ON THE CRITICAL PARAMETER OF INTERLACEMENT PERCOLATION IN HIGH DIMENSION

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Abstract

The vacant set of random interlacements on $\mathbb{Z}^d$, $d \geq 3$, has non-trivial percolative properties. It is known from [18], [16], that there is a non-degenerate critical value u_* such that the vacant set at level u percolates when $u < u_*$ and does not percolate when $u > u_*$. We derive here an asymptotic upper bound on u_*, as d goes to infinity, which complements the lower bound from [21]. Our main result shows that u_* is equivalent to $\log d$ for large d, and thus has the same principal asymptotic behavior as the critical parameter attached to random interlacements on $2d$-regular trees, which has been explicitly computed in [23].

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0 Introduction

Random interlacements have proven useful in understanding how trajectories of random walks can create large separating interfaces, see [19], [20], [4]. In the case of $\mathbb{Z}^d$, $d \geq 3$, it is known that the interlacement at level $u \geq 0$, is a random subset of $\mathbb{Z}^d$, which is connected, ergodic under translations, and infinite when u is positive, see [18]. The density of this set monotonically increases from 0 to 1, as u goes from 0 to ∞. Its complement, the vacant set at level u, displays non-trivial percolative properties. There is a critical value u_* in $(0, \infty)$, such that for $u < u_*$, the vacant set at level u has an infinite connected component, which is unique, see [16], [22], and for $u > u_*$, only has finite connected components, see [18]. Little is known about u_*, and only recently was it shown that u_* diverges when the dimension d tends to infinity, see [21]. The object of the present article is to establish that u_* is equivalent to $\log d$ as d tends to infinity. In particular this result shows that u_* has the same principal asymptotic behavior for large d as the corresponding critical parameter, (which has been explicitly computed in [23]), attached to the percolation of the vacant set of random interlacements on $2d$-regular trees.

We now describe the model. Precise definition and pointers to the literature appear in Section 1. Random interlacements are made of a cloud of paths, which constitute a Poisson point process on the space of doubly infinite $\mathbb{Z}^d$-valued trajectories modulo time-shift, tending to infinity at positive and negative infinite times. The non-negative parameter u mentioned above, roughly comes as a multiplicative factor of the intensity measure of the Poisson point process. Actually, one simultaneously constructs on a suitable probability space $(\Omega, \mathcal{A}, \mathbb{P})$, the whole family $\mathcal{I}^u$, $u \geq 0$, of random interlacements at level $u \geq 0$, cf. (1.30). They are the traces on $\mathbb{Z}^d$ of the trajectories modulo time-shift in the cloud having labels at most u. The complement $\mathcal{V}^u$ of $\mathcal{I}^u$ in $\mathbb{Z}^d$ is the vacant set at level u. It satisfies the identity:

$$(0.1) \qquad \mathbb{P}[\mathcal{V}^u \supseteq K] = \exp\{-u \operatorname{cap}(K)\}, \text{ for all finite } K \subseteq \mathbb{Z}^d .$$

In fact this formula provides a characterization of the law on $\{0,1\}^{\mathbb{Z}^d}$ of the indicator function of $\mathcal{V}^u$, cf. (2.16) of [18]. From Theorem 3.5 of [18] and Theorem 3.4 of [16], one knows that there is a critical value u_* in $(0, \infty)$ such that

(0.2)
- i) for $u > u_*$, $\mathbb{P}$-a.s., all connected components of $\mathcal{V}^u$ are finite,
- ii) for $u < u_*$, $\mathbb{P}$-a.s., there exists an infinite connected component in $\mathcal{V}^u$.

From Theorem 0.1 of [21], one has the following asymptotic lower bound on u_*, as d tends to infinity:

$$(0.3) \qquad \liminf_d \; u_*/\log d \geq 1 .$$

The main object of the present article is to show that the above lower bound does capture the correct asymptotic behavior of u_*, and the following statement holds

Theorem 0.1.

$$(0.4) \qquad \lim_d \; u_*/\log d = 1 .$$

As a by-product this result shows that u_* has the same principal asymptotic behavior as the critical value attached to random interlacements on $2d$-regular trees, when d goes to infinity, see Proposition 5.2 of [23]. We refer the reader to Remark 4.1 for more on this matter. In addition the proof of Theorem 0.1 also shows, cf. Remark 4.1, that

$$\lim_d u_{**}/\log d = 1\,, \tag{0.5}$$

where $u_{**} \in [u_*, \infty)$ is another critical value introduced in [19]. Informally, u_{**} is the critical level above which there is a polynomial decay in L, for the probability of existence of a vacant crossing between a box of side-length L and the complement of a concentric box of double side-length. It is an important and presently unresolved question whether $u_* = u_{**}$ actually holds. However it is known that the connectivity function of the vacant set at level u, i.e. the probability that 0 and a (far away) x are linked by a path in $\mathcal{V}^u$, (i.e. the probability of a vacant crossing at level u between 0 and x), has a stretched exponential decay in x, when u is bigger than u_{**}, see Theorem 0.1 of [17].

We will briefly comment on the proof of Theorem 0.1. In view of (0.3), we only need to show that

$$\limsup_d \; u_*/\log d \le 1\,. \tag{0.6}$$

As for Bernoulli bond or site percolation, similarities between what happens on $\mathbb{Z}^d$ and on $2d$-regular trees for large d, lurk in the background of the proof. The statement corresponding to (0.6) for Bernoulli percolation is an asymptotic lower bound for the critical probability, (a lower bound, and not an upper bound, because the density of $\mathcal{V}^u$ decreases with u). Whereas the required lower bound in the Bernoulli percolation context follows from a short Peierls-type argument, cf. [3], p. 640, [11], p. 222, or [8], p. 25, the proof of (0.6) for random interlacements is quite involved. The long range dependence present in the model is deeply felt.

An important feature of working in high dimension is that the ℓ^1-, the Euclidean, and the ℓ^∞-distances, all behave very differently on $\mathbb{Z}^d$, see (1.1). At large enough scale (i.e. Euclidean distance at least d), the Green function of the simple random walk "feels the invariance principle", and is well-controlled by expressions of the type $(c\sqrt{d}/|\cdot|)^{d-2}$, where c does not depend on d and $|\cdot|$ stands for the Euclidean norm, see Lemma 1.1. However at shorter range the walk feels more of the tree-like nature of the space, and the use of bounds involving the ℓ^1-distance becomes more pertinent, cf. (1.14) and Remark 1.2.

The above dichotomy permeates throughout the proof of (0.6). We use a modification of the renormalization scheme ("for fixed d") employed in [17]. The renormalization scheme enables us to transform certain local controls on the probability of vacant crossings at level $u_0 = (1+5\varepsilon)\log d$, $\varepsilon > 0$ small, into controls on the probability of vacant crossings at arbitrary large scales at a bigger level $u_\infty < (1+10\varepsilon)\log d$.

The local estimates entering the initial step of the renormalization scheme are developed in Section 3. They involve controls on the existence of vacant crossings moving at ℓ^1-distance $c(\varepsilon)d$ from a box of side-length $L_0 = d$, for the interlacement at level u_0. The $2d$-regular tree model lurks behind the control of these local crossings. The key estimates appear in Theorem 3.1 and Corollary 3.4. They result from an enhanced Peierls-type argument involving the consideration of what happens in $\frac{c}{\varepsilon^2}$ many ℓ^1-balls, having each an ℓ^1-radius $c'\varepsilon d$, and lying at mutual ℓ^1-distances at least $c''d$. For this step part of the difficulty stems from the fact that the local estimates need to be strong enough to beat the combinatorial complexity involved in the selection of the dyadic trees entering the renormalization scheme.

The renormalization scheme is developed in Section 2. It propagates along an increasing sequence of levels u_n, with initial value $u_0 = (1 + 5\varepsilon) \log d$ and limiting value $u_\infty < (1 + 10\varepsilon) \log d$, uniform estimates on the probability of events involving the presence of certain vacant crossings at level u_n. Roughly speaking these events correspond to the presence in 2^n boxes of side-length $L_0 (= d)$ of paths in $\mathcal{V}^{u_n}$. The boxes can be thought of as the "bottom leaves" of a dyadic tree of depth n, and are well "spread-out" within a box of side-length $3L_n$, where $L_n = \ell_0^n L_0$, and $\ell_0 = d$. The paths start in each of the 2^n boxes of side-length L_0, and move at Euclidean (and hence ℓ^1-) distance of order $c(\varepsilon)d$ from the boxes. The estimates are conducted uniformly over the possible dyadic trees involved, cf. Propositions 2.1 and 2.3. The main induction step in the above procedure, cf. Proposition 2.1, relies on the sprinkling technique introduced in [18], to control the long range interactions. The rough idea is to introduce more trajectories in the interlacement by letting the levels slightly increase along the convergent sequence u_n. In this fashion one dominates the long range dependence induced by trajectories of the interlacement traveling between far away boxes. In the present context the method uses in an essential way quantitative estimates on Harnack constants in large Euclidean balls, when the dimension d goes to infinity. These estimates crucially enter the proof of Proposition 2.3. The bounds on the Harnack constants are derived in Proposition 1.3 with the help of the general Lemma A.2 from the Appendix, which is an adaptation of Lemma 10.2 in Grigoryan-Telcs [7].

Let us now describe how this article is organized.

In Section 1 we introduce notation and recall several useful facts concerning random walks and random interlacements. An important role is played by the Green function bounds, see Lemma 1.1, and by the bounds on Harnack constants, see Proposition 1.3.

In Section 2 we develop the renormalization scheme. It follows with a number of changes the general line of [17]. The key induction step appears in Proposition 2.1. The main consequences of the renormalization scheme for the proof of Theorem 0.1 are stated in Proposition 2.3.

In Section 3 we derive the crucial local control on the existence of vacant crossings at level u_0 traveling at ℓ^1-distance of order some suitable multiple of d. This local control is stated in Theorem 3.1. It enables to produce the required estimate to initiate the renormalization scheme. This estimate can be found in Corollary 3.4.

Section 4 gives the proof of (0.6). Combined with the lower bound (0.3) from [21], it yields Theorem 0.1. We discuss some further questions concerning the asymptotic behavior of u_* for large d in Remark 4.1.

In the Appendix, we first derive in Lemma A.1 an elementary inequality entering the proof of the Green function bounds from Lemma 1.1. We then present in Lemma A.2 a general result of independent interest providing controls on Harnack constants in terms of killed Green functions, for general nearest neighbor Markov chains on graphs.

Finally let us explain the convention we use concerning constants. Throughout the text c or c' denote positive constants with values changing from place to place. These constants are independent of d. The numbered constants $c_0, c_1, \dots$ are fixed and refer to the value pertaining to their first appearance in the text. Dependence of constants on additional parameters appears in the notation. For instance $c(\varepsilon)$ denotes a constant depending on ε.

1 Notation and random walk estimates

In this section we introduce further notation and gather various useful estimates on simple random walk on $\mathbb{Z}^d$, for large d. Controls on the Green function and on Harnack constants in Euclidean balls play an important role in the sequel. They can be found in Lemma 1.1 and Proposition 1.3. We also recall several useful facts concerning random interlacements.

We let $\mathbb{N} = \{0, 1, 2, \dots\}$ stand for the set of natural numbers. Given a non-negative real number a, we let $[a]$ stand for the integer part of a. We denote with $|\cdot|_1$, $|\cdot|$, and $|\cdot|_\infty$, the ℓ^1-, the Euclidean, and the ℓ^∞-norms on $\mathbb{R}^d$. One has the inequalities:

$$(1.1) \qquad |\cdot|_\infty \le |\cdot| \le |\cdot|_1, \quad |\cdot| \le \sqrt{d}\,|\cdot|_\infty, \quad |\cdot|_1 \le \sqrt{d}\,|\cdot|.$$

Unless explicitly stated otherwise, we tacitly assume that $d \ge 3$.

By finite path we mean a sequence $x_0, \dots, x_N$ in $\mathbb{Z}^d$, with $N \ge 1$, which is such that $|x_{i+1} - x_i|_1 = 1$, for $0 \le i < N$. We sometimes write path in place of finite path, when this causes no confusion. We denote by $B(x, r)$ and $S(x, r)$, the closed ball and the closed sphere with radius $r \ge 0$ and center $x \in \mathbb{Z}^d$. In the case of the ℓ^p-distance, with $p = 1$ or ∞, the corresponding objects are denoted by $B_p(x, r)$ and $S_p(x, r)$. For $A, B \subseteq \mathbb{Z}^d$, we write $A+B$ for the set of $x+y$ with x in A and y in B, and $d(A, B) = \inf\{|x-y|; x \in A, y \in B\}$, for the mutual Euclidean distance between A and B. We write $d_p(A, B)$, with $p = 1$ or ∞, when the ℓ^p-distance is used instead. The notation $K \subset\subset \mathbb{Z}^d$, expresses that K is a finite subset of $\mathbb{Z}^d$. When U is a subset of $\mathbb{Z}^d$, we write $|U|$ for the cardinality of U, $\partial U = \{x \in U^c; \exists y \in U, |x-y|_1 = 1\}$ for the boundary of U and $\partial_{\text{int}} U = \{x \in U; \exists y \in U^c, |x-y|_1 = 1\}$, for the interior boundary of U. We also write $\overline{U}$ in place of $U \cup \partial U$.

We denote with W^+ the set of nearest neighbor $\mathbb{Z}^d$-valued trajectories defined for non-negative times and tending to infinity. We write $\mathcal{W}_+$ and X_n, $n \ge 0$, for the canonical σ-algebra and the canonical process on W_+. We denote by θ_n, $n \ge 0$, the canonical shift on W_+, so that $\theta_n(w) = w(\cdot + n)$, for $w \in W_+$ and $n \ge 0$. Since $d \ge 3$, the simple random walk on $\mathbb{Z}^d$ is transient and we write P_x for the restriction to the set W_+ of full measure of the canonical law of the walk starting at $x \in \mathbb{Z}^d$. When ρ is a measure on $\mathbb{Z}^d$, we denote by P_ρ the measure $\sum_{x \in \mathbb{Z}^d} \rho(x) P_x$ and by E_ρ the corresponding expectation. Given $U \subseteq \mathbb{Z}^d$, we write $H_U = \inf\{n \ge 0; X_n \in U\}$, $\widetilde{H}_U = \inf\{n \ge 1; X_n \in U\}$, and $T_U = \inf\{n \ge 0; X_n \notin U\}$, for the entrance time in U, the hitting time of U, and the exit time from U. In case of a singleton $\{x\}$, we simply write H_x and $\widetilde{H}_x$ for simplicity.

We let $g(\cdot, \cdot)$ stand for the Green function:

$$(1.2) \qquad g(x, x') = \sum_{n \ge 0} P_x[X_n = x'], \text{ for } x, x' \text{ in } \mathbb{Z}^d.$$

The Green function is symmetric in its two variables, and due to translation invariance $g(x, x') = g(x' - x) = g(x - x')$, where

$$(1.3) \qquad g(x) = g(x, 0) = g(0, x), \text{ for } x \in \mathbb{Z}^d.$$

Whereas the ℓ^1-distance is relevant for the description of the short range behavior of $g(\cdot)$ in high dimension, cf. Remark 1.3 1) of [21], and Remark 1.2 below, the Euclidean distance becomes relevant in the description of the "mid-to-long-range" behavior of $g(\cdot)$. The next lemma will be repeatedly used in the sequel. We recall that the convention concerning constants is stated at the end of the Introduction.

Lemma 1.1.

$$g(x) \le (c_0 \sqrt{d}/|x|)^{d-2}, \text{ for } |x| \ge d, \tag{1.4}$$

$$g(x) \ge (c_1 \sqrt{d}/|x|)^{d-2}, \text{ for } |x|^2 \ge d\,|x|_\infty > 0, \text{ (and in particular when } |x| \ge d), \tag{1.5}$$

$$P_x[H_{B(0,L)} < \infty] \le \Big(\frac{cL}{|x|}\Big)^{d-2} \wedge 1, \text{ for } L \ge d, x \in \mathbb{Z}^d, \text{ (with } c \ge 1). \tag{1.6}$$

Proof. We begin with the proof of (1.4), (1.5). To this end we denote with $p_t(u,v), t \ge 0$, $u, v \in \mathbb{Z}$, the transition probability of the simple random walk in continuous time on $\mathbb{Z}$ with exponential jumps of parameter 1. The transition probability of the simple random walk on $\mathbb{Z}^d$ with exponential jumps of parameter d can then be expressed as the product of one-dimensional transition probabilities. Relating the continuous and the discrete time random walks on $\mathbb{Z}^d$, we thus find that

$$g(x) = d \int_0^\infty \prod_{i=1}^d p_t(0, x_i) dt, \text{ for } x = (x_1, \dots, x_d) \in \mathbb{Z}^d\,. \tag{1.7}$$

From Theorem 3.5 of [15], and the fact that the function

$$F(\gamma) = -\log\big(\gamma + \sqrt{\gamma^2+1}\big) + \frac{1}{\gamma}\,\big(\sqrt{\gamma^2+1} - 1\big), \; \gamma > 0\,,$$

that appears in Theorem 3.5 of [15], has derivative $-(1+\sqrt{\gamma^2+1})^{-1}$, tends to 0 in $\gamma = 0$, and thus satisfies the inequality $\log(1+\frac{\gamma}{2}) \le -F(\gamma) \le \log(1+\gamma)$, for $\gamma \ge 0$, we see that for suitable constants $0 < \kappa < 1 < \kappa'$, we have

$$\begin{aligned} &\frac{1}{\kappa'}\,(1 \vee t \vee |u|)^{-\frac{1}{2}} \exp\Big\{-|u| \log\Big(1 + \kappa'\,\frac{|u|}{t}\Big)\Big\} \le p_t(0,u) \le \\ &\frac{1}{\kappa}\,(1 \vee t \vee |u|)^{-\frac{1}{2}} \exp\Big\{-|u| \log\Big(1 + \kappa\,\frac{|u|}{t}\Big)\Big\}, \text{ for } t > 0, u \in \mathbb{Z}. \end{aligned} \tag{1.8}$$

We now prove (1.4) and thus assume $|x| \ge d$. By (1.7), (1.8), we bound $g(x)$ from above as follows, (we also use the inequality $d \le 2^d$ and Lemma A.1 in the Appendix):

$$\begin{aligned} g(x) &\le c^d \int_0^\infty (1 \vee t)^{-\frac{d}{2}} \exp\Big\{-\sum_{i=1}^d |x_i| \log\Big(1 + \kappa\,\frac{|x_i|}{t}\Big)\Big\}\,dt \\ &\overset{(A.1)}{\le} c^d \int_0^\infty (1 \vee t)^{-\frac{d}{2}} \exp\Big\{-|x| \log(1 + \kappa\,\frac{|x|}{t})\Big\}\,dt \\ &\le c^d \int_0^{\kappa|x|} (1 \vee t)^{-\frac{d}{2}} \exp\Big\{-|x| \log\Big(1 + \kappa\,\frac{|x|}{t}\Big)\Big\}\,dt \\ &\quad + c^d \int_{\kappa|x|}^\infty t^{-\frac{d}{2}} \exp\Big\{-\frac{\kappa|x|^2}{2t}\Big\}\,dt\,, \end{aligned} \tag{1.9}$$

where in the last step we used the inequality $\log(1+\gamma) \ge \frac{\gamma}{2}$, for $0 \le \gamma \le 1$. Performing the change of variable $s = \frac{\kappa |x|^2}{2t}$ in the last integral, we see that the last term of (1.9) is smaller than

$$(1.10) \qquad c^d\,|x|^{2-d} \int_0^{\frac{|x|}{2}} s^{\frac{d}{2}-2}\, e^{-s}\, ds \le c^d\,|x|^{2-d}\,\Gamma\Big(\frac{d}{2}-1\Big) \le (c\,\sqrt{d}\,/\,|x|)^{d-2}\,,$$

using the asymptotic behavior of the Gamma function in the last step, cf. [14], p. 88.

As for the first integral in the last line of (1.9), we note that for $1 \le s \le \kappa|x|$, the function $s \to -\frac{d}{2}\,\log s - |x|\,\log(1+\kappa\frac{|x|}{s})$ has derivative

$$-\frac{d}{2s} + \frac{|x|}{s}\;\frac{\kappa|x|}{s+\kappa|x|} \overset{s\le\kappa|x|}{\ge} -\frac{d}{2s} + \frac{|x|}{2s} \overset{|x|\ge d}{\ge} 0\,,$$

and hence is non-decreasing. Thus the first term in the last line of (1.9) is smaller than $c^d(\kappa|x|)^{-(\frac{d}{2}-1)}2^{-|x|}$.

Observe that for $a \ge d$, $\frac{d-2}{2}\,\log a + a\log 2 \ge (d-2)\log\frac{a}{\sqrt{d}}$, (indeed this inequality holds for $a=d$ and $\frac{d-2}{2a} + \log 2 \ge \frac{d-2}{a}$, for $a \ge d$). It follows that the first term in the last line of (1.9) is at most $(c\sqrt{d}/|x|)^{d-2}$. Together with (1.10) this concludes the proof of (1.4).

We now prove (1.5), and assume $x \ne 0$. Since $\log(1+\gamma) \le \gamma$, for $\gamma \ge 0$, and $\kappa' > 1$, it follows from (1.7), (1.8) that

$$(1.11) \qquad \begin{aligned} g(x) &\ge c^d \int_{\kappa'|x|_\infty}^{\infty} t^{-\frac{d}{2}}\,\exp\Big\{-\kappa'\,\frac{|x|^2}{t}\Big\}\,dt \overset{s=\frac{\kappa'|x|^2}{t}}{\ge} c^d|x|^{2-d}\int_0^{\frac{|x|^2}{|x|_\infty}} s^{\frac{d}{2}-2}e^{-s}ds \\ &\ge \Big(\frac{c\sqrt{d}}{|x|}\Big)^{d-2}, \text{ when } |x|^2 \ge d\,|x|_\infty\,, \end{aligned}$$

and (1.5) follows.

Finally (1.6) is a routine consequence of the identity

$$(1.12) \qquad g(x) = E_x[g(X_{H_{B(0,L)}}),\; H_{B(0,L)} < \infty], \text{ for } L \ge 0 \text{ and } x \in \mathbb{Z}^d\,,$$

combined with (1.4), (1.5), and the fact that $\inf_{B(0,L)} g \ge \inf_{\partial B(0,L)} g$. □

Remark 1.2.

1) Although we will not need this fact in the sequel, let us mention that the following lower bound complementing (1.6) holds as well:

$$(1.13) \qquad P_x[H_{B(0,L)} < \infty] \ge \Big(\frac{cL}{|x|}\Big)^{d-2} \wedge 1, \quad \text{for } L \ge d,\; x \in \mathbb{Z}^d,\; (\text{with } c \le 1)\,.$$

Indeed one uses (1.12) together with (1.4), (1.5), and when $d+1 \ge L (\ge d)$, the inequality $\sup_{\partial_{\rm int} B(0,L)} g \le 2d\,\sup_{\partial B(0,L)} g$, which follows from the fact that g is harmonic outside the origin, (the factor $2d$ can then be dominated by $\widetilde{c}^{\,d-2}$).

2) Let us point out that when $x = ([d^\alpha], 0, \dots, 0)$, with $\frac{1}{2} < \alpha < 1$, the upper bound (1.4) does not hold when $d \ge c(\alpha)$. Indeed it follows from (1.7), (1.8) that

$$g(x) \ge d \int_1^2 p_t(0, [d^\alpha])\, p_t(0,0)^{d-1} dt \overset{(1.8)}{\ge} c^d\, d^{-\frac{\alpha}{2}} \exp\{-d^\alpha \log(1 + \kappa' d^\alpha)\}\,,$$

which is much bigger than $(c_0 \sqrt{d}/|x|)^{d-2} \le c^{d-2} \exp\{-(\alpha - \frac{1}{2})(d-2)\log d\}$, for $d \ge c(\alpha)$.

3) We recall from (1.11) of [21], that when $d \ge 5$,

$$g(x) \le \Big(\frac{c_2 d}{|x|_1}\Big)^{\frac{d}{2}-2}, \text{ for } x \in \mathbb{Z}^d\,. \tag{1.14}$$

The inequality is for instance useful when $|x| < d$, but $|x|_1 \ge c_2 d$, a situation where (1.4) is of no help. We will use (1.14) in Section 3, when deriving local bounds on the connectivity function of random interlacements at a level u_0 close to $\log d$, see the proof of Theorem 3.1.

4) The asymptotic behavior of $g(x)$ for d fixed and large x is well-known, see for instance [10], p. 313, or [12], p. 31:

$$\lim_{x\to\infty} \frac{g(x)}{|x|^{d-2}} = \frac{d}{2}\, \Gamma\Big(\frac{d}{2} - 1\Big)\, \pi^{-\frac{d}{2}}\,.$$

The asymptotic behavior of $g(\cdot)$ at the origin, or close to the origin when d tends to infinity is also well-known, see for instance [13], p. 246, or [21], Remark 1.3 1). On the other hand the behavior of $g(\cdot)$ at intermediate scales when d tends to infinity seems much less explored. □

The bounds on the Green function of Lemma 1.1 together with Lemma A.2 from the Appendix enable us to derive quantitative controls on Harnack constants in suitably large Euclidean balls. These bounds will be instrumental for the renormalization scheme developed in the next section, see the proof of Lemma 2.2. First we recall some terminology. When $U \subseteq \mathbb{Z}^d$, we say that a function u defined on $\overline{U}$ is harmonic in U, if for all $x \in U$, $u(x) = \frac{1}{2d} \sum_{|e|=1} u(x+e)$. We can now state:

Proposition 1.3. $(L \ge d)$

Setting $c_3 = 4 + 10\frac{c_0}{c_1}$ (where $c_0 \ge c_1$, see (1.4), (1.5)), there exists $c > 1$, such that when u is a non-negative function defined on $\overline{B(0, c_3 L)}$ harmonic in $B(0, c_3 L)$, one has

$$\max_{B(0,L)} u \le c^d \min_{B(0,L)} u\,. \tag{1.15}$$

Proof. We define $U_1 = B(0,L) \subseteq U_2 = B(0,4L) \subseteq U_3 = B(0, c_3 L)$. In view of Lemma A.2 from the Appendix, any u as above satisfies the inequality

$$\max_{U_1} u \le K \min_{U_1} u\,,$$

where

$$K = \max_{x,y \in U_1}\ \max_{z \in \partial_{\mathrm{int}} U_2} G_{U_3}(x,z)/G_{U_3}(y,z)\,, \tag{1.16}$$

and $G_{U_3}(\cdot,\cdot)$ stands for the Green function of the walk killed outside U_3, cf. (A.8). Applying the strong Markov property at time T_{U_3} and (1.2) we obtain the identity:

$$G_{U_3}(y,z) = G(y,z) - E_y[G(X_{T_{U_3}}, z)], \text{ for } y,z \in \mathbb{Z}^d .$$

Hence when $x, y \in U_1$ and $z \in \partial_{\text{int}} U_2$, we see that

$$(1.17) \qquad G_{U_3}(x,z) \le G(x,z) \overset{(1.4)}{\le} (c_0\sqrt{d}/(2L))^{d-2}, \text{ and}$$

$$\begin{aligned}(1.18) \qquad G_{U_3}(y,z) &\ge (c_1\sqrt{d}/(5L))^{d-2} - \{c_0\sqrt{d}/((c_3-4)L)\}^{d-2} \\ &= \Big(\frac{\sqrt{d}}{L}\Big)^{d-2}\Big(\Big(\frac{c_1}{5}\Big)^{d-2} - \Big(\frac{c_0}{c_3-4}\Big)^{d-2}\Big) \\ &= \Big(\frac{\sqrt{d}}{L}\Big)^{d-2}\Big(\frac{c_1}{5}\Big)^{d-2}\Big(1 - \Big(\frac{1}{2}\Big)^{d-2}\Big) .\end{aligned}$$

We thus find that $K \le 2(\frac{5}{2}\,\frac{c_0}{c_1})^{d-2}$, and the claim (1.15) follows. □

We now briefly review some notation and basic properties concerning the equilibrium measure and the capacity. Given $K \subset\subset \mathbb{Z}^d$, we write e_K for the equilibrium measure of K, and $\text{cap}(K)$ for its total mass the capacity of K:

$$(1.19) \qquad e_K(x) = P_x[\widetilde{H}_K = \infty]\, 1_K(x), x \in \mathbb{Z}^d, \text{cap}(K) = \sum_{x\in K} P_x[\widetilde{H}_K = \infty] .$$

The capacity is subadditive (a straightforward consequence of (1.19)):

$$(1.20) \qquad \text{cap}(K \cup K') \le \text{cap}(K) + \text{cap}(K'), \text{ for } K, K' \subset\subset \mathbb{Z}^d.$$

One can also express the probability to enter K in the following well-known fashion:

$$(1.21) \qquad P_x[H_K < \infty] = \sum_{y \in K} g(x,y)\, e_K(y), \text{ for } x \in \mathbb{Z}^d .$$

Further we have the bound on the capacity of Euclidean balls

$$(1.22) \qquad \text{cap}(B(0,L)) \le \Big(\frac{cL}{\sqrt{d}}\Big)^{d-2}, \text{ for } L \ge d ,$$

which follows from (1.5), (1.6), and (1.21), by letting x tend to infinity.

Remark 1.4. Although we will not need this estimate in the sequel, let us mention that in an analogous way with (1.4), (1.13), and (1.21), one finds that

$$(1.23) \qquad \text{cap}(B(0,L)) \ge \Big(\frac{cL}{\sqrt{d}}\Big)^{d-2}, \text{ for } L \ge d .$$

□

We then turn to the description of random interlacements. We refer to Section 1 of [18] for details. We denote with W the space of doubly infinite nearest-neighbor $\mathbb{Z}^d$-valued trajectories, which tend to infinity at positive and negative infinite times, and by W^* the space of equivalence classes of trajectories in W modulo time-shift. We let π^* stand for the canonical map from W into W^*. We write $\mathcal{W}$ for the canonical σ-algebra on W generated by the canonical coordinates $X_n, n \in \mathbb{Z}$, and by $\mathcal{W}^* = \{A \subseteq W^*; (\pi^*)^{-1}(A) \in \mathcal{W}\}$, the largest σ-algebra on W^* for which $\pi^* : (W, \mathcal{W}) \to (W^*, \mathcal{W}^*)$ is measurable. The canonical probability space for random interlacements is now the following.

We consider the space of point measures on $W^* \times \mathbb{R}_+$:

$$\begin{aligned}\Omega = \Big\{\omega = \sum_{i\geq 0} \delta_{(w_i^*, u_i)}, &\text{ with } (w_i^*, u_i) \in W^* \times \mathbb{R}_+,\ i \geq 0, \text{ and } u \geq 0, \\ &w(W_K^* \times [0,u]) < \infty, \text{ for any } K \subset\subset \mathbb{Z}^d, \text{ and } u \geq 0\Big\},\end{aligned} \tag{1.24}$$

where for $K \subset\subset \mathbb{Z}^d$, $W_K^* \subseteq W^*$ stands for the set of trajectories modulo time-shift that enter K, i.e. $W_K^* = \pi^*(W_K)$, where W_K is the subset of W of trajectories that enter K.

We endow Ω with the σ-algebra $\mathcal{A}$ generated by the evaluation maps: $\omega \to \omega(D)$, where D runs over the σ-algebra $\mathcal{W}^* \times \mathcal{B}(\mathbb{R}_+)$, and with the probability $\mathbb{P}$ on $(\Omega, \mathcal{A})$, which is the Poisson measure with intensity $\nu(d\omega^*)du$ giving finite mass to the sets $W_K^* \times [0,u]$, for $K \subset\subset \mathbb{Z}^d$, $u \geq 0$, with ν the unique σ-finite measure on $(W^*, \mathcal{W}^*)$ such that for any $K \subset\subset \mathbb{Z}^d$, see Theorem 1.1. of [18]:

$$1_{W_K^*}\, \nu = \pi^* \circ Q_K\,, \tag{1.25}$$

where Q_K denotes the finite measure on W_K^0, the subset of W_K of trajectories, which are for the first time in K at time 0, and such that for $A, B \in \mathcal{W}_+$, (we recall that $\mathcal{W}_+$ is defined above (1.2)),and $x \in \mathbb{Z}^d$:

$$Q_K[(X_{-n})_{n\geq 0} \in A,\ X_0 = x,\ (X_n)_{n\geq 0} \in B] = P_x[A|\widetilde{H}_K = x]\, e_K(x)\, P_x[B]\,. \tag{1.26}$$

For $K \subset\subset \mathbb{Z}^d$, $u \geq 0$, one defines on $(\Omega, \mathcal{A})$ the random variable valued in the set of finite point measures on $(W_+, \mathcal{W}_+)$:

$$\mu_{K,u}(dw) = \sum_{i\geq 0} \delta_{(w_i^*)^{K,+}} 1\{w_i^* \in W_K^*, u_i \leq u\}, \text{ for } \omega = \sum_{i\geq 0} \delta_{(w_i^*, u_i)} \in \Omega\,, \tag{1.27}$$

where for $w^* \in W_K^*$, $(w^*)^{K,+}$ stands for the trajectory in W_+, which follows step by step w^* from the first time it enters K.

When $0 \leq u' < u$, one defines $\mu_{K,u',u}(dw)$ in an analogous fashion as in (1.27), replacing the condition $u_i \leq u$ with $u' < u_i \leq u$, in the right-hand side of (1.27). Then for $0 \leq u' < u$, $K \subset\subset \mathbb{Z}^d$, one finds that

(1.28) $\mu_{K,u',u}$ and $\mu_{K,u'}$ are independent Poisson point processes with respective intensity measures $(u - u')\, P_{e_K}$ and $u' P_{e_K}$.

In addition one has the identity

$$\mu_{K,u} = \mu_{K,u'} + \mu_{K,u',u}\,. \tag{1.29}$$

Given $\omega \in \Omega$, the interlacement at level $u \geq 0$ is the subset of $\mathbb{Z}^d$:

$$\begin{aligned}\mathcal{I}^u(\omega) &= \bigcup_{u_i \leq u} \text{range } (w_i^*), \text{ if } \omega = \sum_{i\geq 0} \delta_{(w_i^*, u_i)} \\ &= \bigcup_{K\subset\subset \mathbb{Z}^d}\ \bigcup_{w \in \text{Supp}\, \mu_{K,u}(\omega)} w(\mathbb{N})\,,\end{aligned} \tag{1.30}$$

where for $w^* \in W^*$, range$(w^*) = w(\mathbb{N})$, for any $w \in W$, with $\pi^*(w) = w^*$, and $\mathrm{Supp}\,\mu_{K,u}(\omega)$ refers to the support of the point measure $\mu_{K,u}(\omega)$. The vacant set at level u is the complement of $\mathcal{I}^u(\omega)$:

$$\mathcal{V}^u(\omega) = \mathbb{Z}^d \backslash \mathcal{I}^u(\omega), \text{ for } u \in \Omega, u \geq 0\,. \tag{1.31}$$

One also has, cf. (1.54) of [18],

$$\mathcal{I}^u(\omega) \cap K = \bigcup_{w \in \mathrm{Supp}\,\mu_{K',u}(\omega)} w(\mathbb{N}) \cap K, \text{ for } K \subset K' \subset\subset \mathbb{Z}^d, u \geq 0\,. \tag{1.32}$$

From (1.28) one readily finds that as mentioned in (0.1):

$$\mathbb{P}[\mathcal{V}^u \supseteq K] = \exp\{-u\,\mathrm{cap}(K)\}, \text{ for all } K \subset\subset \mathbb{Z}^d\,, \tag{1.33}$$

an identity that characterizes the law Q_u on $\{0,1\}^{\mathbb{Z}^d}$ of the indicator function of $\mathcal{V}^u(\omega)$, see also Remark 2.2 2) of [18]. This brings us to the conclusion of Section 1 and of this brief review of some useful facts that we will use in the next sections.

2 From local to global: the renormalization scheme

We develop in this section a renormalization scheme that follows in its broad lines the strategy of [17]. We introduce a geometrically increasing sequence of length scales L_n, $n \geq 0$, and an increasing but typically convergent sequence of levels $u_n, n \geq 0$. When the sequence u_n is sufficiently increasing, cf. (2.19), we are able to propagate from scale to scale bounds on the key quantities $p_n(u_n)$ that appear in (2.17). Roughly speaking these controls provide uniform upper bounds on the probability that in a box at scale L_n, 2^n "well-spread" boxes at scale L_0 all witness certain crossing events at Euclidean distance of order cL_0 in the vacant set at level u_n. Interactions are handled by the sprinkling technique originally introduced in Section 3 of [18]. The renormalization scheme enables us to transform local estimates on the existence of vacant crossings at scale L_0 in the vacant set a level u_0 into global estimates on crossings at arbitrary scales in the vacant set at level $u_\infty = \lim u_n$. The difficulty we encounter in the implementation of the scheme stems from the fact that we want both u_0 and u_∞ to be "slightly above" the critical value u_*, see (4.7) and (0.3). However the local controls on vacant crossings at level u_0, which we inject into the renormalization scheme, and develop in the next section, require L_0 to be rather small, i.e. of order d. We are then forced to keep a tight control on the estimates we derive when d goes to infinity. The Green function and entrance probability estimates from Lemma 1.1 together with the bounds on Harnack constants in Euclidean balls from Proposition 1.3 play a pivotal role in this scheme. The fact that the ℓ^∞- and the Euclidean distances behave very differently for large d, see (1.1), also forces upon us some modifications of the geometric constructions in [17], see for instance (2.1) and (2.26). The main results of this section are Proposition 2.1, which contains the main induction step, and Proposition 2.3, which encapsulates the estimates we will use in Section 4.

We consider the length scales

$$L_0 \geq d,\ \widehat{L}_0 = (\sqrt{d} + R)\, L_0, \text{ with } R \geq 1, \tag{2.1}$$

as well as

$$L_n = \ell_0^n\, L_0, \text{ for } n \ge 1, \text{ where } \ell_0 \ge 1000\, \tfrac{c_0}{c_1}\, (\sqrt{d} + R) \text{ is an integer multiple of 100, (we recall that } c_0 \ge c_1, \text{ cf. Lemma 1.1).} \tag{2.2}$$

We organize $\mathbb{Z}^d$ in a hierachical way with L_0 the finest scale and $L_1 < L_2 < \dots$, coarser and coarser scales. Crossing events at the finest scale will involve the length scale $\widehat{L}_0$. We introduce the set of labels of boxes at level $n \ge 0$:

$$I_n = \{n\} \times \mathbb{Z}^d\,. \tag{2.3}$$

To each $m = (n,i) \in I_n$, $n \ge 0$, we attach the box

$$C_m = (iL_n + [0, L_n)^d) \cap \mathbb{Z}^d\,. \tag{2.4}$$

In addition when $n \ge 1$, we define

$$\widetilde{C}_m = \bigcup_{m' \in I_n, d_\infty(C_{m'}, C_m) \le 1} C_{m'} (\supseteq C_m)\,. \tag{2.5}$$

On the other hand when $n = 0$ and $m = (0,i) \in I_0$, we define instead

$$\widetilde{C}_m = B(iL_0, \widehat{L}_0) \overset{(1.1),(2.1)}{\supseteq} \bigcup_{x \in C_m} B(x, RL_0) (\supseteq C_m)\,. \tag{2.6}$$

The above definitions slightly differ from (2.3) in [17] due to the special treatment of the scale $n = 0$. It is relevant here to use Euclidean balls, and thanks to (1.6) of Lemma 1.1, have a good control on the entrance probability of a simple random walk in $\widetilde{C}_m$. The radius of these balls has to be chosen sufficiently large, so that we can show that crossing events at the bottom scale, from C_m to $\partial_{\rm int} \widetilde{C}_m$, are unlikely, (this will be done in the next Section 3).

We then write $S_m = \partial_{\rm int} C_m$ and $\widetilde{S}_m = \partial_{\rm int} \widetilde{C}_m$, for $m \in I_n$, $n \ge 0$. Given $m \in I_n$, with $n \ge 1$, we consider $\mathcal{H}_1(m)$, $\mathcal{H}_2(m) \subseteq I_{n-1}$ defined by:

$$\begin{aligned} \mathcal{H}_1(m) &= \{\overline{m} \in I_{n-1};\; C_{\overline{m}} \subseteq C_m \text{ and } C_{\overline{m}} \cap S_m \neq \emptyset\} \\ \mathcal{H}_2(m) &= \big\{\overline{m} \in I_{n-1};\; C_{\overline{m}} \cap \big\{z \in \mathbb{Z}^d;\; d_\infty(z, C_m) = \tfrac{L_n}{2}\big\} \neq \emptyset\big\}\,. \end{aligned} \tag{2.7}$$

We thus see that for $n \ge 1$, $m \in I_n$, one has

$$\begin{array}{l} \overline{m}_1 \in \mathcal{H}_1(m),\; \overline{m}_2 \in \mathcal{H}_2(m) \text{ implies that} \\ \widetilde{C}_{\overline{m}_1} \cap \widetilde{C}_{\overline{m}_2} = \emptyset, \text{ and } \widetilde{C}_{\overline{m}_1} \cup \widetilde{C}_{\overline{m}_2} \subseteq \widetilde{C}_m\,, \end{array} \tag{2.8}$$

(in the case $n = 1$, we use the lower bound on ℓ_0 in (2.2) as well as (1.1)).

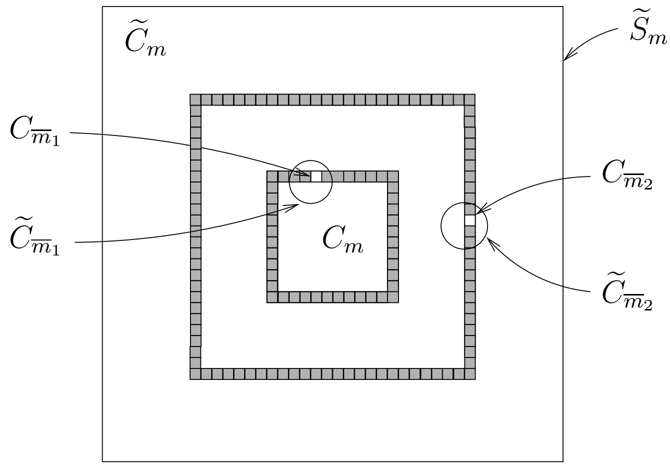

Fig. 1: An illustration of the boxes $C_{\overline{m}_i}$ and balls $\widetilde{C}_{\overline{m}_i}$, $i = 1, 2$, when m belongs to I_1.

Then to each $m \in I_n$, $n \geq 0$, we associate a collection Λ_m of "binary trees of depth n". More precisely we define Λ_m as the collection of subsets $\mathcal{T}$ of $\bigcup_{0 \leq k \leq n} I_k$, such that writing $\mathcal{T}^k = \mathcal{T} \cap I_k$, one has

$$\mathcal{T}^n = \{m\}, \tag{2.9}$$

(2.10) any $m' \in \mathcal{T}^k$, $1 \leq k \leq n$, has two "descendants" $\overline{m}_i(m') \in \mathcal{H}_i(m')$, $i = 1, 2$, such that $\mathcal{T}^{k-1} = \bigcup_{m' \in \mathcal{T}^k} \{\overline{m}_1(m'), \overline{m}_2(m')\}$.

For each $\mathcal{T} \in \Lambda_m$ and $m' \in \mathcal{T}$, one can then define the subtree of "descendants of m' in $\mathcal{T}$" via:

$$\mathcal{T}_{m'} = \{m'' \in \mathcal{T};\ \widetilde{C}_{m''} \subseteq \widetilde{C}_{m'}\} (\in \Lambda_{m'}). \tag{2.11}$$

Given $1 \leq k \leq n$, $m' \in \mathcal{T}^k$, one thus has the partition of $\mathcal{T}_{m'}$:

$$\mathcal{T}_{m'} = \{m'\} \cup \mathcal{T}_{\overline{m}_1(m')} \cup \mathcal{T}_{\overline{m}_2(m')}. \tag{2.12}$$

In addition we have the following rough bound on the collection Λ_m of binary trees attached to $m \in I_n$:

$$|\Lambda_m| \leq (c_4\, \ell_0)^{2(d-1)} (c_4\, \ell_0)^{4(d-1)} \dots (c_4\, \ell_0)^{2^n(d-1)} = (c_4\, \ell_0)^{2(d-1)(2^n-1)}, \tag{2.13}$$

where we used the rough bound for $m' \in I_k$, $1 \leq k \leq n$, and $i = 1, 2$:

$$|\mathcal{H}_i(m')| \leq 2d\Big(c\, \frac{L_k}{L_{k-1}}\Big)^{d-1} = 2d(c\ell_0)^{d-1} \leq (c_4\, \ell_0)^{d-1}, \text{ for some } c_4 > 1.$$

We then introduce for $u \geq 0$, $m \in I_n$, with $n \geq 0$, the event

$$A_m^u = \big\{C_m \stackrel{\mathcal{V}^u}{\longleftrightarrow} \widetilde{S}_m\big\}, \tag{2.14}$$

where the expression in the right-hand side of (2.14) denotes the collection of ω in Ω such that there is a path between C_m and $\widetilde{S}_m$ contained in $\mathcal{V}^u$. In an analogous fashion to

Lemma 2.1 of [17], A^u_m "cascades down to the bottom scale", because any path originating in C_m and ending in $\widetilde{S}_m$ must go through some $C_{\overline{m}_1}$, $\overline{m}_1 \in \mathcal{H}_1(m)$, reach $\widetilde{S}_{\overline{m}_1}$ and then go through some $C_{\overline{m}_2}, \overline{m}_2 \in \mathcal{H}_2(m)$, and reach $\widetilde{S}_{\overline{m}_2}$. Thus similarly to Lemma 2.1 of [17], we find that defining for $u \geq 0$, $n \geq 0$, $m \in I_n$, and $\mathcal{T} \in \Lambda_m$

$$A^u_{\mathcal{T}} = \bigcap_{m' \in \mathcal{T}^0} A^u_{m'}, \text{ (recall } \mathcal{T}^0 = \mathcal{T} \cap I_0) \,, \tag{2.15}$$

one has the inclusion

$$A^u_m \subseteq \bigcup_{\mathcal{T} \in \Lambda_m} A^u_{\mathcal{T}} \,. \tag{2.16}$$

We then introduce the key quantity,

$$p_n(u) = \sup_{\mathcal{T} \in \Lambda_m} \mathbb{P}[A^u_{\mathcal{T}}], \; u \geq 0, n \geq 0, \text{ with } m \in I_n \text{ arbitrary} \,, \tag{2.17}$$

which is well defined due to translation invariance, and find that:

$$\mathbb{P}[A^u_m] \leq |\Lambda_m| \, p_n(u), \text{ for } u \geq 0, n \geq 0 \,. \tag{2.18}$$

The heart of the matter is now to find a recurrence relation bounding $p_{n+1}(u_{n+1})$ in terms of $p_n(u_n)$ for suitably increasing sequences u_n, (we are actually interested in increasing but convergent sequences). We recall that R appears in (2.1).

Proposition 2.1. *There exist positive constants c_5, c_6, c, such that when $\ell_0 \geq c(\sqrt{d} + R)$, then for any increasing sequences u_n, $n \geq 0$, in $(0, \infty)$ and non-decreasing sequences r_n, $n \geq 0$, of positive integers such that*

$$u_{n+1} \geq u_n \Big(1 + \frac{\widehat{L}_0}{L_0} \Big(\frac{c_5}{\ell_0}\Big)^{(n+1)(d-2)}\Big)^{r_n+1}, \; \textit{for all } n \geq 0 \,, \tag{2.19}$$

one has for all $n \geq 0$:

$$p_{n+1}(u_{n+1}) \leq p_n(u_{n+1})(p_n(u_n) + u_n \Big(\frac{\widehat{L}_0}{\sqrt{d}}\Big)^{(d-2)} \Big(4^n \Big(c_6 \, \frac{\widehat{L}_0}{L_0}\Big)^{(d-2)} \ell_0^{-(n+1)(d-2)}\Big)^{r_n}\Big) \,, \tag{2.20}$$

(note that $p_n(\cdot)$ is non-increasing so that $p_n(u_{n+1}) \leq p_n(u_n)$).

Proof. The proof of Proposition 2.1 is an adaptation of the proof of Proposition 2.2 of [17], which will be sketched below, with some modifications, which we will highlight.

One considers some $n \geq 0$, $m \in I_{n+1}$, $\mathcal{T} \in \Lambda_m$, and writes $\overline{m}_1, \overline{m}_2$ for the unique elements of $\mathcal{H}_1(m)$, $\mathcal{H}_2(m)$ in $\mathcal{T}^n$ $(= \mathcal{T} \cap I_n)$. One also writes u' and u, with $0 < u' < u$, in place of u_n and u_{n+1}.

If $\overline{\mathcal{T}} \in \Lambda_{\overline{m}}$, with $\overline{m} \in I_n$, one defines for μ a point process on W_+ defined on Ω, (i.e. a measurable map from Ω into the space of point measures on W_+):

$$A_{\overline{\mathcal{T}}}(\mu) = \bigcap_{m' \in \overline{\mathcal{T}} \cap I_0} \Big\{\omega \in \Omega; \text{ there is a path in } \widetilde{C}_{m'} \backslash \Big(\bigcup_{w \in \mathrm{Supp}\, \mu(\omega)} w(\mathbb{N})\Big) \text{ joining } C_{m'} \text{ with } \widetilde{S}_{m'}\Big\} \,. \tag{2.21}$$

As in (2.19) of [17], using independence we have the bound

$$\mathbb{P}[A^u_{\mathcal{T}}] \le p_n(u)\,\mathbb{P}[A_{\overline{\mathcal{T}}_2}(\mu_{2,2})]\,, \tag{2.22}$$

where $\overline{\mathcal{T}}_2$ stands for $\mathcal{T}_{\overline{m}_2}$, and we have decomposed the point process $\mu_{V,u}$, see (1.27), where

$$V = \widehat{C}_1 \cup \widehat{C}_2, \text{ with} \tag{2.23}$$

$$\widehat{C}_i = \bigcup_{m' \in \overline{\mathcal{T}}_i \cap I_0} \widetilde{C}_{m'} \subseteq \widetilde{C}_{\overline{m}_i}, \text{ for } i = 1,2, \tag{2.24}$$

(i.e. a union of 2^n pairwise disjoint Euclidean balls of radius $\widehat{L}_0$), into a sum of independent Poisson processes via

$$\mu_{V,u} = \mu_{1,1} + \mu_{1,2} + \mu_{2,1} + \mu_{2,2}\,, \tag{2.25}$$

where for $i \neq j$ in $\{1,2\}$, we have set

$$\mu_{i,j} = 1\{X_0 \in \widehat{C}_i, H_{\widehat{C}_j} < \infty\}\,\mu_{V,u} \text{ and } \mu_{i,i} = 1\{X_0 \in \widehat{C}_i, H_{\widehat{C}_j} = \infty\}\,\mu_{V,u}\,.$$

One introduces similar decompositions for $\mu_{V,u'}$ in terms of analogously defined point processes $\mu'_{i,j}$, $1 \le i,j \le 2$, and for $\mu_{V,u',u}$ in terms of $\mu^*_{i,j}$, $1 \le i,j \le 2$.

The heart of the matter is to bound $\mathbb{P}[A_{\overline{\mathcal{T}}_2}(\mu_{2,2})] = \mathbb{P}[A_{\overline{\mathcal{T}}_2}(\mu'_{2,2} + \mu^*_{2,2})]$, which appears in the right-hand side of (2.22), in terms of $p_n(u')$ when $u - u'$ is not too small. For this purpose we employ the sprinkling technique of [18], and loosely speaking establish that $\mu^*_{2,2}$ dominates "up to small corrections" the contribution of $\mu'_{2,1} + \mu'_{1,2}$ in $\mathbb{P}[A^{u'}_{\overline{\mathcal{T}}_2}] = \mathbb{P}[A_{\overline{\mathcal{T}}_2}(\mu'_{2,2} + \mu'_{2,1} + \mu'_{1,2})]$.

With this in mind we define a neighborhood U of $\widetilde{C}_{\overline{m}_2}$, (and in contrast to (2.20) of [17], we do not define U as the ℓ^∞-neighborhood of $\widetilde{C}_{\overline{m}_2}$ of size $\frac{L_{n+1}}{10}$). Instead if $\overline{m}_2 = (n, \overline{i}_2) \in I_n$, see (2.3), we define U as the Euclidean ball, (which is much smaller than the corresponding ℓ^∞-ball of same radius):

$$U = B\Big(\overline{i}_2\, L_n, \frac{L_{n+1}}{10}\Big) \supseteq \widetilde{C}_{\overline{m}_2}, \text{ using (2.1), (2.2), (2.5), (2.6)}\,. \tag{2.26}$$

We then have the following important controls on Euclidean distances:

$$\begin{aligned} d(\partial U, \widehat{C}_2) &\ge \frac{L_{n+1}}{10} - 3\sqrt{d}\, L_n \overset{(2.1),(2.2)}{>} \frac{L_{n+1}}{20}, \text{ when } n \ge 1\,, \\ &\ge \frac{L_{n+1}}{10} - \widehat{L}_0 \overset{(2.1),(2.2)}{>} \frac{L_{n+1}}{20}, \text{ when } n = 0\,, \end{aligned} \tag{2.27}$$

and we have used that $\widehat{C}_2 \subseteq \widetilde{C}_{\overline{m}_2}$, when $m \ge 1$, in the first line, see (2.8). Using similar considerations we find that

$$\begin{aligned} d(\partial U, \widehat{C}_1) &\ge \frac{L_{n+1}}{2} - L_n - L_n - \frac{L_{n+1}}{10} - 1 > \frac{L_{n+1}}{20}, \text{ when } n \ge 1\,, \\ &\ge \frac{L_{n+1}}{2} - \widehat{L}_0 - L_0 - \frac{L_{n+1}}{10} - 1 > \frac{L_{n+1}}{20}, \text{ when } n = 0\,. \end{aligned} \tag{2.28}$$

Since $V = \widehat{C}_1 \cup \widehat{C}_2$, we have established that

$$d(\partial U, V) > \frac{L_{n+1}}{20}\,. \tag{2.29}$$

We then introduce the successive times of return to $\widehat{C}_2$ and departure from U:

$$\begin{array}{c} R_1 = H_{\widehat{C}_2},\ D_1 = T_U \circ \theta_{R_1} + R_1, \text{ and for } k \geq 1, \text{ by induction} \\ R_{k+1} = R_1 \circ \theta_{D_k} + D_k,\ D_{k+1} = D_1 \circ \theta_{D_k} + D_k\,, \end{array} \tag{2.30}$$

so that $0 \leq R_1 \leq D_1 \leq \cdots \leq R_k \leq D_k \leq \cdots \leq \infty$.

Letting $r \geq 1$, play the role of r_n in (2.19), (2.20), we further introduce the decompositions:

$$\begin{array}{l} \mu'_{2,1} = \sum\limits_{1 \leq \ell \leq r} \rho^\ell_{2,1} + \overline{\rho}_{2,1},\ \mu'_{1,2} = \sum\limits_{1 \leq \ell \leq r} \rho^\ell_{1,2} + \overline{\rho}_{1,2}\,, \\ \mu^*_{2,2} = \sum\limits_{1 \leq \ell \leq r} \rho^\ell_{2,2} + \overline{\rho}_{2,2}\,, \end{array} \tag{2.31}$$

where for $i \neq j$ in $\{1,2\}$, and $\ell \geq 1$, we have set

$$\begin{array}{l} \rho^\ell_{i,j} = 1\{R_\ell < D_\ell < R_{\ell+1} = \infty\}\, \mu'_{i,j},\ \overline{\rho}_{i,j} = 1\{R_{r+1} < \infty\}\, \mu'_{i,j}, \text{ and} \\ \rho^\ell_{2,2} = 1\{R_\ell < D_\ell < R_{\ell+1} = \infty\}\, \mu^*_{2,2},\ \overline{\rho}_{2,2} = 1\{R_{r+1} < \infty\}\, \mu^*_{2,2}\,. \end{array}$$

The point processes $\overline{\rho}_{1,2}$ and $\overline{\rho}_{2,2}$ play the role of correction terms, eventually responsible for the last term in the right-hand side of (2.20). The bounds we derive on the intensity measures $\overline{\xi}_{2,1}$ and $\overline{\xi}_{1,2}$ of $\overline{\rho}_{2,1}$ and $\overline{\rho}_{1,2}$ depart from (2.26), (2.27) in [17]. We write

$$\begin{array}{rcl} \overline{\xi}_{2,1}(W_+) & = & u'\, P_{e_V}[X_0 \in \widehat{C}_2,\ H_{\widehat{C}_1} < \infty, R_{r+1} < \infty] \\ & \overset{(1.19)}{\leq} & u' \operatorname{cap}(\widehat{C}_2) \sup\limits_{x \in \widehat{C}_2} P_x[R_{r+1} < \infty] \\ & \overset{\text{strong Markov}}{\leq} & u' \operatorname{cap}(\widehat{C}_2)(\sup\limits_{x \in \partial U} P_x[H_{\widehat{C}_2} < \infty])^r\,. \end{array} \tag{2.32}$$

Combining (1.6) and (2.29) we find that

$$\sup_{x \in \partial U} P_x[H_{\widehat{C}_2} < \infty] \leq 2^n \Big(c\, \frac{\widehat{L}_0}{L_{n+1}}\Big)^{(d-2)} \overset{(2.1),(2.2)}{=} 2^n \Big(c\, \frac{\widehat{L}_0}{L_0}\, \ell_0^{-(n+1)}\Big)^{(d-2)}\,. \tag{2.33}$$

Moreover from (1.20), (1.22), we have

$$\operatorname{cap}(\widehat{C}_2) \leq 2^n \Big(c\, \frac{\widehat{L}_0}{\sqrt{d}}\Big)^{(d-2)}\,, \tag{2.34}$$

and hence

$$\overline{\xi_{2,1}}(W_+) \leq u' \Big(\frac{\widehat{L}_0}{\sqrt{d}}\Big)^{(d-2)} \Big(4^n \Big(c\, \frac{\widehat{L}_0}{L_0}\Big)^{(d-2)}\, \ell_0^{-(n+1)(d-2)}\Big)^r\,. \tag{2.35}$$

In a similar fashion we also obtain:

$$\overline{\xi_{1,2}}(W_+) \leq u' \Big(\frac{\widehat{L}_0}{\sqrt{d}}\Big)^{(d-2)} \Big(4^n \Big(c\, \frac{\widehat{L}_0}{L_0}\Big)^{(d-2)}\, \ell_0^{-(n+1)(d-2)}\Big)^r\,. \tag{2.36}$$

The next objective is to show that the trace on $\widehat{C}_2$ of paths in the support of $\sum_{1 \leq \ell \leq r} \rho^\ell_{2,1}$ and $\sum_{1 \leq \ell \leq r} \rho^\ell_{1,2}$ is stochastically dominated by the corresponding trace on $\widehat{C}_2$ of paths in the support of $\mu^*_{2,2}$, when $u - u'$ is not too small. An important step is the next lemma:

Lemma 2.2. *For $\ell_0 \geq c(\sqrt{d}+R)$, for all $n \geq 0$, $m \in I_{n+1}$, $\mathcal{T} \in \Lambda_m$, $x \in \partial U$, $y \in \partial_{\text{int}} \widehat{C}_2$, one has:*

$$(2.37)\quad P_x[H_{\widehat{C}_1} < R_1 < \infty,\, X_{R_1} = y] \leq \Big(\frac{\widehat{L}_0}{L_0}\Big)^{(d-2)} \Big(\frac{c}{\ell_0}\Big)^{(d-2)(n+1)} P_x[H_{\widehat{C}_1} > R_1,\, X_{R_1} = y]\,,$$

$$(2.38)\quad P_x[H_{\widehat{C}_1} < \infty, R_1 = \infty] \leq \Big(\frac{\widehat{L}_0}{L_0}\Big)^{(d-2)} \Big(\frac{c}{\ell_0}\Big)^{(d-2)(n+1)} P_x[R_1 = \infty = H_{\widehat{C}_1}]\,.$$

Proof. The proof of (2.37) follows closely the proof of (2.30) in Lemma 2.3 of [17]. The difference lies in the control of Harnack constants. Indeed we first observe that the function $h : z \to P_z[R_1 < \infty,\, X_{R_1} = y] = P_z[H_{\widehat{C}_2} < \infty,\, X_{H_{\widehat{C}_2}} = y]$ is a non-negative function harmonic in $\widehat{C}_2^c$. By (2.29) it is therefore harmonic on any $B(z_0, \frac{L_{n+1}}{20})$, with $z_0 \in \partial U$.One can then find c such that for any $\widetilde{z}, \widetilde{z}'$ in ∂U, there exists a sequence z_i, $0 \leq i \leq m$, in ∂U, with $m \leq c$, $z_0 = \widetilde{z}$, $z_m = \widetilde{z}'$, and $|z_{i+1} - z_i| \leq \frac{L_{n+1}}{100 c_3}$, in the notation of Proposition 1.3. Indeed one simply projects $\widetilde{z}, \widetilde{z}'$ on the Euclidean sphere in $\mathbb{R}^d$ of radius $\frac{L_{n+1}}{10}$ with center $\overline{i_2}\, L_n$, the "center" of U, see (2.26), and uses the great circle joining these two points to construct the sequence.

Using (1.15) and a standard chaining argument, it follows that

$$(2.39)\qquad \sup_{z \in \partial U} P_z[R_1 < \infty, X_{R_1} = y] \leq c^d \inf_{z \in \partial U} P_z[R_1 < \infty, X_{R_1} = y]\,.$$

The proof of (2.37) then proceeds as in Lemma 2.3 of [17], (and we use a similar bound to (2.33) above, where $\widehat{C}_1$ replaces $\widehat{C}_2$).

As for (2.38), we first note that for $x \in \partial U$, due to (1.6) and (2.29), we have:

$$(2.40)\quad \inf_{x \in \partial U} P_x[R_1 = \infty,\, H_{\widehat{C}_1} = \infty] \geq 1 - 2\, 2^n \Big(c\, \frac{\widehat{L}_0}{L_{n+1}}\Big)^{(d-2)} \overset{(2.2)}{\geq} 1 - \Big(\frac{c}{\ell_0}\, \frac{\widehat{L}_0}{L_0}\Big)^{(d-2)} \overset{(2.1)}{\geq} \frac{1}{2}\,,$$

when $\ell_0 \geq c'(\sqrt{d} + R)$.

On the other hand a similar calculation leads to:

$$(2.41)\qquad P_x[H_{\widehat{C}_1} < \infty, R_1 = \infty] \leq 2^n \Big(c\, \frac{\widehat{L}_0}{L_{n+1}}\Big)^{(d-2)} \leq \Big(\frac{\widehat{L}_0}{L_0}\Big)^{(d-2)} \Big(\frac{c}{\ell_0}\Big)^{(d-2)(n+1)},$$

and (2.38) follows. □

The proof of Proposition 2.1 then proceeds as the proof of Proposition 2.3 of [17], and yields that under (2.19), (with u' in place of u_n and u in place of u_{n+1}):

$$(2.42)\qquad \begin{aligned} \mathbb{P}[A_{\mathcal{T}_2}(\mu_{2,2})] &\leq p_n(u') + \overline{\xi}_{2,1}(W_+) + \overline{\xi}_{1,2}(W_+) \\ &\overset{(2.35),(2.36)}{\leq} p_n(u') + 2u' \Big(\frac{\widehat{L}_0}{\sqrt{d}}\Big)^{(d-2)} \Big(4^n \Big(c\, \frac{\widehat{L}_0}{L_0}\Big)^{(d-2)} \ell_0^{-(n+1)(d-2)}\Big)^r. \end{aligned}$$

Inserting this inequality into (2.22), we thus infer (2.20) under the assumption on (2.19). □

We assume from now on that $\ell_0 \geq c(\sqrt{d}+R)$, with $c > 2c_5$ sufficiently large so that Proposition 2.1 holds. We then pick the sequences $u_n, n \geq 0$ and $r_n, n \geq 0$, as follows:

$$u_n = u_0 \ \exp\Big\{\Big(\frac{\widehat{L}_0}{L_0}\Big)^{(d-2)} \sum_{0 \leq k < n} (r_k+1)\Big(\frac{c_5}{\ell_0}\Big)^{(k+1)(d-2)}\Big\}, \tag{2.43}$$

$$r_n = r_0\, 2^n\,, \tag{2.44}$$

where $u_0 > 0$ and r_0 is a positive integer. The choice (2.43) ensures that (2.19) is fulfilled and the increasing sequence u_n has the finite limit

$$u_\infty = u_0 \ \exp\Big\{\Big(\frac{c_5\,\widehat{L}_0}{\ell_0\,L_0}\Big)^{(d-2)}\Big(\frac{r_0}{1-2(c_5\,\ell_0^{-1})^{(d-2)}} + \frac{1}{1-(c_5\,\ell_0^{-1})^{(d-2)}}\Big)\Big\}\,. \tag{2.45}$$

The next proposition reduces the task of bounding $p_n(u_n)$ to a set of conditions, which enable us to initiate the induction procedure suggested by Proposition 2.1. We view u_∞ as a function of u_0, r_0, ℓ_0, R, (we introduced R in (2.1)).

Proposition 2.3. *There exists a positive constant c such that when $u_0 > 0$, $r_0 \geq 1$, $\ell_0 \geq c(\sqrt{d}+R)$, $L_0 \geq d$, $\widehat{L}_0 = (\sqrt{d}+R)\,L_0$, $R \geq 1$, $K_0 > \log 2$ satisfy*

$$u_\infty \Big(\frac{\widehat{L}_0}{\sqrt{d}}\Big)^{d-2} \vee e^{K_0} \leq \Big(\frac{\ell_0\,L_0}{c_6\,\widehat{L}_0}\Big)^{\frac{r_0}{2}(d-2)}, \tag{2.46}$$

$$p_0(u_0) \leq e^{-K_0}, \tag{2.47}$$

then

$$p_n(u_n) \leq e^{-(K_0-\log 2)2^n}, \ \textit{for each } n \geq 0\,. \tag{2.48}$$

Proof. The argument is similar to Proposition 2.5 of [17]. We assume as mentioned before $c > 2c_5$ large enough so that Proposition 2.1 applies. Condition (2.46) implies that $c_6\,\widehat{L}_0 \leq \ell_0\,L_0\,(=L_1)$. Thus the last term in the right-hand side of (2.20) satisfies:

$$\begin{aligned} &u_n \Big(\frac{\widehat{L}_0}{\sqrt{d}}\Big)^{(d-2)}\Big(4^n\Big(c_6\,\frac{\widehat{L}_0}{L_0}\Big)^{(d-2)}\,\ell_0^{-(n+1)(d-2)}\Big)^{r_n} \leq \\ &u_\infty \Big(\frac{\widehat{L}_0}{\sqrt{d}}\Big)^{(d-2)}\Big(c_6\,\frac{\widehat{L}_0}{\ell_0\,L_0}\Big)^{(d-2)r_n}\Big(\frac{4}{\ell_0^{d-2}}\Big)^{nr_n} \overset{(2.2),(2.46)}{\leq} \Big(c_6\,\frac{\widehat{L}_0}{\ell_0\,L_0}\Big)^{\frac{r_n}{2}(d-2)}. \end{aligned} \tag{2.49}$$

As a result (2.20) yields that for $n \geq 0$:

$$p_{n+1}(u_{n+1}) \leq p_n(u_n)\Big(p_n(u_n) + \Big(c_6\,\frac{\widehat{L}_0}{\ell_0\,L_0}\Big)^{\frac{r_0}{2}2^n(d-2)}\Big). \tag{2.50}$$

We then define by induction K_n, $n \geq 0$, via the following relation valid for $n \geq 1$,

$$K_n = K_0 - \sum_{0 \leq n' < n} 2^{-(n'+1)} \log\Big(1 + e^{K_{n'}2^{n'}}\Big(c_6\,\frac{\widehat{L}_0}{\ell_0\,L_0}\Big)^{\frac{r_0}{2}2^{n'}(d-2)}\Big), \tag{2.51}$$

so that $K_n \leq K_0$, and hence

$$
(2.52)\qquad
\begin{aligned}
K_n &\geq K_0 - \sum_{n' \geq 0} 2^{-(n'+1)} \log \Big(1 + e^{K_0 2^{n'}} \Big(c_6 \, \frac{\widehat{L}_0}{\ell_0 \, L_0}\Big)^{\frac{r_0}{2} 2^{n'}(d-2)}\Big) \\
&\overset{(2.46)}{\geq} K_0 - \sum_{n' \geq 0} 2^{-(n'+1)} \log 2 = K_0 - \log 2 > 0 \,.
\end{aligned}
$$

As we now show by induction we have $p_n(u_n) \leq e^{-K_n 2^n}$.

Indeed this inequality holds for $n = 0$, due to (2.47), and if it holds for $n \geq 0$, then due to (2.50) we find that

$$
\begin{aligned}
p_{n+1}(u_{n+1}) &\leq e^{-K_n 2^n} \Big(e^{-K_n 2^n} + \Big(c_6 \, \frac{\widehat{L}_0}{\ell_0 \, L_0}\Big)^{\frac{r_0}{2} \, 2^n (d-2)}\Big) \\
&= e^{-K_n 2^{n+1}} \Big(1 + e^{K_n 2^n} \Big(c_6 \, \frac{\widehat{L}_0}{\ell_0 \, L_0}\Big)^{\frac{r_0}{2} \, 2^n (d-2)}\Big) \overset{(2.51)}{=} e^{-K_{n+1} 2^{n+1}} .
\end{aligned}
$$

This proves that $p_n(u_n) \leq e^{-K_n 2^n}$ for all $n \geq 0$, and (2.48) follows. □

Remark 2.4. One of the main issues we now have to face, is proving the local estimate $p_0(u_0) \leq e^{-K_0}$, see (2.47), for large d, with u_0 of order close to $\log d$, (and a posteriori close to u_*). We further need K_0 sufficiently large so that $2^{-(K_0 - \log 2) 2^n}$ beats the combinatorial complexity arising from the choice of the binary trees in the upper bound (2.18), i.e. beats the factor $|\Lambda_n| \overset{(2.13)}{\leq} (c_4 \, \ell_0)^{2(d-1)(2^n - 1)}$. Devising this local estimate will be the object of the next section and will involve aspects of random interlacements at a shorter range, where features reminiscent of random interlacements on $2d$-regular trees, cf. Section 5 of [23], will be manifest. □

3 Local connectivity bounds

The object of this section is to derive exponential bounds on the decay of the probability of existence of a path in the vacant set at level $u_0 = (1+5\varepsilon) \log d$, starting at the origin and traveling at ℓ^1-distance Md, where M is an arbitrary integer and $d \geq c(\varepsilon, M)$, cf. Corollary 3.4. For this purpose we develop an enhanced Peierls-type argument. The main step comes in Theorem 3.1 below. In the present section aspects of random interlacements on $\mathbb{Z}^d$ for large d, reminiscent of random interlacements on $2d$-regular trees, cf. [23], will play a important role. We introduce the parameter

$$
(3.1)\qquad 0 < \varepsilon < \frac{1}{3} \,.
$$

We also introduce in the notation of (1.14)

$$
(3.2)\qquad L = c_7 \, d, \text{ with } c_7 = [e^8 c_2] + 2 \,.
$$

The main result of this section is the following estimate on the connectivity function.

Theorem 3.1. $(d \geq c)$

For any positive integer M, we have

$$\mathbb{P}[0 \overset{\mathcal{V}^{u_0}}{\longleftrightarrow} S_1(0, ML)] \leq \exp\Big\{\frac{M(M-1)}{2}\, L + 3Md - \,\frac{\varepsilon^2}{5}\, Md \log d\Big\}, \tag{3.3}$$

where the notation is similar to (2.14) and

$$u_0 = (1 + 5\varepsilon) \log d\,. \tag{3.4}$$

Proof. Observe that any self-avoiding path from 0 to $S_1(0, ML)$ successively visits the ℓ^1-spheres $S_1(0, iL)$, with $i = 0, \dots, M-1$. Thus considering the first $[\frac{\varepsilon}{10}\, d]$ steps of the path consecutive to the successive entrances in the various spheres $S_1(0, iL)$, we obtain M self-avoiding paths π_i, $i = 0, \dots, M-1$, where π_i starts in $S_1(0, iL)$ and has $[\frac{\varepsilon}{10}\, d]$ steps for each i. Denoting by z_i, $i = 0, \dots, M-1$, the respective starting points of these paths, we find that:

$$\mathbb{P}[0 \overset{\mathcal{V}^{u_0}}{\longleftrightarrow} S_1(0, ML)] \leq \sum_{z_i, \pi_i} \mathbb{P}[\mathcal{V}^{u_0} \supseteq \text{range}\, \pi_i, \text{ for } i = 0, \dots, M-1]\,, \tag{3.5}$$

where the above sum runs over $z_i \in S(0, iL)$ and self-avoiding paths π_i with $[\frac{\varepsilon}{10}\, d]$ steps and starting points z_i, for $i = 0, \dots, M-1$. The next lemma provides a very rough bound on the cardinality of ℓ^1-spheres and ℓ^1-balls. Crucially it shows that ℓ^1-spheres and balls of radius cd are "rather small": their cardinality grows at most geometrically in d.

Lemma 3.2. $(\ell \in \mathbb{N})$

$$\begin{aligned} &\text{i)} \quad |S_1(0, \ell)| \leq 2^d\, e^{\ell + d} \\ &\text{ii)} \quad |B_1(0, \ell)| \leq 2^d\, e^{\ell + 1 + d}\,. \end{aligned} \tag{3.6}$$

Proof. We express the generating function of $|S_1(0, k)|$, $k \geq 0$, as follows. Given $|t| < 1$, we have:

$$\begin{aligned} \sum_{k \geq 0} t^k |S_1(0, k)| &= \sum_{k \geq 0} t^k \sum_{\substack{m_1, \dots, m_d \geq 0 \\ m_1 + \dots + m_d = k}} 2^{|\{i \in \{1, \dots, d\};\, m_i \neq 0\}|} \\ &= \sum_{m_1, \dots, m_d \geq 0} t^{m_1 + \dots + m_d} 2^{|\{i \in \{1, \dots, d\};\, m_i \neq 0\}|} = \Big(1 + 2 \sum_{m \geq 1} t^m\Big)^d \\ &= \Big(\frac{1+t}{1-t}\Big)^d \leq \frac{2^d}{(1-t)^d}\,. \end{aligned} \tag{3.7}$$

As a result we see that for $0 < t < 1$, $\ell \geq 0$,

$$|S_1(0, \ell)| \leq 2^d (1-t)^{-d}\, t^{-\ell}\,.$$

Choosing $t = \ell/(d + \ell)$, we find that

$$|S_1(0, \ell)| \leq 2^d \Big(1 + \frac{\ell}{d}\Big)^d \Big(1 + \frac{d}{\ell}\Big)^\ell \leq 2^d\, e^{\ell + d}\,, \tag{3.8}$$

where we used the inequality $1 + u \leq e^u$ in the last step. This proves (3.6) i). As for the inequality (3.6) ii), by (3.6) i) we can write:

$$|B_1(0, \ell)| \leq 2^d\, e^d \sum_{k=0}^{\ell} e^k = 2^d\, e^d\, \frac{e^{\ell+1} - 1}{e - 1} \leq 2^d\, e^{\ell + 1 + d}\,, \tag{3.9}$$

and our claims follows. □

We now come back to (3.5). By a very rough counting argument for the number of possible choices of π_i, we have a Peierls-type bound:

$$
\begin{aligned}
&\mathbb{P}[0 \overset{\mathcal{V}^{u_0}}{\longleftrightarrow} S_1(0, ML)] \le \\
&\Big(\prod_{k=0}^{M-1} |S_1(0,kL)|\,(2d)^{\frac{\varepsilon}{10}d}\Big) \sup_{z_i,\pi_i} \mathbb{P}[\mathcal{V}^{u_0} \supseteq \mathrm{range}\,\pi_i, i=0,\dots,M-1] \overset{(3.6)\,i)}{\le} \\
&\Big(\prod_{k=0}^{M-1} 2^d\, e^{kL+d}\Big)(2d)^{\frac{\varepsilon}{10}Md} \sup_{z_i,\pi_i} \mathbb{P}[\mathcal{V}^{u_0} \supseteq \mathrm{range}\,\pi_i, i=0,\dots,M-1] \le \\
&e^{\frac{M(M-1)}{2}\,L+2Md}\,(2d)^{\frac{\varepsilon}{10}Md} \sup_{z_i,\pi_i} \mathbb{P}[\mathcal{V}^{u_0} \supseteq \mathrm{range}\,\pi_i, i=0,\dots,M-1]\,,
\end{aligned} \tag{3.10}
$$

where the supremum runs over a similar collection as the sum in (3.5).

The next objective is to bound the probability in the last line of (3.10). For this purpose for each x in the set

$$
B \overset{\mathrm{def}}{=} \bigcup_{i=0}^{M-1} B_1\Big(z_i, \frac{\varepsilon}{10}\,d\Big), \;\text{(pairwise disjoint ℓ^1-balls appear in this union)}, \tag{3.11}
$$

we write z_x for the unique z_i such that $x \in B\Big(z_i, \dfrac{\varepsilon}{10}\,d\Big)$. We then define for any x in B the subset W_x^* of W^*, see above (1.24), (not to be confused with $W^*_{\{x\}}$):

$$
\begin{aligned}
W_x^* = {} & \text{the image under } \pi^* \text{ of } \Big\{w \in W;\ \text{the minimum of } d_1(z_x, w(n)),\ n \in \mathbb{Z}, \\
& \text{is reached for the first time at } w(n) = x, \text{ and } w \text{ does not enter any} \\
& B_1\Big(z_i, \frac{\varepsilon}{10}\,d\Big), \text{ with } z_i \neq z_x\Big\}.
\end{aligned} \tag{3.12}
$$

Note that clearly $W_x^* \subseteq W^*_{\{x\}}$, and that

$$
W_x^*,\ x \in B, \text{ are pairwise disjoint measurable subsets of } W^*. \tag{3.13}
$$

It then follows that for $z_i, \pi_i, 0 \le i \le M-1$, as in (3.10), we have

$$
\begin{aligned}
&\mathbb{P}[\mathcal{V}^{u_0} \supseteq \mathrm{range}\ \pi_i,\ i=0,\dots,M-1] \le \\
&\mathbb{P}\Big[\omega\Big(\bigcup_{i=0}^{M-1} \bigcup_{x \in \mathrm{range}\ \pi_i} W_x^* \times [0,u_0]\Big) = 0\Big] = \exp\Big\{-u_0 \sum_{i=0}^{M-1} \sum_{x \in \mathrm{range}\,\pi_i} \nu(W_x^*)\Big\} \le \\
&\exp\Big\{-u_0\, M\, \frac{\varepsilon d}{10} \times \inf_{x \in B}\ \nu(W_x^*)\Big\}.
\end{aligned} \tag{3.14}
$$

We will now seek a lower bound on $\nu(W_x^*)$, for $x \in B$.

Choosing $K = \{x\}$ in (1.25), (1.26), by (1.19) we see that for any x in B

$$
\begin{aligned}
\nu(W_x^*) = {} & P_x\big[|X_n - z_x|_1 \ge |x - z_x|_1, \text{ for } n \ge 0, \text{ and } H_{\bigcup_{z_i \neq z_x} B_1(z_i, \frac{\varepsilon}{10}\,d)} = \infty\big] \times \\
& P_x\big[|X_n - z_x|_1 > |x - z_x|_1, \text{ for } n > 0, \text{ and } H_{\bigcup_{z_i \neq z_x} B_1(z_i, \frac{\varepsilon}{10}\,d)} = \infty\big] \\
\ge {} & \big(P_x[|X_n - z_x|_1 > |x - z_x|_1, \text{ for } n > 0] - \sum_{z_i \neq z_x} P_x\big[H_{B_1(z_i, \frac{\varepsilon}{10}\,d)} < \infty\big]\big)_+^2.
\end{aligned} \tag{3.15}
$$

In view of (1.14) and the choice of L in (3.2) we see that when $d \geq 8$, we have:

$$\begin{aligned}(3.16)\qquad &\sum_{z_i \neq z_x} P_x\big[H_{B_1(z_i, \frac{\varepsilon}{10}\, d)} < \infty\big] \leq \sum_{z_i \neq z_x} \sup_{y \in B(z_i, \frac{\varepsilon}{10}\, d)} g(y-x) \Big| B_1\Big(0, \frac{\varepsilon}{10}\, d\Big)\Big| \overset{(1.14),(3.6)ii)}{\leq} \\ &2 \sum_{j \geq 1} \Big(\frac{c_2\, d}{jL - \frac{\varepsilon}{5}\, d}\Big)^{\frac{d}{2}-2} 2^d\, e^{\frac{\varepsilon}{10}\, d+1+d} \overset{(3.2)}{\leq} 2e^{-8(\frac{d}{2}-2)+3d+1} \sum_{j \geq 1} j^{-(\frac{d}{2}-2)} \overset{d \geq 8}{\leq} c\, e^{-d}\,.\end{aligned}$$

The next lemma yields a lower bound on the first term in the last line of (3.16).

Lemma 3.3. $(d \geq c)$

When $|y|_1 \leq \frac{d}{2}$, *one has*

$$(3.17)\qquad P_y[|X_n|_1 > |y|_1,\ \textit{for all}\, n > 0] \geq 1 - \frac{4(|y|_1 \vee 1)}{2d - (|y|_1 \vee 1)}\,.$$

Proof. We first note that for $z = (z_1, \dots, z_d)$ in $\mathbb{Z}^d$, P_z-a.s., $\big||X_1|_1 - |z|_1\big| = 1$, and

$$\begin{aligned}(3.18)\qquad &P_z[|X_1|_1 = |z|_1 + 1] = \frac{1}{2d}\Big(2d - \sum_{k=1}^{d} 1\{z_k \neq 0\}\Big) \geq p_{|z|_1}, \text{ where} \\ &p_m \overset{\text{def}}{=} \Big(\frac{1}{2} + \frac{1}{2}\Big(1 - \frac{m}{d}\Big)_+\Big), \text{ for } m \geq 0\,.\end{aligned}$$

We then introduce the canonical Markov chain N_n on $\mathbb{N}$ that jumps to $m+1$ with probability p_m and to $m-1$ with probability $q_m = 1 - p_m$, when located at m. We denote with Q_m the canonical law of this Markov chain starting in m. In view of (3.18), a coupling argument shows that we can construct X_n and N_n on the same probability space so that a.s. $|X_n|_1 \geq N_n$, for all $n \geq 0$, and $X_0 = y \in \mathbb{Z}^d$, $N_0 = |y|_1$. Consequently we see that when $y \neq 0$, we have the bound, (with $m = |y|_1 \leq \frac{d}{2}$):

$$\begin{aligned}(3.19)\qquad P_y[H_{S_1(0,d^2)} < \widetilde{H}_{B_1(0,|y|_1)}] &\geq Q_{|y|_1}[H_{d^2} < \widetilde{H}_{|y|_1}] \\ &= p_m(1 + \rho_{m+1} + \rho_{m+1}\,\rho_{m+2} + \dots + \rho_{m+1}\dots\rho_{d^2-1})^{-1},\end{aligned}$$

where $\rho_\ell = \frac{q_\ell}{p_\ell}$, for $\ell \geq 0$, and we have used [5], (5), p. 73.

Note that the expression in the right-hand side of (3.19) is a decreasing function of each ρ_ℓ, $m+1 < \ell < d^2$. If we further observe that $\rho_\ell \leq (\frac{1}{2} - \frac{1}{2} \times \frac{1}{4})(\frac{1}{2} + \frac{1}{2} \times \frac{1}{4})^{-1} = \frac{3}{5}$, for $m+1 < \ell \leq \frac{3}{4}\, d$, and $\rho_\ell \leq 1$, for $\frac{3}{4}\, d < \ell \leq d^2 - 1$, we see that the above expression is bigger than:

$$\begin{aligned}&\Big(1 - \frac{m}{2d}\Big)\Big(1 + \frac{m}{2d-m} \sum_{k \geq 0} \Big(\frac{3}{5}\Big)^k + \Big(\frac{3}{5}\Big)^{[\frac{3}{4}\, d]-m} d^2\Big)^{-1} \overset{m \leq \frac{d}{2}}{\geq} \\ &\Big(1 - \frac{m}{2d}\Big)\Big(1 + \frac{5}{2}\,\frac{m}{2d-m} + \frac{5}{3}\Big(\frac{3}{5}\Big)^{\frac{d}{4}} d^2\Big)^{-1} \underset{1 \leq m \leq \frac{d}{2}}{\overset{d \geq c}{\geq}} \Big(1 - \frac{m}{2d}\Big)\Big(1 + 3\frac{m}{2d-m}\Big)^{-1} \geq \\ &\Big(1 - \frac{m}{2d}\Big)\Big(1 - 3\frac{m}{2d-m}\Big)\ \Big(\geq 0, \text{ since } m \leq \frac{d}{2}\Big).\end{aligned}$$

By the strong Markov property at time $H_{S_1(0,d^2)}$, we thus find that for $d \ge c$, $1 \le |y|_1 \le \frac{d}{2}$, we have:

$$
\begin{aligned}
&P_y[|X_n|_1 > |y|_1, \text{ for all } n > 0] \ge \\
&\Big(1 - \frac{|y|_1}{2d}\Big)\Big(1 - \frac{3|y|_1}{2d - |y|_1}\Big)\big(1 - \sup_{|z|_1 = d^2} P_z[H_{B_1(0,\frac{d}{2})} < \infty]\big) \overset{(1.1)}{\ge} \\
&\Big(1 - \frac{|y|_1}{2d}\Big)\Big(1 - \frac{3|y|_1}{2d - |y|_1}\Big) - \sup_{|z| \ge d^{3/2}} P_z[H_{B(0,d)} < \infty] \overset{(1.6)}{\ge} \\
&1 - \frac{|y|_1}{2d} - \frac{3|y|_1}{2d - |y|_1} + \frac{3|y|_1^2}{2d(2d - |y|_1)} - \Big(\frac{c}{\sqrt{d}}\Big)^{(d-2)} \underset{y \neq 0}{\overset{d \ge c}{\ge}} 1 - \frac{4|y|_1}{2d - |y|_1}\,.
\end{aligned} \tag{3.20}
$$

This completes the proof of (3.17) when $y \neq 0$. The extension to the case $y = 0$ is immediate. □

We use the above lemma to bound the first term in the last line of (3.15) from below. In view of (3.16) and (3.17) we thus find that for $d \ge c$, and any $x \in B$, (see (3.11)):

$$
\nu(W_x^*) \ge \Big(1 - 5\,\frac{|x - z_x|_1 \vee 1}{2d - (|x - z_x|_1 \vee 1)}\Big)^2 \ge 1 - 10\,\frac{\varepsilon/10}{2 - \varepsilon/10} \ge 1 - \varepsilon\,. \tag{3.21}
$$

Coming back to (3.14) we thus find that

$$
\mathbb{P}[\mathcal{V}^{u_0} \supseteq \text{range } \pi_i, \; i = 0, \dots, M-1] \le \exp\Big\{ - \frac{u_0}{10}\, M\,\varepsilon(1-\varepsilon)\,d\Big\}\,. \tag{3.22}
$$

Inserting this bound in the last line of (3.10) we obtain

$$
\begin{aligned}
&\mathbb{P}[0 \overset{\mathcal{V}^{u_0}}{\longleftrightarrow} S_1(0, ML)] \le \exp\Big\{\frac{M(M-1)}{2}\, L + 2Md - \frac{u_0}{10}\,\varepsilon(1-\varepsilon)\,Md\Big\}(2d)^{\frac{\varepsilon}{10}Md} \\
&\overset{(3.4)}{\le} \exp\Big\{\frac{M(M-1)}{2}\, L + 3Md + \frac{\varepsilon}{10}\,Md\log d - \frac{1}{10}\,(\varepsilon + 4\varepsilon^2 - 5\varepsilon^3)\,Md\log d\Big\}\,.
\end{aligned}
$$

Since $5\varepsilon^3 \le 2\varepsilon^2$, due to (3.1), the claim (3.3) follows. □

We will use the following corollary in the proof of Theorem 0.1 in the next section.

Corollary 3.4. *(with (3.1), (3.2))*

Given $M \ge 1$, then for $d \ge c(M, \varepsilon)$,

$$
\mathbb{P}[0 \overset{\mathcal{V}^{u_0}}{\longleftrightarrow} S_1(0, ML)] \le \exp\Big\{ - \frac{\varepsilon^2}{10}\, dM\log d\Big\}\,. \tag{3.23}
$$

Proof. This is an immediate consequence of (3.3). □

Remark 3.5. One should note that the bound of Theorem 3.1 deteriorates when M becomes large. One can view Theorem 3.1 as a Peierls-type bound, (slightly enhanced due to the role of M in the proof). In the next section we will pick M as a large constant depending on ε, and use Corollary 3.4 to produce the local estimate, which will enable us to initiate the renormalization scheme of Section 2. In this fashion the local estimate on crossings in $\mathcal{V}^{u_0}$ at ℓ^1-distance of order $c(\varepsilon)d$ will be transformed into an estimate on crossings at all scales in $\mathcal{V}^{u_\infty}$, where $u_\infty \le (1 + 10\varepsilon)\log d$. □

4 Denouement

In this section we prove Theorem 0.1. We combine the local bound on the connectivity function at level u_0 of the last section, cf. Corollary 3.4, with the renormalization scheme of Section 2, cf. Proposition 2.3, in order to produce a bound on vacant crossings at a level $u_\infty \in [(1+5\varepsilon)\log d,\ (1+10\varepsilon)\log d]$, valid at arbitrary large scales.

Proof of Theorem 0.1: We pick ε and u_0 as in (3.1), (3.4). For the renormalization scheme of Section 2, we choose, (the constant c_7 appears in (3.2)):

(4.1) $$L_0 = d,\ \widehat{L}_0 = (\sqrt{d} + R)\, L_0, \text{ with } R = 300 c_7\, \varepsilon^{-2}, \text{ and}$$

(4.2) $$\ell_0 = d\,.$$

In the notation of Proposition 2.3 and (2.13) we pick

(4.3) $$r_0 = 24, \text{ and}$$

(4.4) $$K_0 = \log(4(c_4\, \ell_0)^{2(d-1)}) \overset{(4.2)}{=} \log(4(c_4\, d)^{2(d-1)})\,.$$

In the application of Corollary 3.4, we choose

(4.5) $$M = [100\varepsilon^{-2}] + 1\,,$$

so that in the notation of (3.2), (4.1)

(4.6) $$ML + 1 \le RL_0\,.$$

We will now check that the assumptions of Proposition 2.3 hold for $d \ge c(\varepsilon)$. By (2.45) we see that for $d \ge c(\varepsilon)$,

(4.7) $$u_0 = (1+5\varepsilon)\log d < u_\infty < (1+10\varepsilon)\log d\,,$$

and also that

(4.8) $$\widehat{L}_0 \le 2d^{\frac{3}{2}}\,.$$

As a result we find that

(4.9) $$u_\infty \Big(\frac{\widehat{L}_0}{\sqrt{d}}\Big)^{(d-2)} \le (1+10\varepsilon)\,(\log d)(2d)^{(d-2)}\,,$$

and that

(4.10) $$e^{K_0} = 4(c_4\, d)^{2(d-1)}\,,$$

whereas on the other hand

(4.11) $$\Big(\frac{\ell_0\, L_0}{c_6\, \widehat{L}_0}\Big)^{\frac{r_0}{2}\,(d-2)} \ge (cd)^{6(d-2)}\,.$$

Since $2(d-1) < 6(d-2)$, we see that for $d \ge c(\varepsilon)$, the expression in the left-hand side of (4.11) dominates the corresponding expressions in (4.9) and (4.10), i.e. (2.46) holds.

There remains to check (2.47). For this purpose we apply Corollary 3.4, and find that for $d \ge c(\varepsilon)$, since $\widehat{L}_0 \ge \sqrt{d}\, L_0 + ML + 1$, cf. (4.1), (4.6), we have

$$\begin{aligned} p_0(u_0) &= \mathbb{P}\big[[0, L_0 - 1]^d \overset{\mathcal{V}^{u_0}}{\longleftrightarrow} \partial_{\rm int} B(0, \widehat{L}_0)\big] \\ &\le L_0^d\, \mathbb{P}[0 \overset{\mathcal{V}^{u_0}}{\longleftrightarrow} S_1(0, ML)] \underset{(4.5)}{\overset{(3.23)}{\le}} \exp\{d \log d - 10 d \log d\} = d^{-9d}. \end{aligned} \tag{4.12}$$

We thus find that for $d \ge c(\varepsilon)$, $p_0(u_0) \le e^{-K_0}$, i.e. (2.47) holds as well. It now follows from Proposition 2.3 that for $d \ge c(\varepsilon)$:

$$p_n(u_\infty) \le e^{-(K_0 - \log 2) 2^n}, \text{ for all } n \ge 0\,. \tag{4.13}$$

Taking (2.13), (2.18) into account yields that for all $n \ge 1$,

$$\begin{aligned} &\mathbb{P}\big[[0, L_n - 1]^d \overset{\mathcal{V}^{u_\infty}}{\longleftrightarrow} \partial_{\rm int}[-L_n, 2L_n - 1]^d\big] \le \\ &(c_4\, \ell_0)^{2(d-1)(2^n - 1)} e^{-(K_0 - \log 2) 2^n} \overset{(4.10)}{\le} 2^{-2^n}\,. \end{aligned} \tag{4.14}$$

In particular the above inequality implies that $\mathbb{P}[0 \overset{\mathcal{V}^{u_\infty}}{\longleftrightarrow} \infty] = 0$, and hence $u_* \le u_\infty < (1 + 10\varepsilon) \log d$, for $d \ge c(\varepsilon)$. The claim (0.6) readily follows. Combining this upper bound with the lower bound (0.3), we have thus proved Theorem 0.1. □

Remark 4.1.

The inequality (4.14) together with the fact that $L_n = L_0\, \ell_0^n$, for $n \ge 0$, is more than enough to show that for ε as in (3.1) and $d \ge c(\varepsilon)$,

$$\lim_{L \to \infty} L^\gamma\, \mathbb{P}[B_\infty(0, L) \overset{\mathcal{V}^{(1+10\varepsilon)\log d}}{\longleftrightarrow} S_\infty(0, 2L)] = 0\,,$$

for some, and in fact all $\gamma > 0$. From the definition of the critical parameter u_{**} in [19]:

$$\begin{aligned} u_{**} &= \inf\{u \ge 0;\ \alpha(u) > 0\}, \text{ where} \\ \alpha(u) &= \sup\{\alpha \ge 0;\ \lim_{L \to \infty} L^\alpha\, \mathbb{P}[B_\infty(0, L) \overset{\mathcal{V}^u}{\longleftrightarrow} S_\infty(0, 2L)] = 0\}\,, \end{aligned} \tag{4.15}$$

(the supremum is by convention equal to zero, when the set is empty), we thus find that for $d \ge c(\varepsilon)$,

$$u_{**} \le (1 + 10\varepsilon) \log d\,. \tag{4.16}$$

Since $u_* \le u_{**}$, it follows that we have also proved that

$$\lim_d u_{**} / \log d = 1\,. \tag{4.17}$$

It is presently open whether $u_* = u_{**}$, however one knows that $0 < u_* \le u_{**} < \infty$, for all $d \ge 3$, cf. [21], and that for $u > u_{**}$ the connectivity function has a stretched exponential decay, cf. [17].

2) One may wonder whether the following reinforcement of (0.4) actually holds:

$$\mathbb{P}[0 \in \mathcal{V}^{u_*}] = e^{-u_*/g(0)} \sim (2d)^{-1}, \text{ as } d \to \infty\,.$$

This would indicate a similar high-dimensional behavior as for Bernoulli percolation, see [1], [2], [6], [9], [11]. In the case of interlacement percolation on a $2d$-regular tree, such an asymptotic behavior is known to hold, cf. [22]. □

A Appendix

In this appendix we prove an elementary inequality, which enters the proof of the Green function estimate (1.14), see Lemma A.1 below. We then prove in Lemma A.2 a bound on Harnack constants in terms of killed Green functions, for nearest neighbor Markov chains on graphs. The result is stated in a rather general formulation, due to its independent interest. It is an adaptation of Lemma 10.2 of [7]. We recall that Lemma A.2 enters the proof of Proposition 1.3.

Lemma A.1.

$$\text{(A.1)} \qquad \textit{for } a, b \geq 0,\ \sqrt{a^2+b^2}\ \log(1+\sqrt{a^2+b^2}) \leq a\log(1+a) + b\log(1+b)\,.$$

Proof. We introduce $\psi(u) = u\log(1+u)$, $u \geq 0$, as well as $\varphi_b(a) = \sqrt{a^2+b^2}$ and $\chi_b(a) = \psi(a) + \psi(b) - \psi(\varphi_b(a))$, for $a, b \geq 0$. We want to show that

$$\text{(A.2)} \qquad \chi_b(a) \geq 0, \text{ for } a, b > 0\,.$$

We note that $\chi_b(0) = 0$, and that

$$\chi_b'(a) = \log(1+a) + 1 - \frac{1}{1+a} - \Big(\log(1+\varphi_b(a)) + 1 - \frac{1}{1+\varphi_b(a)}\Big)\ \frac{a}{\varphi_b(a)}\,.$$

The claim (A.2) will follow once we show that

$$\text{(A.3)} \qquad \chi_b'(a) \geq 0, \text{ for } a, b > 0\,.$$

To this end we note that for $a > 0$, $\chi_0'(a) = 0$, and that

$$\text{(A.4)} \qquad \begin{aligned} \frac{\partial}{\partial b}\ \chi_b'(a) &= -\Big(\frac{1}{1+\varphi_b(a)}\ \frac{b}{\varphi_b(a)} + \frac{1}{(1+\varphi_b(a))^2}\ \frac{b}{\varphi_b(a)}\Big)\ \frac{a}{\varphi_b(a)} \\ &\quad + \Big(\log(1+\varphi_b(a)) + 1 - \frac{1}{1+\varphi_b(a)}\Big)\ \frac{ab}{\varphi_b(a)^3} \\ &= \frac{ab}{(1+\varphi_b(a))\varphi_b(a)^3}\ \Big\{\log(1+\varphi_b(a))(1+\varphi_b(a)) - \frac{\varphi_b(a)}{1+\varphi_b(a)}\Big\}\,. \end{aligned}$$

Introduce the function $\rho(u) = \log(1+u)(1+u) - \frac{u}{1+u}$, $u \geq 0$. Observe that $\rho(0) = 0$, and $\rho'(u) = \log(1+u) + 1 - \frac{1}{(1+u)^2} \geq 0$, so that $\rho(u) \geq 0$, for $u \geq 0$. Coming back to the last line of (A.4), we find that for $a > 0$, $\frac{\partial}{\partial b}\ \chi_b'(a) \geq 0$, for $b \geq 0$. This shows (A.3) and the claim (A.1) follows. □

We then turn to the second result of this appendix. We consider a connected graph Γ with an at most countable vertex set E, and edge set $\mathcal{E}$, (a subset of the collection of unordered pairs of E). Given $U \subseteq E$, we define ∂U, $\partial_{\text{int}} U$ and $\overline{U}$ similarly to what is described at the beginning of the Section 1, (with obvious modifications). We consider an irreducible Markov chain on E, nearest-neighbor in the wide sense, (i.e. at each step the Markov chain moves to a vertex, which is at graph-distance at most one from its current location). We write X_n, $n \geq 0$, for the canonical process, P_x for the canonical law starting from $x \in E$, and otherwise use similar notation as described at the beginning of Section 1.

We denote with $p(x,y)$, $x,y \in E$, the transition probability. We assume that the Markov chain satisfies the ellipticity condition:

$$p(x,y) > 0, \text{ when } x, y \text{ are neighbors, (i.e. } \{x,y\} \in \mathcal{E}\}) . \tag{A.5}$$

For f a bounded function on E, we define

$$Lf(x) = E_x[f(X_1)] - f(x) = \sum_{y \sim x} p(x,y)(f(y) - f(x)), \text{ for } x \in E , \tag{A.6}$$

where $y \sim x$ means that $y = x$ or y is a neighbor of x. Given $U \subseteq E$, a bounded function on $\overline{U}$ is said to be harmonic in U when (with a slight abuse of notation):

$$Lf(x) = 0, \text{ for } x \in U . \tag{A.7}$$

When U is a finite strict subset of E, the Green function killed outside U is defined as follows, (the notation is similar to Section 1):

$$G_U(x,y) = E_x\Big[\sum_{k \geq 0} 1\{X_k = y, T_U > k\}\Big], \ x, y \in E . \tag{A.8}$$

It follows from the ellipticity assumption (A.5), that when U is connected, $G_U(x,y) > 0$, for all $x, y \in U$. The next lemma is an adaptation of Lemma 10.2 of [7].

Lemma A.2. *Assume that $\emptyset \neq U_1 \subseteq U_2 \subseteq U_3$ are finite strict subsets of E, with U_3 connected, and that u is a bounded non-negative function on $\overline{U}_3$, which is harmonic in U_3. Then one has:*

$$\max_{U_1} u \leq K \min_{U_1} u , \tag{A.9}$$

where

$$K = \max_{x,y \in U_1} \max_{z \in \partial_{\text{int}} U_2} G_{U_3}(x,z)/G_{U_3}(y,z) . \tag{A.10}$$

Proof. We define for $x \in E$,

$$v(x) = E_x[u(X_{H_{U_2}}), \ H_{U_2} < T_{U_3}] . \tag{A.11}$$

We first note that

$$u(x) \geq v(x), \text{ for } x \in \overline{U}_3, \text{ and } u(x) = v(x), \text{ for } x \in U_2 . \tag{A.12}$$

Indeed in view of (A.11), u and v agree on U_2, and thanks to our assumptions, $u(X_{n \wedge T_{U_3}})$, $n \geq 0$, is a bounded martingale under P_x, $x \in \overline{U}_3$, so that by the stopping theorem we find

$$\begin{aligned} u(x) = E_x[u(X_{H_{U_2} \wedge T_{U_3}})] &= v(x) + E_x[u(X_{T_{U_3}}), T_{U_3} < H_{U_2}] \\ &\geq v(x), \text{ for } x \in \overline{U}_3 . \end{aligned}$$

The claim (A.12) follows.

Applying the simple Markov property at time 1 in (A.11), when $x \in U_3 \backslash U_2$, we see that

$$v \text{ is harmonic in } U_3 \backslash U_2 \,. \tag{A.13}$$

In addition we have for $x \in U_2$:

$$v(x) = u(x) = \sum_{y \sim x} p(x,y)\, u(y) \overset{(A.12)}{\geq} \sum_{y \sim x} p(x,y)\, v(y)\,,$$

and the last inequality is an equality when $x \in U_2 \backslash \partial_{\rm int} U_2$. We have thus shown that

$$Lv = 1_{\partial_{\rm int} U_2} Lv \leq 0, \text{ on } U_3\,. \tag{A.14}$$

Applying the stopping theorem, we see that under any P_x

$$v(X_{n \wedge T_{U_3}}) - \sum_{0 \leq k < n \wedge T_{U_3}} Lv(X_k),\ n \geq 0, \text{ is a martingale}\,.$$

Taking expectations and letting n tend to infinity, we obtain the identity:

$$\begin{aligned} v(x) &= E_x[v(X_{T_{U_3}})] - E_x\Big[\sum_{0 \leq k < T_{U_3}} Lv(X_k)\Big] \\ &= -\sum_{z \in E} G_{U_3}(x,z)\, Lv(z) \overset{(A.14)}{=} \sum_{z \in \partial_{\rm int} U_2} G_{U_3}(x,z)(-Lv)(z), x \in E\,. \end{aligned} \tag{A.15}$$

Since v and u agree on $U_2 \supseteq U_1$, (A.9) is a direct consequence of the above representation formula for v. □

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