Lecture 24:

Explosions of solutions of stochastic differential equations: an application of Girsanov transformations.

We consider $c(\cdot): \mathbb{R}^d \to \mathbb{R}^d$ a locally Lipschitz function:

(7.108) $\forall M>0,\ \exists K_M>0$, such that $|c(x)-c(y)| \le K_M |x-y|$, for $|x|,|y|\le M$.

If we now consider $X_t, t\ge 0$, the canonical process on $(C(\mathbb{R}_+,\mathbb{R}^d), \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, W_x)$, i.e. canonical Brownian motion starting from x, we know from (6.27) that

(7.109) $Z_t = \exp\{\int_0^t c(X_s).dX_s - \frac{1}{2}\int_0^t |c(X_s)|^2 ds\}$, $t\ge 0$,

is a continuous local martingale.

However when $c(\cdot)$ "grows too fast at infinity", it need not be a martingale. The key quantity, cf. (6.84), is

(7.110) $e_t = 1 - E_x[Z_t] \ge 0$,

since we know from (6.84), that $e_t = 0$, implies that $Z_s, 0\le s\le t$, is a martingale.

We will now provide an interpretation of $e_t, t\ge 0$, in terms of the possible "explosions" of the SDE:

(7.111) $\begin{cases} dY_t = c(Y_t)dt + dB_t, \\ Y_0 = x. \end{cases}$

For this purpose we choose a sequence of bounded, Lipschitz functions $c_N(\cdot)$ on $\mathbb{R}^d$, such that

(7.112) $c_N(\cdot) = c(\cdot)$ on $\bar{B}_N = \{z\in\mathbb{R}^d;\ |z|\le N\}$.

Using the canonical Brownian motion $X_\cdot$ as "driving noise", we have (7.3), (7.4) a unique solution $Y_t^N, t\ge 0$, of

(7.113) $Y_t^N = X_t + \int_0^t c_N(Y_s^N)ds$,

(since $\sigma(\cdot) = Id$, $Y_\cdot^N$ is even $(\mathcal{F}_t)$-adapted and a deterministic function of $X_\cdot$, cf. (7.21))

We then define the $(\mathcal{F}_t)$-stopping time:

(7.114) $T_N = \inf\{u \geq 0;\ |Y^N_u| \geq N\}$.

<u>Lemma:</u> $(N, k \geq 0)$

(7.115) W_x-a.s., for $t \geq 0$, $\quad Y^{N+k}_{t\wedge T_N} = Y^N_{t \wedge T_N}$.

<u>Proof:</u>

Define $T^k_N = \inf\{u \geq 0;\ |Y^{N+k}_u| \geq N\}$. As below (7.5) we find that
$Y^{N+k}_{t\wedge T_N \wedge T^k_N} - Y^N_{t\wedge T_N\wedge T^k_N} \overset{(7.111)}{=} \int_0^{t\wedge T_N \wedge T^k_N} c_{N+k}(Y^{N+k}_s) - c_N(Y^N_s)\, ds$
and since $|Y^{N+k}_s|, |Y^N_s| \leq N$ for $0 < s \leq T_N \wedge T^k_N$, and $c_{N+k}(\cdot) = c_N(\cdot) = c(\cdot)$, on $\overline{B}_N$, we see that for $t_0 \geq 0$,

(7.116) $\sup\limits_{t\leq t_0} |Y^{N+k}_{t\wedge T_N \wedge T^k_N} - Y^N_{t\wedge T_N\wedge T^k_N}| \overset{(7.108)}{\leq} K_N \int_0^{t_0} |Y^{N+k}_{s\wedge T_N\wedge T^k_N} - Y^N_{s\wedge T_N \wedge T^k_N}|\, ds$.

By Gronwall's lemma, cf. (7.9), we see that

(7.117) for all $t_0 \geq 0$, $\quad \sup\limits_{t\leq t_0} |Y^{N+k}_{t\wedge T_N\wedge T^k_N} - Y^N_{t\wedge T_N \wedge T^k_N}| = 0$.

This last equality immediately implies that $T_N = T^k_N$, and the claim (7.115) follows.

□

Thanks to (7.115), we see that

(7.118) T_N is a non-decreasing sequence of $(\mathcal{F}_t)$-stopping times.

We can now define the <u>explosion time</u> of the SDE
$Y_t = X_t + \int_0^t c(Y_s)\, ds$, $t \geq 0$,
as the $(\mathcal{F}_t)$-stopping time:

(7.119) $T = \lim\limits_{N\to\infty} T_N \in (0, \infty]$.

The relation between the explosion time and e_t in (7.110) comes in the following:

<u>Theorem:</u>

(7.120) for $t \geq 0$, $\quad e_t = W_x[T \leq t]$.

<u>Proof:</u>

The case $t=0$ is obvious and we thus assume $t > 0$.

By (7.106), (7.107), we know that if we define the probability

(7.121) $Q_N = \exp\{\int_0^t c_N(X_s).dX_s - \frac{1}{2}\int_0^t |c_N(X_s)|^2 ds\}\, W_x$,

under Q_N,

(7.122) $X_s = x + \beta_s + \int_0^s c_N(X_u)\,du$, for $0 \le s \le t$,

where $(\beta_s)_{0\le s\le t}$ is a d-dimensional Brownian motion, and the law of $(X_s)_{0\le s\le t}$ under Q_N is that of the restriction to time $[0,t]$ of the unique solution of the martingale problem attached to $L = \frac{1}{2}\Delta + c_N.\nabla$, and x. By (7.42) the law of $(Y^N_s)_{0\le s\le t}$, under W_x with $Y^N_\cdot$ as in (7.113) coincides with the law of $(X_s)_{0\le s\le t}$ under Q_N. As a result we find that setting

$S_N = T_{B_N} \left(\overset{\text{def}}{=} \inf\{s \ge 0;\ |X_s| \ge N\}\right)$,

we have

(7.123) $W_x[T_N > t] = Q_N[S_N > t] = E_x[S_N > t, \exp\{\int_0^t c_N(X_s).dX_s - \frac{1}{2}\int_0^t |c_N(X_s)|^2 ds\}]$

Note that W_x-a.s. on $\{S_N > t\}$,

$\int_0^t c_N(X_s).dX_s = \int_0^{t\wedge S_N} c_N(X_s).dX_s \underset{(7.112)}{\overset{(5.60)}{=}} \int_0^{t\wedge S_N} c(X_s).dX_s = \int_0^t c(X_s).dX_s$,

and therefore:

(7.124) $W_x[T_N > t] = E_x[S_N > t, Z_t]$.

Now W_x-a.s. $S_n \uparrow \infty$, and thus letting $N \to \infty$, we find

$W_x[T > t] = \lim_N W_x[T_N > t] = E_x[Z_t]$,

and in view of (7.110) the claim (7.120) follows.

$\square$