-a.s.),

(7.75) $\langle \tilde{N}^1, \tilde{N}^2\rangle_t^Q = \langle N^1, N^2\rangle_t^P$, for $0\le t\le T$.

<u>Proof:</u>

Without loss of generality we assume $N_0 = 0$. We first claim that:

(7.76) $\tilde{N}_t Z_t$, $0\le t\le T$, is a local martingale under P.

Indeed it follows from Ito's formula (6.22) and from (6.28) that P-a.s, for $t\ge 0$,

$\tilde{N}_t Z_t = \int_0^t Z_s\, d\tilde{N}_s + \int_0^t \tilde{N}_s\, dZ_s + \langle \tilde{N}, Z\rangle_t$, and

$Z_t = 1 + \int_0^t Z_s\, dM_s$, so $\langle \tilde{N}, Z\rangle_t = \langle N - \langle N,M\rangle, Z\rangle_t = \langle N, Z\rangle_t \overset{(5.90)}{=} \int_0^t Z_s\, d\langle N, M\rangle_s$.

As a result

(7.77) $\tilde{N}_t Z_t = \int_0^t Z_s\, dN_s - \int_0^t Z_s\, d\langle N,M\rangle_s + \int_0^t \tilde{N}_s\, dZ_s + \int_0^t Z_s\, d\langle N, M\rangle_s$
$= \int_0^t Z_s\, dN_s + \int_0^t \tilde{N}\, dZ_s$ and (7.76) follows.

We will now see that $\tilde{N}_t$, $0\le t\le T$, is a local martingale under Q.

We define:

(7.78) $T_n = \inf\{u\ge 0;\ |\tilde{N}_u| \ge n\}$, $(\uparrow\infty$, as $n\to\infty)$,
$S_m = \inf\{u\ge 0;\ |M_u|\ge m$ or $\langle M\rangle_u \ge m\}$, $(\uparrow\infty$, as $m\to\infty)$.

The calculation (7.77) applied to $M_{t\wedge S_m}$, $Z_{t\wedge S_m}$, $N_{t\wedge T_n\wedge S_m}$, $\tilde{N}_{t\wedge T_n\wedge S_m}$ shows that:

(7.79) $\tilde{N}_{t\wedge T_n\wedge S_m} Z_{t\wedge S_m}$ is a continuous local martingale, which is bounded, and hence a martingale.

As a result we see that for $0\le s\le t\le T$, and $A\in\mathcal{G}_s$, we have:

(7.80) $E^P[\underbrace{1_A \tilde{N}_{t\wedge T_n\wedge S_m}}_{\mathcal{G}_t\text{-meas.}} \underbrace{Z_{T\wedge S_m}}_{\text{martingale}}] = E^P[\underbrace{1_A \tilde{N}_{t\wedge T_n\wedge S_m}}_{\mathcal{G}_s\text{-meas.}} \underbrace{Z_{t\wedge S_m}}_{\text{martingale}}] =$

$E^P[\underbrace{1_A \tilde{N}_{s\wedge T_n\wedge S_m}}_{\mathcal{G}_s\text{-meas.}} \underbrace{Z_{s\wedge S_m}}_{\text{martingale}}] = E^P[1_A \tilde{N}_{s\wedge T_n\wedge S_m} Z_{T\wedge S_m}]$.

Letting $m\to\infty$, and observing that $Z_{T\wedge S_m} \xrightarrow[m\to\infty]{L^1(P)} Z_T$ by uniform integrability and a.s. convergence, and keeping in mind that $|\tilde{N}_{\cdot\wedge T_n}|\le n$, we find that

$$E^P[1_A \tilde{N}_{t\wedge T_n} Z_T] = E^P[1_A \tilde{N}_{s\wedge T_n} Z_T]$$

$$\qquad \| \qquad\qquad\qquad \|$$

(7.81) $E^Q[1_A \tilde{N}_{t\wedge T_n}] = E^Q[1_A \tilde{N}_{s\wedge T_n}]$

Since $T_n \uparrow \infty$, P and Q-a.s., this proves that

(7.82) $\tilde{N}_t$, $0\le t\le T$, is a continuous local martingale under Q.

We then turn to the proof of (7.75). We introduce

$I_t \overset{\text{def}}{=} N_t^2 - \langle N\rangle_t = 2\int_0^t N_s\, dN_s$,

using Ito's formula (6.22) and the notation $\langle\ \rangle = \langle\ \rangle^P$.

By (7.82) we find that

(7.83) $\tilde{I}_t \overset{(7.74)}{=} I_t - \langle I, M\rangle_t = N_t^2 - \langle N\rangle_t - 2\int_0^t N_s\, d\langle N,M\rangle_s$ is a continuous local martingale under Q.

As a result,

(7.84) $\tilde{N}_t^2 - \langle N\rangle_t = N_t^2 - 2N_t\langle N,M\rangle_t + \langle N,M\rangle_t^2 - \langle N\rangle_t$

$\overset{(7.83)}{=} \tilde{I}_t + \langle N,M\rangle_t^2 + 2\int_0^t N_s\, d\langle N,M\rangle_s - 2N_t\langle N,M\rangle_t$.

It follows from Ito's formula that:

$J_t = 2N_t\langle N,M\rangle_t - 2\int_0^t N_s\, d\langle N,M\rangle_s = 2\int_0^t \langle N,M\rangle_s\, dN_s =$ continuous local martingale under P,

and

(7.85) $\tilde{J}_t \overset{(7.74)}{=} J_t - 2\int_0^t \langle N,M\rangle_s\, d\langle N,M\rangle_s = J_t - \langle N,M\rangle_t^2$.

We can then come back to (7.84) to conclude that

(7.86) $\tilde{N}_t^2 - \langle N\rangle_t = \tilde{I}_t - \tilde{J}_t \overset{(7.74)}{=}$ continuous local martingale under Q.

This shows that

(7.87) $\langle \tilde{N} \rangle_t^Q = \langle N \rangle_t^P$, $0 \le t \le T$,

and the claim (7.75) now follows by polarization. □

We will now apply the above theorem in order to construct the solution of certain martingale problems.

Theorem:

Assume that $b(\cdot), c(\cdot): \mathbb{R}^d \to \mathbb{R}^d$ and $a(\cdot): \mathbb{R}^d \to M^+_{d\times d}$ (i.e. the set of $d\times d$ non-negative matrices) are measurable locally bounded, with b, a, ${}^t cac$ bounded. Then there is a bijective correspondence between the solutions of the martingale problem attached to

(7.88) $L = \frac{1}{2} \sum_{i,j=1}^d a_{ij}\, \partial^2_{ij} + \sum_{i=1}^d b_i \partial_i$, $x \in \mathbb{R}^d$,

and

(7.89) $\tilde{L} = \frac{1}{2} \sum_{i,j=1}^d a_{ij}\, \partial^2_{ij} + \sum_{i=1}^d (b_i + (ac)_i)\, \partial_i$, $x \in \mathbb{R}^d$.

This bijective correspondence is the following. To each P solution of the martingale problem attached to (L, x), one associates the law $\tilde{P}$ on $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F})$, which is specified by the fact that

(7.90) $\frac{d\tilde{P}}{dP}\Big|_{\mathcal{F}_t} = \exp\{\int_0^t \underbrace{c(X_s) . d\bar{X}_s}_{\mathbb{R}^d\text{- scalar product}} - \frac{1}{2}\int_0^t {}^t cac(X_s)\, ds\}$, for $t \ge 0$,

where $\bar{X}_t = X_t - \int_0^t b(X_s)\, ds$, $t \ge 0$.

Proof:

Note that under P, the process $\bar{X}$ is a continuous local martingale and

(7.91) $\langle \bar{X}^i, \bar{X}^j \rangle_t \overset{(7.53)}{=} \int_0^t a_{ij}(X_s)\, ds$, for $t \ge 0$.

As a result $\int_0^t c(X_s). d\bar{X}_s$ is a continuous local martingale and P-a.s.:

(7.92) $\langle \int_0^\cdot c(X_s). d\bar{X}_s \rangle_t = \int_0^t {}^t cac(X_s)\, ds$, for $t \ge 0$.

As a result the expression in (7.90) is the stochastic exponential of $\int_0^\cdot c(X_s). d\bar{X}_s$. From Novikov's criterion (6.72) or from (6.72), we know that:

(7.93) $Z_t = \exp\{\int_0^t c(X_s).d\bar{X}_s - \frac{1}{2}\int_0^t {}^t c a c(X_s) ds\}$, $t \geq 0$, is a continuous martingale.

We will now use the following

Lemma:

(7.94) There is a unique probability Q on $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F})$ such that $\frac{dQ}{dP}\Big|_{\mathcal{F}_t} = Z_t$, for each $t \geq 0$.

Proof:

Uniqueness follows from Dynkin's lemma.

Existence:

Consider for $n \geq 2$ the laws

π_n on $\Sigma_n \overset{def}{=} C([0,1], \mathbb{R}^d) \times \underbrace{C_0([0,1], \mathbb{R}^d) \times \dots \times C_0([0,1], \mathbb{R}^d)}_{(n-1) \text{ copies}}$, of $((X_\cdot)_{0 \leq \cdot \leq 1}, (X_{1+\cdot} - X_1)_{0 \leq \cdot \leq 1}, \dots, (X_{n-1+\cdot} - X_{n-1})_{0 \leq \cdot \leq 1})$ under $Z_n P$,

(here $C_0([0,1], \mathbb{R}^d) = \{w \in C([0,1], \mathbb{R}^d);\ w(0) = 0\}$).

The laws π_n, $n \geq 2$, are consistent, i.e. the image of π_{n+1} on Σ_n under the "projection" from $\Sigma_{n+1} \to \Sigma_n$, which drops the last component, is π_n, for all $n \geq 2$, thanks to the martingale property of Z_t, $t \geq 0$, under P.

By Kolmogorov's extension theorem (see for instance Stroock's book: "Probability theory: an analytic view", p.129), there is a (unique) probability π on $\Sigma = C([0,1], \mathbb{R}^d) \times \prod_1^\infty C_0([0,1], \mathbb{R}^d)$, such that for each $n \geq 2$, the image of π under the "projection" $\Sigma \to \Sigma_n$ is π_n. If one now defines the map

$\varphi: \Sigma \to C(\mathbb{R}_+, \mathbb{R}^d)$, such that

(7.95) $\varphi((w_1, w_2, \dots)) = w$ such that

$w(t) = w_1(t)$, for $0 \leq t \leq 1$,

$w(t) = w_1(1) + w_2(t-1)$, for $1 \leq t \leq 2$,

$w(t) = w_1(1) + w_2(1) + \dots + w_n(1) + w_{n+1}(t-n)$, for $n \leq t \leq n+1$,

then the image Q of π under φ satisfies (by the definition of π_n):

(7.96) $\frac{dQ}{dP}\Big|_{\mathcal{F}_n} = Z_n$, for any $n \geq 2$.

The martingale property (7.93) now implies the property (7.94). □

We will now see that the unique Q constructed by the lemma satisfies:

(7.97) Q solves the martingale problem $\tilde{L}, x$.

To this end we first note by (7.74) that under $Z_n P$

$$\overline{X}^i_t - \langle \overline{X}^i, \int_0^\cdot c(X_s).d\overline{X}_s \rangle_t = X^i_t - \int_0^t b_i(X_s)ds - \sum_{j=1}^d \int_0^t c_j(X_s)d\langle \overline{X}^j, \overline{X}^i\rangle_s$$
$$\overset{(7.91)}{=} X^i_t - \int_0^t b_i(X_s) + (a.c)_i(X_s)\,ds, \quad 0\le t\le n,$$

is a local martingale.

If we now define $T_m = \inf\{u\ge 0; |X_u - \int_0^u (b+ac)(X_s)ds| \ge m\}$, so that $T_m \uparrow \infty$, as $m\to\infty$, ($ac = a^{1/2}(a^{1/2}c)$ is bounded, since a and ${}^t c\, a c$ are bounded). So

(7.98) $X_{t\wedge T_m} - \int_0^{t\wedge T_m} (b+ac)(X_s)ds$, $0\le t\le n$, is a martingale under $Z_n P$, (and also under Q).

As a result we see that for each m, the $\mathbb{R}^d$-valued

(7.99) $M_t = X_t - \int_0^t (b+ac)(X_s)ds$ is a local martingale under Q.

By (7.75) and (7.91) we see that Q-a.s.

(7.100) $\langle M^i, M^j\rangle_t = \int_0^t a_{ij}(X_s)ds$, $t\ge 0$.

The application of Itô's formula yields that for any f in $C^2(\mathbb{R}^d)$, Q-a.s., for $t\ge 0$:

(7.101)
$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \partial_i f(X_s)dX^i_s + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial^2_{ij} f(X_s) d\langle X^i, X^j\rangle_s$$

$$\underset{(7.100)}{\overset{(7.99)}{=}} f(X_0) + \int_0^t \nabla f(X_s).(b+ac)(X_s) + \int_0^t \nabla f(X_s).dM_s + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial^2_{ij} f(X_s) a_{ij}(X_s)ds$$

$$= f(X_0) + \int_0^t \tilde{L} f(X_s)ds + \text{continuous local martingale under } Q.$$

This completes the proof of (7.97).

Since Q coincides with $\tilde{P}$, we see that for any x in $\mathbb{R}^d$

(7.102) $P \to \tilde{P}$ sends solutions of the martingale problem L, x into solutions of the martingale problem $\tilde{L}, x$.

There now remains to see that the correspondence is bijective.

(7.103) The correspondence $P \to \tilde{P}$ is injective.

Assume that $P_1 \to \tilde{P}$ and $P_2 \to \tilde{P}$. Then for $t \geq 0$, $P_1 \sim P_2 \sim \tilde{P}$ on $\mathcal{F}_t$, and the integral $\int_0^s c(X_u).d\bar{X}_u$, $0 \leq s \leq t$, is identical $P_1, P_2, \tilde{P}$-a.s. (this feature goes back to (4.50), (4.57), (4.59) and the explicit approximating sequences used to construct stochastic integrals, see exercise 2) below). Now the equalities $\tilde{P} = Z_t P_1 = Z_t P_2$ on $\mathcal{F}_t$, implies that $P_1 = P_2$ on $\mathcal{F}_t$, for $t \geq 0$, and (7.103) follows, by Dynkins lemma.

(7.104) The correspondence $P \to \tilde{P}$ is surjective.

Indeed consider Q a solution of the martingale problem $(\tilde{L}, x)$. The above shows that there is a unique P on $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F})$ such that for any $t \geq 0$

(7.105) $$\frac{dP}{dQ}\Big|_{\mathcal{F}_t} = \exp\Big\{-\int_0^t c(X_s).d\Big(X_s - \int_0^s (b+ac)(X_u)du\Big) - \frac{1}{2}\int_0^t {}^t c\,a\,c(X_u)du\Big\} \overset{\text{def}}{=} \tilde{Z}_t,$$

and P is a solution of the martingale problem (L, x), (this is an application of (7.102) with Q playing the role of P).

Note that for $t \geq 0$,

$$\tilde{Z}_t = \exp\Big\{-\int_0^t c(X_s).d\bar{X}_s + \frac{1}{2}\int_0^t ({}^t c\,a\,c)(X_u)du\Big\} = Z_t^{-1},$$

so that for $t \geq 0$,

$$\frac{d\tilde{P}}{dP}\Big|_{\mathcal{F}_t} = Z_t = \frac{dQ}{dP}\Big|_{\mathcal{F}_t},$$

and hence $\tilde{P} = Q$, thus completing the proof of (7.104). □

Example:

We know from (7.60), that the martingale problem attached to $L = \frac{1}{2}\Delta$ is well posed (this is the case $b=0$, $\sigma =$ Identity matrix $(d \times d)$). The solution to the martingale problem attached to (L, x) is W_x, "Wiener measure starting from x".

Consider now $b(\cdot): \mathbb{R}^d \to \mathbb{R}^d$, bounded measurable.

When $\tilde{L} = \frac{1}{2}\Delta + b.\nabla$, we see by (7.90) that the martingale problem attached to $(\tilde{L}, x)$ has a unique solution, which is a probability $\tilde{W}_x$ on $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F})$ such that

(7.106) $$\left.\frac{d\tilde{W}_x}{dW_x}\right|_{\mathcal{F}_t} = \exp\left\{\int_0^t b(X_s).dX_s - \frac{1}{2}\int_0^t |b(X_s)|^2 ds\right\}, \text{ for any } t \geq 0.$$

Note that $\tilde{W}_x$-a.s.,

(7.107) $X_t = x + \int_0^t b(X_s)\,ds + \beta_t$, for $t \geq 0$,

where $\beta_t, t \geq 0$, is an $(\mathcal{F}_t)$-Brownian motion under $\tilde{W}_x$.

In particular this yields a (weak) solution to the stochastic differential equation attached to $b(\cdot)$ (only bounded measurable!) and $\sigma(\cdot) = Id$. □

Exercises:

1) Show that all solutions to (7.107) have the same law, (hint: use the theorem p. 142).

2) Consider P a solution of the martingale problem (L, x), where L is as in (7.88) and $\tilde{P}$ such that (7.90) holds, (with the same assumptions on $a(\cdot), b(\cdot), c(\cdot)$).

a) Show that when H is a bounded progressively measurable process, $\int_0^t H_s\, d\bar{X}_s$ is well-defined regardless of whether one uses that under P, $\bar{X}_\cdot$ is a continuous martingale or that under $\tilde{P}$, $\bar{X}_\cdot$ is a continuous semi-martingale, (hint: use the approximating sequences from (4.57), with $A_s = s$, and (4.59), alternatively use (5.82), (7.75)).

b) Show that $\int_0^t c(X_s).d\bar{X}_s$ is well-defined regardless of whether one works with P or $\tilde{P}$ to interpret the stochastic integral.

3) When $b = 1$, in (7.106), show that although the restrictions of W_x and $\tilde{W}_x$ to $\mathcal{F}_t$ are equivalent, for each $t \geq 0$, one has $W_x \perp \tilde{W}_x$, (i.e. there is $A \in \mathcal{F}$ with $W_x(A) = 1 = \tilde{W}_x(A^c)$). □