RANDOM INTERLACEMENTS AND THE GAUSSIAN FREE FIELD

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Abstract

We consider continuous time random interlacements on $\mathbb{Z}^d$, $d \geq 3$, and characterize the distribution of the corresponding stationary random field of occupation times. When $d = 3$, we relate this random field to the two-dimensional Gaussian free field pinned at the origin, by looking at scaled differences of occupation times of long rods by random interlacements at appropriately tuned levels. In the main asymptotic regime, a scaling factor appears in the limit, which is independent of the free field, and distributed as the time-marginal of a zero-dimensional Bessel process. For arbitrary $d \geq 3$, we also relate the field of occupation times at a level tending to infinity, to the d-dimensional Gaussian free field.

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0 Introduction

In this article we consider continuous time random interlacements on $\mathbb{Z}^d$, $d \geq 3$, where each doubly infinite trajectory modulo time-shift in the interlacement is decorated by i.i.d. exponential variables with parameter 1 which specify the time spent by the trajectory at each step. We are interested in the random field of occupation times, i.e. the total time spent at each site of $\mathbb{Z}^d$ by the collection of trajectories with label at most u in the interlacement point process.

When $d = 3$, we relate this stationary random field to the *two-dimensional* Gaussian free field pinned at the origin, by looking at the properly scaled field of differences of occupation times of long rods of size N, when the level u is either proportional to $\log N/N$ or much larger than $\log N/N$. The choice of u proportional to $\log N/N$ corresponds to a non-degenerate probability that the interlacement at level u meets a given rod. In the asymptotic regime it brings into play an independent proportionality factor of the Gaussian free field, which is distributed as a certain time-marginal of a zero-dimensional Bessel process. This random factor disappears from the description of the limiting random field, when instead $uN/\log N$ tends to infinity .

For arbitrary $d \geq 3$, we also relate the properly scaled field of differences of occupation times of sites by the interlacement at a level u tending to infinity, with the Gaussian free field on $\mathbb{Z}^d$.

Rather than discussing our results any further, we first present the model and refer to Section 1 for additional details. We consider the spaces $\widehat{W}_+$ and $\widehat{W}$ of infinite and doubly infinite $\mathbb{Z}^d \times (0,\infty)$-valued sequences, with $d \geq 3$, such that the $\mathbb{Z}^d$-valued components form an infinite, respectively doubly infinite, nearest neighbor trajectory spending finite time in any finite subset of $\mathbb{Z}^d$, and such that the $(0,\infty)$-valued components have an infinite sum in the case of $\widehat{W}_+$, and infinite "forward" and "backward" sums, when restricted to positive and negative indices, in the case of $\widehat{W}$.

We write X_n, σ_n, with $n \geq 0$, or $n \in \mathbb{Z}$, for the respective $\mathbb{Z}^d$- and $(0,\infty)$-valued canonical coordinates on $\widehat{W}_+$ and $\widehat{W}$. We denote by P_x, $x \in \mathbb{Z}^d$, the law on $\widehat{W}_+$ endowed with the canonical σ-algebra, under which X_n, $n \geq 0$, are distributed as simple random walk starting at x, and σ_n, $n \geq 0$, are i.i.d. exponential variables with parameter 1, independent from the X_n, $n \geq 0$. We write $\widehat{W}^*$ for the space $\widehat{W}$ modulo time-shift, i.e. $\widehat{W}^* = \widehat{W}/\sim$, where for $\widehat{w}, \widehat{w}'$ in $\widehat{W}$, $\widehat{w} \sim \widehat{w}'$ means that $\widehat{w}(\cdot) = \widehat{w}'(\cdot + k)$ for some $k \in \mathbb{Z}$. We denote by π^*: $\widehat{W} \to \widehat{W}^*$ the canonical map, and endow $\widehat{W}^*$ with the σ-algebra consisting of sets with inverse image under π^* belonging to the canonical σ-algebra of $\widehat{W}$.

The continuous time interlacement point process on $\mathbb{Z}^d$, $d \geq 3$, is a Poisson point process on $\widehat{W} \times \mathbb{R}_+$. Its intensity measure has the form $\widehat{\nu}(d\widehat{w}^*)du$, where $\widehat{\nu}$ is the σ-finite measure on $\widehat{W}^*$ such that for any finite subset K of $\mathbb{Z}^d$, the restriction of $\widehat{\nu}$ to the subset of $\widehat{W}^*$ made of $\widehat{w}^*$ for which the $\mathbb{Z}^d$-valued trajectory modulo time-shift enters K, is equal to $\pi^* \circ \widehat{Q}_K$, the image of $\widehat{Q}_K$ under π^*, where $\widehat{Q}_K$ is the finite measure specified by:

(0.1)

i) $\widehat{Q}_K(X_0 = x) = e_K(x)$, with e_K the equilibrium measure of K, see (1.4),

ii) when $e_K(x) > 0$, conditionally on $X_0 = x$, $(X_n)_{n\geq 0}$, $(X_{-n})_{n\geq 0}$, $(\sigma_n)_{n\in\mathbb{Z}}$ are independent, respectively distributed as simple random walk starting at x, as simple random walk starting at x conditioned never to return to K, and as a doubly infinite sequence of independent exponential variables with parameter 1.

The existence and uniqueness of such a measure $\widehat{\nu}$ can be shown just as in Section 1 of [19]. The canonical continuous time interlacement point process is then constructed on a space $(\Omega, \mathcal{A}, \mathbb{P})$, similar to (1.16) of [19], with $\omega = \sum_{i\geq 0} \delta_{(\widehat{w}_i^*, u_i)}$ denoting a generic element of the set Ω. We also refer to Remark 2.4 4) which explains how $\mathbb{Z}^d$, $d \geq 3$, can be replaced with a transient weighted graph, and continuous time random interlacements on a transient weighted graph are constructed.

In the present work our main interest focuses on the collection of (continuous) occupation times:

$$
\begin{aligned}
&L_{x,u}(\omega) = \sum_{i\geq 0} \sum_{n\in\mathbb{Z}} \sigma_n(\widehat{w}_i)\, 1\{X_n(\widehat{w}_i) = x,\, u_i \leq u\}, \text{ for } x \in \mathbb{Z}^d, u \geq 0, \\
&\text{where } \omega = \sum_{i\geq 0} \delta_{(\widehat{w}_i^*, u_i)} \in \Omega, \text{ and } \pi^*(\widehat{w}_i) = \widehat{w}_i^*, \text{ for each } i \geq 0.
\end{aligned} \tag{0.2}
$$

We compute the Laplace functional of this random field, and show in Theorem 2.3 that when V is a non-negative function on $\mathbb{Z}^d$ with finite support one has the identity:

$$
\mathbb{E}\Big[\exp\Big\{-\sum_{x\in\mathbb{Z}^d} V(x) L_{x,u}\Big\}\Big] = \exp\left\{-u \frac{\sum_{\phi\neq I} c_I \,\Pi_{x\in I} V(x)}{\sum_I g_I \,\Pi_{x\in I} V(x)}\right\}, \ u \geq 0, \tag{0.3}
$$

where in the above formula I runs over the collection of subsets of the support of V, g_I denotes the determinant of the Green function $g(\cdot,\cdot)$ restricted to $I \times I$, see (1.1), and c_I the sum of the coefficients of the matrix of cofactors of the above matrix. Both quantities are positive and, see (2.13), their ratio c_I/g_I coincides with the capacity of I, i.e. the total mass of the equilibrium measure e_I of I. We refer to (2.26) for the extension of this formula to the case where $\mathbb{Z}^d$ is replaced by a transient weighted graph. One can also consider the discrete occupation times, where σ_n is replaced by 1 in (0.2), however this random field turns out to be somewhat less convenient to handle than $(L_{x,u})_{x\in\mathbb{Z}^d}$ for the kind of questions we investigate here, see Remark 2.4 5).

The continuous time interlacement point process is related to the Poisson point process of Markov loops initiated in [18], which later found various incarnations, see for instance Theorem 2.1 of [3], Section 4 and 3 of [5], [9], Chapter 9 of [8], and was extensively analyzed in [10], [11]. Heuristically random interlacements correspond to a "restriction to loops going through infinity" of this Poisson point process, see [11], p. 85. It has been shown in Theorem 13 of [10], see also [11], p. 61, that the field of occupation times of the Poisson point process of Markov loops on a finite weighted graph with non-degenerate killing, at a suitable choice of the level is distributed as half the square of the Gaussian free field on the finite graph. No such identity holds in our context when considering a fixed level u, see Remark 2.4. However, and this is the main object of this article, we present limiting procedures which relate the field of occupation times of random interlacements to the Gaussian free field.

The link with the two-dimensional Gaussian free field comes as follows. We look at the occupation times of long vertical rods in $\mathbb{Z}^3$, by random interlacements at properly tuned levels. Let us incidentally mention that the consideration of long rods in the context of random interlacements has been helpful in several instances, e.g. Section 3 of [15], or Section 5 of [20]. They typically have been used as a tool in the detection of long $*$-crossings in planes, left by the trajectories of the random interlacements at level u, and have enabled to quantify the rarity of such crossings when u is small. Here the rods in question are the subsets of $\mathbb{Z}^3$:

$$J_y = \{x = (y,k) \in \mathbb{Z}^3;\ 1 \le k \le N\}, \text{ for } y \in \mathbb{Z}^2, \text{ and } N > 1, \tag{0.4}$$

and the corresponding $\mathbb{Z}^2$-stationary field of occupation times is given by

$$\mathcal{L}_{y,u} = \sum_{x \in J_y} L_{x,u},\ y \in \mathbb{Z}^2,\ u \ge 0. \tag{0.5}$$

We choose the levels $(u_N)_{N>1}$ and $(u'_N)_{N>1}$, so that

$$\text{i) } u_N = \alpha\, \frac{\log N}{N}, \text{ with } \alpha > 0, \qquad \text{ii) } \frac{\log N}{N} = o(u'_N). \tag{0.6}$$

The choice in (0.6) i) corresponds to a non-degenerate limiting probability $\exp\{-\frac{\pi}{3}\,\alpha\}$ that the interlacement at level u_N does not meet any given rod J_y, see (4.74), whereas the choice in (0.6) ii) induces a vanishing limit for the corresponding probability.

If we now introduce the Gaussian free field pinned at the origin, or more precisely, see (1.29), a centered Gaussian field $(\psi_y)_{y\in\mathbb{Z}^2}$, with covariance $3(a(y)+a(y')-a(y'-y))$, $y, y' \in \mathbb{Z}^2$, where $a(\cdot)$ is the potential kernel of the two-dimensional simple random walk, see (1.6), and R an independent non-negative random variable, having the law $BES^o(\sqrt{\alpha}, \frac{3}{2\pi})$ of a zero-dimensional Bessel process at time $\frac{3}{2\pi}$ starting in $\sqrt{\alpha}$ at time 0, see (1.30), we show in Theorems 4.2 and 4.9 that when N tends to infinity,

$$\Big(\frac{\mathcal{L}_{y,u_N}}{\log N}\Big)_{y\in\mathbb{Z}^2} \text{ converges in distribution to the flat field with value } R^2, \tag{0.7}$$

and that

$$\Big(\frac{\mathcal{L}_{y,u_N} - \mathcal{L}_{0,u_N}}{\sqrt{\log N}}\Big)_{y\in\mathbb{Z}^2} \text{ converges in distribution to the random field } (R\psi_y)_{y\in\mathbb{Z}^2}. \tag{0.8}$$

In the case (0.6) ii) we instead find that when N goes to infinity,

$$\Big(\frac{\mathcal{L}_{y,u'_N}}{N u'_N}\Big)_{y\in\mathbb{Z}^2} \text{ converges in distribution to the flat field with value } 1, \tag{0.9}$$

and that

$$\Big(\frac{\mathcal{L}_{y,u'_N} - \mathcal{L}_{0,u'_N}}{\sqrt{N u'_N}}\Big)_{y\in\mathbb{Z}^2} \text{ converges in distribution to } (\psi_y)_{y\in\mathbb{Z}^2}. \tag{0.10}$$

There is an important connection between random interlacements and the structure left locally by random walk on a large torus, see [23], [22]. In this light one may wonder whether some of the above results have counterparts in the case of simple random walk on a large two-dimensional torus. We refer to Remark 4.10 1) for more on this issue. Some consequences of the above limit results for discrete occupation times of long rods can also be found in Remark 4.10 2).

In this article, we yet provide a further link between random interlacements and the Gaussian free field, by considering the occupation times of random interlacements at a level u tending to infinity. If $(\gamma_x)_{x\in\mathbb{Z}^d}$ stands for the Gaussian free field on $\mathbb{Z}^d$, $d \geq 3$, i.e. the centered Gaussian field with covariance function $E[\gamma_x \gamma_{x'}] = g(x,x')$, $x, x' \in \mathbb{Z}^d$, we show in Theorem 5.1 that when u tends to infinity,

$$\Big(\frac{1}{u}\, L_{x,u}\Big)_{x\in\mathbb{Z}^d} \text{ converges in distribution towards the flat field with value 1,} \tag{0.11}$$

and that

$$\Big(\frac{L_{x,u} - L_{x,0}}{\sqrt{2u}}\Big)_{x\in\mathbb{Z}^d} \text{ converges in distribution towards } (\gamma_x - \gamma_0)_{x\in\mathbb{Z}^d}. \tag{0.12}$$

We refer to Remark 5.2 for the extension of these results to the case of random interlacements on a transient weighted graph, and to discrete occupation times.

Let us say a few words concerning proofs. We provide in Theorem 2.1 an expression for the characteristic function of $\sum_{x\in\mathbb{Z}^d} V(x) L_{x,u}$, with V finitely supported, which shows that close to the origin it can be expressed as the exponential of an analytic function. This identity on the one hand leads to (0.3), see Theorem 2.3. On the other hand, this identity underlies the general line of attack, which we employ when proving the limit theorems corresponding to (0.7) - (0.10), and (0.11) - (0.12). Namely we investigate the asymptotic behavior of the power series representing the above mentioned analytic functions. The proof of (0.8) is by far the most delicate. We analyze the large N asymptotics of the power series expressing the logarithm of the characteristic function of $\sum_{y\in\mathbb{Z}^2} W(y)\mathcal{L}_{y,u_N}$ close to the origin, with W finitely supported on $\mathbb{Z}^2$, and such that $\sum_y W(y) = 0$. This asymptotic analysis relies on certain cancellations, which take place and enable to control the coefficients of the power series. In the crucial Theorem 4.1 we bound these coefficients, show the asymptotic vanishing of odd coefficients, and compute the (non-vanishing) limit of even coefficients. This theorem contains enough information to yield both (0.8) and (0.10), see Theorem 4.2. Once (0.8), (0.10) are proved, (0.7), (0.8) follow in a simpler fashion and in essence only require the consideration of one single rod, say J_0. The proof of (0.11), (0.12) in Theorem 5.1 follows a similar pattern, but is substantially simpler.

Let us now describe how this article is organized.

In Section 1 we provide additional notation and collect some results concerning potential theory, the two-dimensional free field, and zero-dimensional Bessel processes.

Section 2 contains the identity for the characteristic functional of the field of occupation times in Theorem 2.1 and the proof of formula (0.3) for the Laplace functional in Theorem 2.3. The extension of these results to the set-up of weighted graphs can be found in Remark 2.4 4).

In Section 3 we collect estimates as preparation for the study in the next section of occupation times of long rods in $\mathbb{Z}^3$.

Section 4 presents the limiting results (0.7) - (0.10) , see Theorems 4.2 and 4.9, relating random interlacements in $\mathbb{Z}^3$ to the two-dimensional free field. The heart of the matter lies in Theorem 4.1, where controls over the relevant power series are derived.

In Section 5 we prove (0.11), (0.12) in Theorem 5.1 and provide in Remark 5.2 the extension of these results to the case of transient weighted graphs, and to discrete occupation times.

Finally let us explain our convention concerning constants. We denote with $c, c', \widetilde{c}, \overline{c}$ positive constants changing from place to place. Numbered constants refer to the value corresponding to their first appearance in the text. In Sections 1, 2 and 5 constants only depend on d. In Section 3, where $d = 3$, they depend on Λ in (3.1), and in Section 3, where $d = 3$ as well, on Λ and W, see (4.3). Otherwise dependence of constants on additional parameters appears in the notation.

1 Notation and some useful facts

In this section we provide some additional notation and recall various useful facts concerning random walks, discrete potential theory, the two-dimensional free field, and zero-dimensional Bessel processes.

We let $\mathbb{N} = \{0, 1, \dots\}$ denote the set of natural numbers. When u is a non-negative real number we let $[u]$ stand for the integer part of u. Given a finite set A, we denote by $|A|$ its cardinality. We write $|\cdot|$ for the Euclidean norm on $\mathbb{R}^d$, $d \ge 1$. For $A, A' \subseteq \mathbb{Z}^d$, we denote by $d(A, A') = \inf\{|x - x'|;\, x \in A,\, x' \in A'\}$ the mutual distance of A and A'. When $A = \{x\}$, we write $d(x, A')$ in place of $d(A, A')$ for simplicity. We write $U \subset\subset \mathbb{Z}^d$, to indicate that U is a finite subset of $\mathbb{Z}^d$. Given f, g square summable functions on $\mathbb{Z}^d$ we write $(f, g) = \sum_{x \in \mathbb{Z}^d} f(x) g(x)$ for their scalar product. When $U \subseteq \mathbb{Z}^d$, and f is a function on U, we routinely identify f with the function on $\mathbb{Z}^d$, which vanishes outside U and coincides with f on U. We denote the sup-norm of such a function with $\|f\|_{L^\infty(U)}$, and sometimes with $\|f\|_\infty$, when there is no ambiguity.

Given $U \subseteq \mathbb{Z}^d$, we write $H_U = \inf\{n \ge 0; X_n \in U\}$, $\widetilde{H}_U = \inf\{n \ge 1; X_n \in U\}$, and $T_U = \inf\{n \ge 0; X_n \notin U\}$ for the entrance time of U, the hitting time of U, and the exit time from U. When ρ is a measure on $\mathbb{Z}^d$, we denote by P_ρ the measure $\sum_{x \in \mathbb{Z}^d} \rho(x) P_x$, and by E_ρ the corresponding expectation. So far P_x, $x \in \mathbb{Z}^d$, has only been defined when $d \ge 3$, see above (0.1). When $d = 1$ or 2, this notation simply stands for the canonical law of simple random walk starting at x, and X_n, $n \ge 0$, for the canonical process.

When $d \ge 3$, we denote by $g(\cdot, \cdot)$ the Green function

$$g(x, x') = \sum_{n \ge 0} P_x[X_n = x'], \text{ for } x, x' \text{ in } \mathbb{Z}^d. \tag{1.1}$$

It is a symmetric function, and due to translation invariance one has

$$g(x, x') = g(x' - x) = g(x - x'), \text{ where } g(\cdot) = g(\cdot, 0). \tag{1.2}$$

Classically one knows that $g(\cdot) \le g(0)$, and that, see [7], p. 31,

$$c'(1 \vee |x|)^{-(d-2)} \le g(x) \le c(1 \vee |x|)^{-(d-2)}, \text{ for } x \in \mathbb{Z}^d\,. \tag{1.3}$$

When $K \subset\subset \mathbb{Z}^d$, the equilibrium measure of K and the capacity of K, i.e. the total mass of e_K, are denoted by

$$(1.4) \qquad e_K(x) = P_x[\widetilde{H}_K = \infty]\, 1_K(x), \text{ for } x \in \mathbb{Z}^d, \text{ and } \mathrm{cap}(K) = \sum_{x \in \mathbb{Z}^d} P_x[\widetilde{H}_K = \infty].$$

One can express the probability to enter K via the formula

$$(1.5) \qquad P_x[H_K < \infty] = \sum_{x' \in K} g(x, x')\, e_K(x'), \text{ for } x \in \mathbb{Z}^d.$$

We will also consider the two-dimensional potential kernel, see (1.40), p. 37 of [7], or p. 121, 122, 148 of [17]:

$$(1.6) \qquad a(y) = \lim_{n \to \infty} \sum_{j=0}^{n} P_0[X_j = 0] - P_0[X_j = y], \text{ for } y \in \mathbb{Z}^2\,.$$

It is a non-negative function on $\mathbb{Z}^2$, which is symmetric and satisfies, cf. P2, p. 123 of [17]:

$$(1.7) \qquad \lim_{y' \to \infty} a(y + y') - a(y') = 0, \text{ for any } y \in \mathbb{Z}^2.$$

In Sections 3 and 4, see also (0.4), we consider long vertical rods, which are the subsets of $\mathbb{Z}^3$ defined for $y \in \mathbb{Z}^2$ and $N > 1$, by

$$(1.8) \qquad J_y = \{y\} \times J \subseteq \mathbb{Z}^3, \text{ where } J = \{1, \dots N\}.$$

The next lemma collects limit statements concerning the potentials of long rods, and in particular relates the difference of such potentials to the two-dimensional potential kernel.

Lemma 1.1. $(d = 3, N > 1, y \in \mathbb{Z}^2)$

$$(1.9) \qquad \lim_N \frac{1}{2 \log N} \sum_{|z| \le N} g((0, z)) = \frac{3}{2\pi}, \text{ (with } z \in \mathbb{Z} \text{ and } (0, z) \in \mathbb{Z}^3\text{)}.$$

For $x = (0, z)$ in J_0 and $y \in \mathbb{Z}^2$, one has

$$(1.10) \qquad \sum_{x' \in J_0} g(x, x') - \sum_{x'' \in J_y} g(x, x'') = \frac{3}{2}\, a(y) - b_N(y, z)\,,$$

where b_N is a non-negative function on $\mathbb{Z}^2 \times J$ such that

$$(1.11) \qquad b_N(y, z) \le \psi_y(d(z, J^c)), \text{ where } \lim_{r \to \infty} \psi_y(r) = 0, \text{ for each } y \in \mathbb{Z}^2.$$

Proof. The claim (1.9) is an immediate consequence of the fact that $\sum_1^N \frac{1}{k} \sim \log N$, as N goes to infinity and, cf. Theorem 1.5.4, p. 31 of [7]:

$$(1.12) \qquad g(x) \sim \frac{3}{2\pi}\, |x|^{-1}, \text{ as } x \to \infty\,.$$

We now turn to the proof of (1.10), (1.11). We denote by $\widetilde{Y}_.$ and $\widetilde{Z}_.$ independent continuous time random walks on $\mathbb{Z}^2$ and $\mathbb{Z}$ with respective jump rates 2 and 1, starting at the respective origins of $\mathbb{Z}^2$ and $\mathbb{Z}$. So $(\widetilde{Y}_., \widetilde{Z}_.)$ is a continuous time random walk on $\mathbb{Z}^3$, starting at the origin, with jump rate equal to 3, and the left-hand side of (1.10) equals:

$$\tag{1.13} \begin{aligned} &3E\Big[\int_0^\infty 1\{\widetilde{Y}_s = 0,\ \widetilde{Z}_s + z \in J\}\, ds - \int_0^\infty 1\{\widetilde{Y}_s = y,\ \widetilde{Z}_s + z \in J\} ds\Big] \overset{\text{independence}}{=} \\ &3\int_0^\infty (P[\widetilde{Y}_s = 0] - P[\widetilde{Y}_s = y])\, P[\widetilde{Z}_s + z \in J]\, ds = 3(I_1 - I_2), \end{aligned}$$

where we have set

$$\tag{1.14} \begin{aligned} I_1 &= \int_0^\infty P[\widetilde{Y}_s = 0] - P[\widetilde{Y}_s = y]\, ds, \\ I_2 &= \int_0^\infty (P[\widetilde{Y}_s = 0] - P[\widetilde{Y}_s = y])\, P[\widetilde{Z}_s + z \notin J]\, ds, \end{aligned}$$

and note that the integrand in I_1 is non-negative as a direct application of the Chapman-Kolmogorov equation at time $\frac{s}{2}$ and the Cauchy-Schwarz inequality. If we let $Y_k, k \geq 0$, and $T_k, k \geq 0$, (with $T_0 = 0$), stand for the discrete skeleton of $\widetilde{Y}_.$ and its successive jump times, we see that for $T > 0$,

$$\tag{1.15} \begin{aligned} &\int_0^T P[\widetilde{Y}_s = 0] - P[\widetilde{Y}_s = y]\, ds = \sum_{k \geq 0} E[(T_{k+1} \wedge T - T_k \wedge T)\, 1\{Y_k = 0\}] - \\ &E[(T_{k+1} \wedge T - T_k \wedge T)\, 1\{Y_k = y\}] \overset{\text{independence}}{=} \\ &\sum_{k \geq 0} E[T_{k+1} \wedge T - T_k \wedge T](P[Y_k = 0] - P[Y_k = y]). \end{aligned}$$

Observe that $T_{k+1} - T_k$ is an exponential variable with parameter 2, which is independent from T_k, so that for $k \geq 0$

$$\tag{1.16} a_{k,T} \overset{\text{def}}{=} E[T_{k+1} \wedge T - T_k \wedge T] = E\Big[T_k \leq T,\ 2\int_0^\infty s \wedge (T - T_k)\, e^{-2s} ds\Big]$$

decreases to zero as k tends to infinity and increases to $\frac{1}{2}$ as T tends to ∞. We set $s_k = \sum_{0 \leq j \leq k} P[Y_j = 0] - P[Y_j = y]$, for $k \geq 0$, so that by (1.6), $\lim_k s_k = a(y)$. After summation by parts in the last member of (1.15), we find that:

$$\tag{1.17} \int_0^T P[\widetilde{Y}_s = 0] - P[\widetilde{Y}_s = y]\, ds = \sum_{k \geq 0} (a_{k,T} - a_{k+1,T}) s_k\, .$$

Using the observations below (1.16) we see that the left-hand side of (1.17) tends to $\frac{1}{2}a(y)$ as T goes to infinity, so that

$$\tag{1.18} I_1 = \frac{1}{2}\, a(y)\, .$$

As for I_2, which is non-negative due to the remark below (1.14), we see that $3I_2 \leq \psi_y(d(z, J^c))$, where we have set

$$\tag{1.19} \psi_y(r) = 3\int_0^\infty (P[\widetilde{Y}_s = 0] - P[\widetilde{Y}_s = y])\, P[|\widetilde{Z}_s| \geq r]\, ds, \text{ for } r \geq 0.$$

If is plain that ψ_y is a non-increasing function, which tends to zero at infinity by dominated convergence. This concludes the proof of Lemma 1.1. □

We now turn to the discussion of the two-dimensional massless Gaussian free field pinned at the origin. For this purpose we begin by the consideration of the more traditional two-dimensional massless Gaussian free field with Dirichlet boundary conditions outside the square $U_L = [-L, L]^2$, with $L \ge 1$, see for instance [2]. It is a centered Gaussian field $\varphi_{y,L}$, $y \in \mathbb{Z}^2$, with covariance function

$$E[\varphi_{y,L}\,\varphi_{y',L}] = g_L(y, y'), \text{ for } y, y' \in \mathbb{Z}^2, \tag{1.20}$$

where $g_L(\cdot, \cdot)$ stands for the Green function of the two-dimensional random walk killed when exiting U_L:

$$g_L(y, y') = E_y\Big[\sum_{k\ge 0} 1\{X_k = y', k < T_{U_L}\}\Big], \text{ for } y, y' \in \mathbb{Z}^2. \tag{1.21}$$

Writing H_0 in place of $H_{\{0\}}$, see above (1.1), it follows from the strong Markov property and (1.20), (1.21), that for any $y \in \mathbb{Z}^2$,

$$\varphi_{y,L} - P_y[H_0 < T_{U_L}]\,\varphi_{0,L} \text{ is orthogonal to } \varphi_{0,L}.$$

Hence defining for any $\gamma \in \mathbb{R}$,

$$\Phi_{y,L}(\gamma) = \varphi_{y,L} - P_y[H_0 < T_{U_L}]\,\varphi_{0,L} + P_y[H_0 < T_{U_L}]\,\gamma, \; y \in \mathbb{Z}^2\,, \tag{1.22}$$

the law of the above random field is a regular conditional probability for the law of $(\varphi_{y,L})$ given its value at the origin $\varphi_{0,L} = \gamma$. The next lemma will provide two possible interpretations for the centered Gaussian field we consider in the sequel, in terms of the two-dimensional massless Gaussian free field.

Lemma 1.2. *For y, y' in $\mathbb{Z}^2$, one has*

$$\begin{aligned} a(y) + a(y') - a(y' - y) &= \lim_{L\to\infty} E[(\varphi_{y,L} - \varphi_{0,L})(\varphi_{y',L} - \varphi_{0,L})] \\ &= \lim_{L\to\infty} E[\Phi_{y,L}(0)\,\Phi_{y',L}(0)]\,. \end{aligned} \tag{1.23}$$

Proof. By (1.20) we see that

$$\begin{aligned} &E[(\varphi_{y,L} - \varphi_{0,L})(\varphi_{y',L} - \varphi_{0,L})] = g_L(y, y') + g_L(0, 0) - g_L(0, y) - g_L(0, y') = \\ &g_L(y, y) - g_L(0, y) + g_L(0, 0) - g_L(0, y') - (g_L(y, y) - g_L(y, y')). \end{aligned} \tag{1.24}$$

From Proposition 1.6.3, p. 39 of [7], one knows that for y_1, y_2 in U_L:

$$g_L(y_1, y_2) = \sum_{y'\in\partial U_L} P_{y_1}[X_{T_{U_L}} = y']\,a(y' - y_2) - a(y_1 - y_2), \tag{1.25}$$

so that by (1.7) and $a(0) = 0$, we find that

$$\lim_{L\to\infty} g_L(y_1, y_1) - g_L(y_1, y_2) = a(y_1 - y_2), \text{ for } y_1, y_2 \in \mathbb{Z}^2. \tag{1.26}$$

Coming back to (1.24) and keeping in mind the symmetry of $g_L(\cdot, \cdot)$, the first equality of (1.23) follows. As for the second equality, we note that

$$\Phi_{y,L}(0) = \varphi_{y,L} - \varphi_{0,L} + P_y[H_0 > T_{U_L}]\,\varphi_{0,L}. \tag{1.27}$$

By the strong Markov property and the symmetry of $g_L(\cdot,\cdot)$ one has

$$P_y[H_0 > T_{U_L}] = (g_L(0,0) - g_L(0,y))/g_L(0,0) \overset{(1.26)}{\leq} c(y)/g_L(0,0),$$

and by (1.20) one finds that

$$E[\varphi_{0,L}^2] = g_L(0,0)\,. \tag{1.28}$$

Since $\lim_L g_L(0,0) = \infty$, it follows that the last term of (1.27) converges to 0 in L^2 as L tends to infinity, and the second equality of (1.23) now follows. □

We thus introduce on some auxiliary probability space

(1.29) $\psi_y, y \in \mathbb{Z}^2$, a centered Gaussian field with covariance function $E[\psi_y \psi_{y'}] = 3(a(y) + a(y') - a(y'-y))$, $y, y' \in \mathbb{Z}^2$.

Up to an inessential multiplicative factor $\sqrt{3}$, we can thus interpret $\psi_y, y \in \mathbb{Z}^2$, as the field of "increments at the origin" of the two-dimensional massless free field, or as the two-dimensional massless free field pinned at the origin.

The last topic of this section concerns zero-dimensional Bessel processes. We denote by $BES^0(a,\tau)$ the law at time $\tau \geq 0$ of a zero-dimensional Bessel process starting from $a \geq 0$. If R is a random variable with distribution $BES^0(a,\tau)$, the Laplace transform of R^2 is given by the formula, see [14], p. 411, or [6], p. 239:

$$E[e^{-\lambda R^2}] = \exp\Big\{ - \frac{\lambda a^2}{1 + 2\tau\lambda}\Big\}, \text{ for } \lambda \geq 0\,. \tag{1.30}$$

We also denote by $BESQ^0(a^2,\tau)$ the law of R^2; this is the distribution of a zero-dimensional square Bessel process at time τ, starting from a^2 at time 0.

2 Laplace functional of occupation times

In this section we obtain a formula for the Laplace functional of the occupation times $L_{x,u}$, which proves (0.3), see Theorem 2.3. As a by-product we note the absence for fixed u of a global factorization for the field $L_{x,u} - L_{0,u}$, $x \in \mathbb{Z}^d$, similar to that of the limit law in (0.8), even through each individual variable $L_{x,u} - L_{0,u}$ is distributed as the product of a time-marginal of a zero-dimensional Bessel process with an independent centered Gaussian variable, see Remark 2.4 2). The preparatory Theorem 2.1 will be repeatedly used in the sequel, and shows in particular that the characteristic function of a finite linear combination of the variables $L_{x,u}$, $x \in \mathbb{Z}^d$, is analytic in the neighborhood of the origin. This will play an important role in Section 4.

We denote by G the linear operator

$$Gf(x) = \sum_{x' \in \mathbb{Z}^d} g(x,x')\, f(x'),\ x \in \mathbb{Z}^d\,, \tag{2.1}$$

which is well defined when $\sum_{x'} g(x,x')|f(x')| < \infty$, and in particular when f vanishes outside a finite set. When V is a function on $\mathbb{Z}^d$ vanishing outside a finite set, we write GV for the composition of G with the multiplication operator by V, so that GV naturally operates on $L^\infty(\mathbb{Z}^d)$, (we recall that $\|\cdot\|_\infty$ denotes the corresponding sup-norm, see the beginning of Section 1).

Theorem 2.1. *If V has support in $K \subset\subset \mathbb{Z}^d$, then for $\|V\|_\infty \le c(K)$,*

$$\|GV\|_{L^\infty \to L^\infty} < 1\,, \tag{2.2}$$

and for any $u \ge 0$,

$$\mathbb{E}\Big[\exp\Big\{\sum_{x \in \mathbb{Z}^d} V(x) L_{x,u}\Big\}\Big] = \exp\{u(V, (I - GV)^{-1} 1)\}. \tag{2.3}$$

Proof. The claim (2.2) is immediate. As for (2.3), we note that defining for $x \in \mathbb{Z}^d$, $u \ge 0$, the function on $\widehat{W}^*$, (see above (0.1) for notation):

$$\gamma_x(\widehat{w}^*) = \sum_{n \in \mathbb{Z}} \sigma_n(\widehat{w})\, 1\{X_n(\widehat{w}) = x\}, \text{ for any } \widehat{w} \in \widehat{W} \text{ with } \pi^*(\widehat{w}) = \widehat{w}^*,$$

we have the identity

$$L_{x,u}(\omega) = \sum_i \gamma_x(\widehat{w}_i^*)\, 1\{u_i \le u\}, \text{ for } \omega = \sum_i \delta_{(\widehat{w}_i^*, u_i)} \in \Omega,\ x \in \mathbb{Z}^d,\ u \ge 0\,. \tag{2.4}$$

The interlacement point process ω is Poisson with intensity measure $\widehat{\nu}(d\widehat{w}^*)du$ under $\mathbb{P}$, and hence when V is supported in $K \subset\subset \mathbb{Z}^d$ and $\|V\|_\infty \le c(K)$, we have

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big\{\sum_{x \in \mathbb{Z}^d} V(x) L_{x,u}\Big\}\Big] \\
&= \mathbb{E}\Big[\exp\Big\{\int_{\widehat{W}^* \times \mathbb{R}_+} \sum_{x \in \mathbb{Z}^d} V(x) \gamma_x(\widehat{w}^*)\, 1\{v \le u\}\, d\omega(\widehat{w}^*, v)\Big\}\Big] \\
&= \exp\Big\{u \int_{\widehat{W}^*} \big(e^{\sum_{x \in \mathbb{Z}^d} V(x)\gamma_x(\widehat{w}^*)} - 1\big)\, d\widehat{\nu}(\widehat{w}^*)\Big\} \\
&\overset{(0.1)}{=} \exp\Big\{u\, E_{e_K}\big[e^{\sum_{x \in \mathbb{Z}^d} V(x) \sum_{k \ge 0} \sigma_k 1\{X_k = x\}} - 1\big]\Big\} \\
&= \exp\Big\{u\, E_{e_K}\Big[\exp\Big\{\int_0^\infty V(\overline{X}_s)\, ds\Big\} - 1\Big]\Big\},
\end{aligned} \tag{2.5}$$

where for $s \ge 0$, $\widehat{w} \in \widehat{W}_+$, we have set

$$\overline{X}_s(\widehat{w}) = X_k(\widehat{w}), \text{ when } \sigma_0(\widehat{w}) + \dots + \sigma_{k-1}(\widehat{w}) \le s < \sigma_0(\widehat{w}) + \dots + \sigma_k(\widehat{w}),$$

(by convention the term bounding s from below vanishes when $k = 0$), i.e. $\overline{X}_.$ is the natural continuous time random walk on $\mathbb{Z}^d$ with jump parameter 1 defined on $\widehat{W}_+$. Thus for $\|V\|_\infty \le c(K)$ we find by a classical calculation that

$$\begin{aligned}
&E_{e_K}\big[e^{\int_0^\infty V(\overline{X}_s)ds}\big] = E_{e_K}\Big[\sum_{n \ge 0} \frac{1}{n!}\Big(\int_0^\infty V(\overline{X}_s) ds\Big)^n\Big] = \\
&\sum_{n \ge 0} E_{e_K}\Big[\int_{0 < s_1 < \dots < s_n < \infty} V(\overline{X}_{s_1}) \dots V(\overline{X}_{s_n})\, ds_1 \dots ds_n\Big] = \\
&\mathrm{cap}(K) + \sum_{n \ge 1} \sum_{x \in K} e_K(x)[(GV)^n\, 1](x)
\end{aligned} \tag{2.6}$$

using Fubini's theorem and the Markov property in the last step. Since V vanishes outside K, it also follows from (1.5) that for $n \ge 1$, one has:

$$\sum_x e_K(x)[(GV)^n\, 1](x) = \sum_{x'} V(x')[(GV)^{n-1} 1](x') = (V, (GV)^{n-1} 1). \tag{2.7}$$

As a result we see that when $\|V\|_\infty \le c(K)$,

$$E_{e_K}\big[e^{\int_0^\infty V(\overline{X}_s)ds} - 1\big] = \sum_{n\ge 1} (V, (GV)^{n-1}1) = (V, (I - GV)^{-1}1). \tag{2.8}$$

Inserting this identity in the last line of (2.5) concludes the proof of Theorem 2.1. □

Remark 2.2. As a staightforward consequence of Theorem 2.1 see that for any finitely supported real valued function V on $\mathbb{Z}^d$ and $u \ge 0$, the random variable $\sum_{x\in\mathbb{Z}^d} V(x)L_{x,u}$ has a characteristic function which coincides in the neighborhood of the origin with the exponential of an analytic function. So this characteristic function is analytic in the sense of Chapter 7 of [12]. In particular with Theorem 7.1.1, p. 193 of [12], one has the identity

$$\mathbb{E}\Big[\exp\big\{z \sum_x V(x)\, L_{x,u}\big\}\Big] = \Phi_{V,u}(z),\ z \in S, \tag{2.9}$$

where in the above formula S stands for the maximal vertical strip in $\mathbb{C}$ to which the function $z \to \exp\{u \sum_{n\ge 1} z^n (V, (GV)^{n-1}1)\}$ can be an analytically extended, and $\Phi_{V,u}$ for this extension, (that is, for $z \in S$, $\exp\{z \sum_x V(x)L_{x,u}\}$ is integrable, and the equality(2.9) holds). This fact will be very helpful and repeatedly used in the sequel. □

We now derive an alternative expression for the right-hand side of (2.3), and need some additional notation for this purpose.

For $I \subset\subset \mathbb{Z}^d$ non-empty, we denote by G_I the matrix $g(x, x')$, $x, x' \in I$. It is well-known to be positive definite, see for instance Lemma 3.3.6 of [13], and we introduce

$$g_I = \det(G_I) > 0, \tag{2.10}$$

where the right-hand side does not depend on the identification of I with $\{1, \dots, |I|\}$ we use. We also set by convention $g_I = 1$, when $I = \phi$. Further we introduce

$$c_I = \text{the sum of all coefficients of the matrix of cofactors of } G_I, \tag{2.11}$$

and note that c_I does not depend on the identification of I with $\{1, \dots, |I|\}$ we employ; for instance c_I/g_I coincides with the sum of all coefficients of the inverse matrix of G_I. The above also shows that

$$c_I > 0, \text{ when } I \subset\subset \mathbb{Z}^d \text{ is non-empty.} \tag{2.12}$$

We extend the notation to the case $I = \phi$ with the convention $c_\phi = 0$. It is known, see [17], p. 301, that

$$\text{cap}(I) = c_I/g_I, \text{ for all } I \subset\subset \mathbb{Z}^d. \tag{2.13}$$

We are now ready for the main result of this section.

Theorem 2.3. *When V has support in $K \subset\subset \mathbb{Z}^d$, then for $\|V\|_\infty \le c(K)$, $u \ge 0$,*

$$\mathbb{E}\Big[\exp\big\{-\sum_x V(x)\, L_{x,u}\big\}\Big] = \exp\Big\{-u\, \frac{\sum_{I\subset K} c_I\, V_I}{\sum_{I\subset K} g_I\, V_I}\Big\}, \tag{2.14}$$

where $V_I \stackrel{\text{def}}{=} \Pi_{x\in I} V(x)$, ($V_I = 1$, by convention when $I = \phi$).

In addition (2.14) *holds whenever V is non-negative and vanishes outside K.*

Proof. By Theorem 2.1 we know that for $\|V\|_\infty \le c(K)$, the left-hand side of (2.14) equals $\exp\{-u(V,(I+GV)^{-1}1)\}$. With no loss of generality we assume that $|K| \ge 2$. We identify K with $\{1,\dots,n\}$, where $n = |K|$, via an enumeration $x_1,\dots,x_n$ of K. Writing $v_\ell = V(x_\ell)$, we see that

$$(V,(I+GV)^{-1}1) = \sum_{k,\ell=1}^{n} \boldsymbol{C}_{k,\ell}\, v_\ell / \det(\boldsymbol{I}+\boldsymbol{GV}), \tag{2.15}$$

where $\boldsymbol{G}$ stands for the $n\times n$ matrix $g(x_k,x_\ell)$, $1\le k,\ell\le n$, $\boldsymbol{V}$ for the diagonal matrix with coefficients v_ℓ, $1\le \ell\le n$, on the diagonal, $\boldsymbol{I}$ for the identity matrix, and $\boldsymbol{C}$ for the matrix of cofactors of $\boldsymbol{I}+\boldsymbol{GV}$. Observe that for $1\le k,\ell\le n$, one has

$$\boldsymbol{C}_{k,\ell} = \det((\boldsymbol{I}+\boldsymbol{GV})^{k,\ell}), \tag{2.16}$$

where $(\boldsymbol{I}+\boldsymbol{GV})^{k,\ell}$ stands for the $n\times n$ matrix, where the k-th line and the ℓ-th column of $\boldsymbol{I}+\boldsymbol{GV}$ have been replaced by 0, except for the coefficient at their intersection, which is replaced by 1.

Given an $n\times n$ matrix $\boldsymbol{A}=(a_{k,\ell})$, we develop the determinant of $\boldsymbol{A}$ according to the classical formula

$$\det \boldsymbol{A} = \sum_\sigma \operatorname{sign}(\sigma) \prod_{k=1}^{n} a_{k,\sigma(k)}, \tag{2.17}$$

where σ runs over the permutations of $\{1,\dots,n\}$ and $\operatorname{sign}(\sigma)$ denotes the signature of σ.

We now develop the determinant $\det(\boldsymbol{I}+\boldsymbol{GV})$. For each subset $J\subseteq\{1,\dots,n\}$ we collect the terms corresponding to permutations σ of $\{1,\dots,n\}$ such that $\sigma(k)=k$, for $k\in J$, the choice of 1 in each term $(1+g(0)v_k)$, $k\in J$, and for any $k\notin J$ such that $\sigma(k)=k$, the choice of $g(0)v_k$ instead. Thus for each such J, setting $\widetilde{J}=\{1,\dots,n\}\backslash J$, the sum of these terms equals $\det(\boldsymbol{G}_{|\,\widetilde{J}\times\widetilde{J}})\prod_{\ell\in\widetilde{J}} v_\ell$. Thus summing over all subsets J of $\{1,\dots,n\}$ we find:

$$\det(\boldsymbol{I}+\boldsymbol{GV}) = \sum_{I\subseteq K} g_I V_I\,. \tag{2.18}$$

We now turn to the numerator of the right-hand side of (2.15). We use the convention $\{k,\ell\}=\{k\}=\{\ell\}$, when $k=\ell$. As above we develop the determinant $\det((\boldsymbol{I}+\boldsymbol{GV})^{k,\ell})$, see (2.16), (2.17). We can assume that the permutations σ of $\{1,\dots,n\}$ entering the development satisfy $\sigma(k)=\ell$. For each $J\subseteq\{1,\dots,n\}\backslash\{k,\ell\}$, we collect the terms corresponding to permutations σ such that $\sigma(m)=m$, for $m\in J$, the choice of 1 in each term $(1+g(0)v_m)$, for $m\in J$, and for any $m\notin J\cup\{k,\ell\}$ with $\sigma(m)=m$, the choice of $g(0)v_m$ instead. Setting $\widetilde{J}=\{1,\dots,n\}\backslash J$, we see that the sum of these terms for a fixed given J as above equals $\Pi_{m\in\widetilde{J}\backslash\{\ell\}} v_m \det(\boldsymbol{G}^{k,\ell}_{\widetilde{J}\times\widetilde{J}})$, where $\boldsymbol{G}^{k,\ell}_{\widetilde{J}\times\widetilde{J}}$ stands for the matrix where the k-th line and the ℓ-th column of the matrix $\boldsymbol{G}_{|\,\widetilde{J}\times\widetilde{J}}$, (i.e. $\boldsymbol{G}$ restricted to $\widetilde{J}\times\widetilde{J}$), are replaced by zero except for the coefficient at their intersection, which is replaced by 1. Thus summing over all possible $J\subseteq\{1,\dots,n\}\backslash\{k,\ell\}$ and all k,ℓ in $\{1,\dots,n\}$, we obtain

$$\begin{aligned}\sum_{k,\ell=1}^{n} \boldsymbol{C}_{k,\ell}\, v_\ell &= \sum_{k,\ell=1}^{n}\ \sum_{H\supseteq\{k,\ell\}}\ \prod_{m\in H} v_m \det(\boldsymbol{G}^{k,\ell}_{H\times H}) \\ &= \sum_{\phi\neq H\subseteq\{1,\dots,n\}}\ \prod_{m\in H} v_m \sum_{k,\ell\in H} \det(\boldsymbol{G}^{k,\ell}_{H\times H}) \overset{(2.11)}{=} \sum_{I\subseteq K} V_I\, c_I\,,\end{aligned} \tag{2.19}$$

(using the convention $c_\phi = 0$ in the last equality).

Combining (2.15), (2.16), (2.18) we obtain (2.14). Finally in the case of a non-negative V with support in K, we note that $\mathbb{E}[\exp\{-z\sum_x V(x)L_{x,u}\}]$ is analytic in the strip $Re\, z > 0$, and coincides for small positive z with the function $\exp\{-u\sum_{I\subseteq K} c_I V_I z^{|I|}/\sum_{I\subseteq K} g_I V_I\, z^{|I|}\}$, which is analytic in the neighborhood of the positive half-line. Both functions thus coincide for $z = 1$, and our last claim follows. □

Remark 2.4.

1) Choosing $V = \lambda 1_K$, with $\lambda \ge 0$ and $K \subset\subset \mathbb{Z}^d$, we deduce from (2.14) by letting λ tend to infinity that

$$\mathbb{P}[L_{x,u} = 0, \text{ for all } x \in K] = \exp\Big\{-u\,\frac{c_K}{g_K}\Big\}, \text{ for } u \ge 0.$$

Introducing the interlacement at level u:

$$\mathcal{I}^u(\omega) = \{x \in \mathbb{Z}^d; \text{ for some } i \ge 0 \text{ such that } u_i \le u,\ \widehat{w}_i^* \text{ enters } x\}, \text{ if } \omega = \sum_{i\ge 0} \delta_{(\widehat{w}_i^*, u_i)},$$

and taking (2.13) into account, we recover the well-known formula, see (2.16) of [19]:

$$\mathbb{P}[\mathcal{I}^u \cap K = \phi] = \exp\{-u\,\mathrm{cap}\,(K)\}, \text{ for } u \ge 0,\ K \subset\subset \mathbb{Z}^d. \tag{2.20}$$

2) Choosing $V = \lambda 1_{\{x\}}$, with $\lambda \ge 0$ and $x \in \mathbb{Z}^d$, Theorem 2.3 now yields that

$$E[\exp\{-\lambda L_{x,u}\}] = \exp\Big\{-\frac{\lambda u}{1 + g(0)\lambda}\Big\}, \text{ for } \lambda \ge 0, \tag{2.21}$$

and in view of (1.30) we find that

$$L_{x,u} \text{ is } BESQ^0\Big(u, \frac{g(0)}{2}\Big) \text{ distributed.} \tag{2.22}$$

If $x, x' \in \mathbb{Z}^d$ are distinct, choosing $V = z(1_{\{x\}} - 1_{\{x'\}})$ in (2.14) with z small and real and extending the identity to $z = it$, $t \in \mathbb{R}$, with the help of (2.9), we find that

$$E[\exp\{it(L_{x',u} - L_{x,u})\}] = \exp\Big\{-2u\frac{(g(0) - g(x'-x))t^2}{1 + (g(0)^2 - g(x'-x)^2)t^2}\Big\}, \text{ for } t \in \mathbb{R}. \tag{2.23}$$

In view of (1.30) we thus find that

(2.24) $L_{x,u} - L_{x',u}$ has the law of $R\psi$, where R and ψ are independent respectively $BES^0(\sqrt{u}, \frac{g(0)+g(x-x')}{4})$ and centered Gaussian with variance $4(g(0) - g(x-x'))$ distributed.

Let us however point out that in the case of three distinct points x, x', x'' in $\mathbb{Z}^d$, the law of the random vector $(L_{x',u} - L_{x,u}, L_{x'',u} - L_{x,u})$ does not coincide with that of the scalar multiplication of a two-dimensional Gaussian vector by an independent $BES^0(a,\tau)$-variable, when $u > 0$. Indeed one has

$$\mathbb{P}[L_{x',u} - L_{x,u} > 0,\ L_{x'',u} - L_{x,u} = 0] \ge \mathbb{P}[\mathcal{I}^u \ni x', \mathcal{I}^u \cap \{x, x''\} = \phi] > 0,$$

as a consequence of (2.20) and the fact that $\mathrm{cap}(\{x, x', x''\}) > \mathrm{cap}(\{x, x''\})$. But for the above mentioned distribution both components necessarily vanish simultaneously on a set of full measure, and the above probability would equal zero if such an identity in law was to hold. We will however see in Section 5 how $L_{x,u}$, $x \in \mathbb{Z}^d$, can be related to the d-dimensional Gaussian free field, by letting u tend to infinity, instead of keeping u fixed.

3) Random interlacements can be related to the Poissonian gas of Markov loops, see [10], [11]. Heuristically they correspond to "loops passing through infinity", see [11], p. 85. The identity for Markov loops corresponding to (2.3) of Theorem 2.1 above can be found in Corollary 1 of Chapter 4 §1 and Proposition 7 of Chapter 2 §4 of [11]. The presence of a logarithm and a trace in the expressions leading to Proposition 7 of [11] is emblematic of the Markov loop measure, and can be contrasted with the expression in (2.8) for random interlacements, (which is then inserted in the last line of (2.5)). In the case of a Poissonian gas of Markov loops on a finite weighted graph with a suitable killing, it is shown in Theorem 13 of [10] that the occupation field of the gas of loops at level $\frac{1}{2}$, (playing the role of u in the context of [10]), is distributed as half the square of a centered Gaussian free field with covariance the corresponding Green density. For similar reasons as in 2) above, no such identity holds for random interlacements at any fixed level u. We will however present in the next two sections limiting procedures that relate random interlacements to the Gaussian free field.

4) As we now explain, the results of this section can be extended to the case of continuous time random interlacements on a transient weighted graph. One considers a countable connected graph E which is locally finite and endowed with non-negative symmetric weights $\rho_{x,x'} = \rho_{x',x}$, which are positive exactly when $\{x, x'\}$ belongs to the edge set $\mathcal{E}$ of E. One assumes that the induced random walk with transition probability $p_{x,x'} = \rho_{x,x'}/\rho(x)$, where $\rho(x) = \sum_{x' \in E} \rho_{x,x'}$, is transient. Random interlacements can be constructed on such a transient weighted graph, see [19], Remark 1.4 and [21]. Continuous time random interlacements can also be constructed, in essence by the same procedure described in the Introduction, endowing the discrete doubly infinite paths with i.i.d. exponential variables of parameter 1. The corresponding expression for the measure $\widehat{Q}_K$, for K finite subset of E, remains the same as in (0.1), simply the expression for $e_K(\cdot)$ the equilibrium measure of K, which appears in (1.4), now has to be multiplied by the factor $\rho(x)$ in the present context.

The occupation time variables $L_{x,u}$, $x \in E$, $u \geq 0$, are defined by a similar formula as in (0.2), but the expression on the right-hand side of (0.2) is now divided by $\rho(x)$. The linear operator G corresponding to (2.1) operates say on functions f on E with finite support, via the formula:

$$Gf(x) = \sum_{x' \in E} g(x, x') f(x') \rho(x'), \; x \in E,$$

where $g(\cdot, \cdot)$ now stands for the Green density, which is obtained by dividing the expression corresponding to the right-hand side of (1.1) by $\rho(x')$.

The proof of Theorem 2.1 can be adapted to this context to show that when K is a finite subset of E, and V has support in K, then for $\|V\|_{L^\infty(E)}$ sufficiently small, $\|GV\|_{L^\infty(E) \to L^\infty(E)} < 1$, and for any $u \geq 0$,

$$\mathbb{E}\Big[\exp\Big\{\sum_{x \in E} V(x) L_{x,u}\, \rho(x)\Big\}\Big] = \exp\{u(V, (I - GV)^{-1} 1)\}, \tag{2.25}$$

where now (f, g) stands for $\sum_{x \in E} f(x) g(x) \rho(x)$, (whenever this sum is absolutely convergent). Likewise the proof of Theorem 2.3 is easily adapted and one finds that for V as above, $u \geq 0$,

$$\mathbb{E}\Big[\exp\big\{-\sum_{x\in E} V(x) L_{x,u}\big\}\Big] = \exp\Big\{-u\,\frac{\sum_{I\subseteq K} c_I \Pi_{x\in I} V(x)}{\sum_{I\subseteq K} g_I \Pi_{x\in I} V(x)}\Big\}, \tag{2.26}$$

with g_I and c_I defined as in (2.10), (2.11), (with $g(\cdot,\cdot)$ now denoting the Green density).

5) One can define the stationary field of discrete occupation times $\ell_{x,u}$, $x \in \mathbb{Z}^d$, $u \geq 0$, analogously to $L_{x,u}$, simply replacing σ_n by 1 in (0.2). When V is a function on $\mathbb{Z}^d$ with support contained in $K \subset\subset \mathbb{Z}^d$, it follows that $1 - e^{-V}$ is a function supported in K with values in $(-\infty, 1)$, and one has the identity

$$\mathbb{E}\Big[\exp\big\{\sum_x V(x)\ell_{x,u}\big\}\Big] = \mathbb{E}\Big[\exp\big\{\sum_x (1-e^{-V(x)}) L_{x,u}\big\}\Big],\ u \geq 0, \tag{2.27}$$

as can be seen by integrating out the exponential variables in the right member of (2.27), (of course both members of the above equality may be infinite). As a result Theorem 2.1 and Theorem 2.3 also yield identities concerning the Laplace functional of $(\ell_{x,u})_{x\in\mathbb{Z}^d}$.

□

3 Preparation for the study of long rods

In this section we introduce notation specific to $\mathbb{Z}^3$ and provide estimates in Lemmas 3.1 and 3.2, which will be recurrently used in the next section, when we investigate the occupation times spent by interlacements at a suitably scaled level in long rods. These controls will play an important role in the asymptotic analysis of the power series entering the characteristic functions of these occupation times. Throughout this section we assume that $d = 3$, and constants depend on the finite subset Λ of $\mathbb{Z}^2$ introduced in (3.1) below. The notation $\|\cdot\|_\infty$ refers to the supremum norm $\|\cdot\|_{L^\infty(B)}$, where B appears in (3.1).

We consider $\Lambda \subset\subset \mathbb{Z}^2$ containing 0 and $N > 1$. We also define

$$B = \Lambda \times J, \text{ where } J = \{1, \dots, N\}. \tag{3.1}$$

We write $\pi_{\mathbb{Z}^2}$ and $\pi_{\mathbb{Z}}$ for the respective $\mathbb{Z}^2$- and $\mathbb{Z}$-projections on $\mathbb{Z}^3$ identified with $\mathbb{Z}^2 \times \mathbb{Z}$. Given a function F on B, we write $\langle F\rangle$ for the function obtained by averaging F on horizontal layers and $\langle F\rangle_z$ for the average of F on the layer $\Lambda \times \{z\}$, so that

$$\langle F\rangle(x) = \langle F\rangle_z = \frac{1}{|\Lambda|}\sum_{y\in\Lambda} F((y,z)),\ \text{for } x \in B \text{ with } \pi_{\mathbb{Z}}(x) = z. \tag{3.2}$$

We also introduce the function

$$[F]_0(x) = F((0,z)),\ \text{for } x \text{ in } B \text{ with } \pi_{\mathbb{Z}}(x) = z. \tag{3.3}$$

It is plain that for any function F on B,

$$\langle F - \langle F\rangle\rangle = 0, \tag{3.4}$$

and that

$$F = \langle F\rangle, \text{ when } F \text{ only depends on the } \mathbb{Z}\text{-component}. \tag{3.5}$$

We consider non-empty sub-intervals of J:

$$I_0 \subseteq I_1 \subsetneq J, \text{ with } L = d(I_0, J\backslash I_1) \geq 1\,, \tag{3.6}$$

(we refer to the beginning of Section 1 for notation). We write

$$C_0 = \Lambda \times I_0 \subseteq C_1 = \Lambda \times I_1 \subseteq B. \tag{3.7}$$

We recall the convention concerning constants and the notation $\|\cdot\|_\infty$ stated at the beginning of this section. The estimates in the next lemma reflect the decay at infinity of the Green function, see (1.12), and the fact that the discrete gradient of $g(\cdot)$ has an improved decay at infinity, see (3.15) below.

Lemma 3.1. *For any function F on B, one has:*

$$\|GF\|_\infty \leq c \log N\, \|F\|_\infty, \tag{3.8}$$

$$\|1_{C_0} G 1_{B\backslash C_1} F\|_\infty \leq c \log\Big(\frac{N+1}{L}\Big)\, \|F\|_\infty, \tag{3.9}$$

$$\|GF\|_\infty \leq c\, \|F\|_\infty, \text{ when } \langle F\rangle = 0, \tag{3.10}$$

$$\|1_{C_0} G 1_{B\backslash C_1} F\|_\infty \leq \frac{c}{L}\, \|1_{B\backslash C_1} F\|_\infty, \text{ when } \langle F\rangle = 0, \tag{3.11}$$

$$\|GF - [GF]_0\|_\infty \leq c\, \|F\|_\infty, \tag{3.12}$$

$$\|1_{C_0}(G 1_{B\backslash C_1} F - [G 1_{B\backslash C_1} F]_0)\|_\infty \leq \frac{c}{L}\, \|1_{B\backslash C_1} F\|_\infty\,. \tag{3.13}$$

Proof. We begin with (3.9) and note that for $x \in C_0$,

$$|(G 1_{B\backslash C_1} F)(x)| = \Big|\sum_{x' \in B\backslash C_1} g(x,x')\, F(x')\Big| \overset{(1.3)}{\leq} c\Big(\sum_{L\leq k\leq N} \frac{1}{k}\Big)\, \|F\|_\infty \leq c \log\Big(\frac{N+1}{L}\Big) \|F\|_\infty,$$

whence (3.9). The bound (3.8) is proved in the same fashion.

We then turn to the proof of (3.11), and note that when $\langle F\rangle = 0$, for $x \in C_0$ one has with the notation $x' = (y', z')$, $\overline{x} = (\overline{y}, z')$ (so that $\pi_{\mathbb{Z}}(x') = \pi_{\mathbb{Z}}(\overline{x}) = z'$):

$$\begin{aligned} (G 1_{B\backslash C_1} F)(x) &= \sum_{x' \in B\backslash C_1} g(x,x') F(x') = \sum_{z' \in J\backslash I_1} \sum_{y' \in \Lambda} g(x,x') F(x') \\ &\overset{\langle F\rangle = 0}{=} \frac{1}{|\Lambda|} \sum_{z' \in J\backslash I_1} \sum_{y', \overline{y} \in \Lambda} g(x,x')(F(x') - F(\overline{x})) \\ &= \frac{1}{|\Lambda|} \sum_{z' \in J\backslash I_1} \sum_{y', \overline{y} \in \Lambda} (g(x,x') - g(x,\overline{x})) F(x'). \end{aligned} \tag{3.14}$$

From Theorem 1.5.5, p. 32 of [7], one knows that

$$|g(x+a) - g(x)| \leq \frac{c\,|a|}{1+|x|^2}\,, \text{ for } x \in \mathbb{Z}^3,\ |a| \leq \text{diam}(\Lambda), \tag{3.15}$$

where $\text{diam}(\Lambda)$ stands for the diameter of Λ. As a result we see that

$$|(G 1_{B\backslash C_1} F)(x)| \leq c\, \|1_{B\backslash C_1} F\|_\infty \sum_{k\geq L} \frac{1}{1+k^2} \leq \frac{c}{L}\, \|1_{B\backslash C_1} F\|_\infty\,, \tag{3.16}$$

whence (3.11). One proves (3.10) analogously.

As for (3.13) we note that with the notation $x_0 = (0, z)$ when $x = (y, z)$, we have for $x \in C_0$:

$$
\begin{aligned}
|(G1_{B\setminus C_1}F - [G1_{B\setminus C_1}F]_0)(x)| &= \Big| \sum_{x' \in B\setminus C_1} (g(x,x') - g(x_0,x'))F(x') \Big| \\
&\overset{(3.15)}{\le} c \sum_{L \ge k} \frac{1}{1+k^2} \, \|1_{B\setminus C_1}F\|_\infty \le \frac{c}{L} \, \|1_{B\setminus C_1}F\|_\infty,
\end{aligned}
\tag{3.17}
$$

whence (3.13). The bound (3.12) is proved analogously. □

We conclude this section with the following lemma.

Lemma 3.2. *For F, H functions on B, one has*

$$
\|\langle F(GH)\rangle\|_\infty \le c \, \|F\|_\infty \, \|H\|_\infty, \; \textit{when } \langle F\rangle = 0 \textit{ or } \langle H\rangle = 0. \tag{3.18}
$$

Proof. The case $\langle H\rangle = 0$ is immediate thanks to (3.10). In the case where $\langle F\rangle = 0$, we write for $z \in J$, with the notation $x = (y, z)$, $\overline{x} = (\overline{y}, z)$,

$$
\begin{aligned}
\langle F(GH)\rangle_z &= \frac{1}{|\Lambda|} \sum_{y \in \Lambda} F(x) \sum_{x' \in B} g(x,x') \, H(x') \\
&\overset{\langle F\rangle = 0}{=} \frac{1}{|\Lambda|^2} \sum_{y, \overline{y} \in \Lambda} (F(x) - F(\overline{x})) \sum_{x' \in B} g(x,x') \, H(x') \\
&= \frac{1}{|\Lambda|^2} \sum_{y, \overline{y} \in \Lambda, x' \in B} (g(x,x') - g(\overline{x},x')) \, F(x) \, H(x').
\end{aligned}
\tag{3.19}
$$

By (3.15) we thus find that with the notation $\pi_{\mathbb{Z}}(x') = z'$,

$$
|\langle F(GH)\rangle|_z \le \frac{c}{|\Lambda|^2} \sum_{y, \overline{y} \in \Lambda, x' \in B} \frac{1}{1 + |z - z'|^2} \, \|F\|_\infty \, \|H\|_\infty \le c \, \|F\|_\infty \, \|H\|_\infty, \tag{3.20}
$$

and (3.18) follows. □

4 Occupation times of long rods in $\mathbb{Z}^3$

In this section we relate the field of occupation times of long rods in $\mathbb{Z}^3$ by random interlacements at a suitably scaled level, see (4.1) below, with the two-dimensional free field pinned at the origin introduced in (1.29). The main results are stated in Theorems 4.2 and 4.9. The approach is roughly the following. By Theorem 2.1 we can express the characteristic functionals of the scaled fields of occupation times of the long rods as exponentials of certain power series. The main task is to control the asymptotic behavior of these power series. This analysis is carried out in the central Theorem 4.1 as well as in the simpler Theorem 4.8. Throughout this section we assume that $d = 3$. The constants depend on the finite subset Λ of $\mathbb{Z}^2$, cf. (3.1) and above (4.3), as well as on the function W with support in Λ that appears in (4.3). As in Section 3 we denote by $\|\cdot\|_\infty$ the supremum norm $\|\cdot\|_{L^\infty(B)}$, with B as in (3.1).

We consider $\alpha > 0$, and a positive sequence γ_N tending to infinity. We will analyze the random fields of occupation times of the long rods J_y, $y \in \mathbb{Z}^2$, with $J_y = \{y\} \times J = \{y\} \times \{1, \dots, N\} \subseteq \mathbb{Z}^3$, by random interlacements at the scaled levels

$$u_N = \alpha \,\frac{\log N}{N} \text{ and } u'_N = \gamma_N \,\frac{\log N}{N}, \text{ with } N > 1. \tag{4.1}$$

The corresponding occupation times of the rods J_y, $y \in \mathbb{Z}^2$, are

$$\mathcal{L}_{y,N} = \sum_{x \in J_y} L_{x,u_N}, \ \mathcal{L}'_{y,N} = \sum_{x \in J_y} L_{x,u'_N}. \tag{4.2}$$

Let us point out that sequences of levels converging faster to zero than u_N are not interesting in the present context, see Remark 4.3 below.

As in Section 3, we consider some $\Lambda \subset\subset \mathbb{Z}^2$ containing 0. Further we introduce a function W on $\mathbb{Z}^2$ such that

$$\text{i) } W(y) = 0 \text{ outside } \Lambda, \qquad \text{ii) } \sum_y W(y) = 0. \tag{4.3}$$

We define the functions on $\mathbb{Z}^3$,

$$V_N(x) = \frac{1}{\sqrt{\log N}}\, W(y)\, 1_J(z), \quad V'_N(x) = \frac{1}{\sqrt{\gamma_N}}\, V_N(x), \text{ with } x = (y,z), \tag{4.4}$$

so that

$$\mathcal{L}_N \overset{\text{def}}{=} \sum_{y \in \mathbb{Z}^2} W(y)\, \frac{\mathcal{L}_{y,N}}{\sqrt{\log N}} = \sum_{x \in \mathbb{Z}^3} V_N(x)\, L_{x,u_N}, \tag{4.5}$$

and similarly

$$\mathcal{L}'_N \overset{\text{def}}{=} \sum_{y \in \mathbb{Z}^2} W(y)\, \frac{\mathcal{L}'_{y,N}}{\sqrt{\gamma_N \log N}} = \sum_{x \in \mathbb{Z}^3} V'_N(x)\, L_{x,u'_N}. \tag{4.6}$$

It follows from Theorem 2.1 and Remark 2.2 that

$$\mathbb{E}[\exp\{z\, \mathcal{L}_N\}] = \exp\Big\{\sum_{n \ge 1} a_N(n)\, z^n\Big\}, \text{ for } |z| < r_N \text{ in } \mathbb{C}, \tag{4.7}$$

with $r_N > 0$ and

$$a_N(n) = u_N(V_N, (GV_N)^{n-1} 1) \text{ for } n \ge 1. \tag{4.8}$$

As a result of the centering condition (4.3) ii) we have

$$a_N(1) = 0. \tag{4.9}$$

In a similar fashion we have

$$\mathbb{E}[\exp\{z\, \mathcal{L}'_N\}] = \exp\Big\{\sum_{n \ge 1} a'_N(n)\, z^n\Big\}, \text{ for } |z| < r'_N \text{ in } \mathbb{C}, \tag{4.10}$$

with $r'_N > 0$ and

$$(4.11) \qquad a'_N(n) = u'_N(V'_N, (GV'_N)^{n-1}1) \overset{(4.1),(4.4)}{=} \frac{1}{\alpha}\, \gamma_N^{1-\frac{n}{2}} a_N(n), \text{ for } n \geq 1.$$

The heart of the matter for the proof of Theorem 4.2 lies in the analysis of the large N behavior of the coefficients $a_N(n)$, $n \geq 1$. There is a dichotomy between the case of odd n, with an asymptotic vanishing of $a_N(n)$, and even n, with a positive limit of $a_N(n)$, as N goes to infinity. The crucial controls are contained in the next theorem. We recall the convention concerning constants stated at the beginning of this section.

Theorem 4.1.

$$(4.12) \qquad |a_N(n)| \leq \alpha\, c_0^n, \textit{ for all } n \geq 1, N > 1,$$

$$(4.13) \qquad \textit{for any } k \geq 0, \lim_N a_N(2k+1) = 0,$$

$$(4.14) \qquad \textit{for any } k \geq 1, \lim_N a_N(2k) = \frac{\alpha}{2}\, \mathcal{E}(W) \Big(\frac{3}{2\pi}\, \mathcal{E}(W)\Big)^{k-1},$$

where we have set, see (1.6) *for notation,*

$$(4.15) \qquad \mathcal{E}(W) = -3 \sum_{y,y'} W(y)\, W(y')\, a(y'-y).$$

Note that due to (4.3) ii) we can express $\mathcal{E}(W)$ in terms of the two-dimensional Gaussian free field ψ_y, $y \in \mathbb{Z}^2$, introduced in (1.29), via the formula:

$$(4.16) \qquad \mathcal{E}(W) = E\Big[\Big(\sum_{y \in \mathbb{Z}^2} W(y)\, \psi_y\Big)^2\Big].$$

Before turning to the proof of Theorem 4.1, we first explain how this theorem enables us to derive the convergence in law of the appropriately scaled fields $\mathcal{L}_{y,N} - \mathcal{L}_{0,N}$, $y \in \mathbb{Z}^2$, and $\mathcal{L}'_{y,N} - \mathcal{L}'_{0,N}$, $y \in \mathbb{Z}^2$. We tacitly endow $\mathbb{R}^{\mathbb{Z}^2}$ with the product topology, so that the convergence stated in Theorem 4.2 actually corresponds to the convergence in distribution of all finite dimensional marginals of the relevant random fields. The main result of this section is the next theorem, which proves (0.8) and (0.10).

Theorem 4.2.

(4.17) *As N goes to infinity, $\big(\frac{\mathcal{L}_{y,N} - \mathcal{L}_{0,N}}{\sqrt{\log N}}\big)_{y \in \mathbb{Z}^2}$, converges in distribution to the random field $(R\psi_y)_{y \in \mathbb{Z}^2}$,*

where R and $(\psi_y)_{y \in \mathbb{Z}^2}$ are independent and

$$(4.18) \qquad R \textit{ is } BES^0\Big(\sqrt{\alpha}, \frac{3}{2\pi}\Big)\textit{-distributed,}$$

(4.19) $\qquad (\psi_y)_{y \in \mathbb{Z}^2}$ *is the centered Gaussian field introduced in* (1.29).

Moreover,

(4.20) *as N goes to infinity, $\big(\frac{\mathcal{L}'_{y,N} - \mathcal{L}'_{0,N}}{\sqrt{N u'_N}}\big)_{y \in \mathbb{Z}^2}$ converges in distribution to $(\psi_y)_{y \in \mathbb{Z}^2}$.*

Proof of Theorem 4.2 *(assuming Theorem* 4.1*).* We begin with the proof of (4.17). We consider W as in (4.3) and $\mathcal{L}_N$ as in (4.5). By (4.7), (4.12) and Remark 2.2, we see that $\exp\{z\mathcal{L}_N\}$ is integrable for any z with $c_0|Re\, z| < 1$, and that

$$\mathbb{E}[\exp\{z\mathcal{L}_N\}] = \exp\Big\{\sum_{n\ge 1} a_N(n) z^n\Big\}, \text{ for any } |z| < c_0^{-1} \text{ in } \mathbb{C}. \tag{4.21}$$

In particular (4.12) implies that

$$\sup_N \mathbb{E}[\cosh(r\mathcal{L}_N)] < \infty, \text{ when } r < c_0^{-1}. \tag{4.22}$$

As a result the laws of the variables $\mathcal{L}_N$ are tight, and the variables $\exp\{z\mathcal{L}_N\}$, $N > 1$, with $|Re\, z| \le r < c_0^{-1}$, are uniformly integrable. If along some subsequence N_k, $k \ge 1$, the variables $\mathcal{L}_{N_k}$ converge in distribution to $\mathcal{L}$, it follows from Theorem 5.4, p. 32 in [1], that for $|z| < c_0^{-1}$,

$$\begin{aligned} E[\exp\{z\mathcal{L}\}] &= \lim_k \ \mathbb{E}[\exp\{z\mathcal{L}_{N_k}\}] \\ &= \lim_k \ \exp\Big\{\sum_{n\ge 1} a_{N_k}(n) z^n\Big\} \\ &= \exp\Big\{\frac{\alpha}{2}\,\mathcal{E}(W)\, z^2 \big/ \Big(1 - \frac{3}{2\pi}\,\mathcal{E}(W) z^2\Big)\Big\}, \end{aligned} \tag{4.23}$$

using Theorem 4.1 in the last equality. This determines the characteristic function of the law of $\mathcal{L}$, and by (1.30) shows that $\mathcal{S}$ has same distribution as $R\psi$ where R, ψ are independent variables with R $BES^0(\sqrt{\alpha}, \frac{3}{2\pi})$-distributed, and ψ a centered Gaussian variable with zero mean and variance $\mathcal{E}(W)$. This proves that for any W as in (4.3), $\mathcal{L}_N$ converges in distribution to $R\psi$ as above, when N tends to infinity. In view of (4.16), this completes the proof of (4.17).

The proof of (4.20) is analogous. Due to (4.11), we know that $a'_N(n) = \frac{1}{\alpha}\, \gamma_N^{1-\frac{n}{2}} a_N(n)$, and in particular $a'_N(2) = \frac{1}{\alpha}\, a_N(2)$ converges to $\frac{1}{2}\,\mathcal{E}(W)$, whereas for $n \ne 2$, $a'_N(n)$ converges to zero as N goes to infinity. We can use the same arguments as above, and find that $\mathcal{L}'_N$ converges in distribution to a variable $\mathcal{L}'$ such that for small $|z|$ in $\mathbb{C}$, $\mathbb{E}[\exp\{z\mathcal{L}'\}] = \exp\{\frac{1}{2}\,\mathcal{E}(W) z^2\}$. The claim (4.20) then follows immediately. □

Remark 4.3. For the kind of limit theorems discussed here, sequences of levels converging to zero faster than u_N lead to trivial results, as we now explain. In a standard way, see for instance Remark 3.1 3) in [16], one has the bound $\text{cap}(J_y) \le c\, \frac{N}{\log N}$, for all $y \in \mathbb{Z}^2$. If we pick u''_N so that $u''_N = o(\frac{\log N}{N})$, then (2.20) implies that for all $y \in \mathbb{Z}^2$, with probability tending to 1 as N goes to infinity, the interlacement at level u''_N does not intersect J_y. In particular, if we define $\mathcal{L}''_{y,N}$ in analogy to $\mathcal{L}_{y,N}$ in (4.2) with u''_N in place of u_N, the random field $(\mathcal{L}''_{y,N})_{y\in\mathbb{Z}^2}$ converges in distribution to the constant field equal to zero, as N goes to infinity. □

Proof of Theorem 4.1. We recall the convention concerning constants and the notation $\|\cdot\|_\infty$ stated at the beginning of this section. The linear operators under consideration throughout the proof will be restricted to the space of functions vanishing outside B. The centering condition (4.3) ii) and the ensuing identity $\langle V_N \rangle = 0$ play a crucial role. We will first prove (4.12), (4.13). Our first step is to control the norm of the operators $(GV_N)^2$.

Lemma 4.4.

$$\|(GV_N)^2\|_{L^\infty(B)\to L^\infty(B)} \le c_1 \,. \tag{4.24}$$

Proof. Given a function F on B, we write in the notation of (3.2), (3.3):

$$\begin{aligned}(GV_N)^2 F &= A_1 + A_2 + A_3, \text{ where}\\ A_1 &= GV_N\, G(V_N\, F - \langle V_N F\rangle)\\ A_2 &= GV_N\, [G\langle V_N\, F\rangle]_0\\ A_3 &= GV_N(G\langle V_N\, F\rangle - [G\langle V_N\, F\rangle]_0).\end{aligned} \tag{4.25}$$

With the help of Lemma 3.1 we see that:

$$\|A_1\|_\infty \overset{(3.8),(4.4)}{\le} c\sqrt{\log N}\, \|G(V_N\, F - \langle V_N\, F\rangle)\|_\infty \overset{(3.10),(4.4)}{\le} c\,\|F\|_\infty \,. \tag{4.26}$$

Since $\langle V_N[G\langle V_N\, F\rangle]_0\rangle = 0$, we also find that

$$\|A_2\|_\infty \overset{(3.10),(4.4)}{\le} \frac{c}{\sqrt{\log N}}\, \|G\langle V_N\, F\rangle\|_\infty \overset{(3.8),(4.4)}{\le} c\,\|F\|_\infty \,. \tag{4.27}$$

Finally we have

$$\|A_3\|_\infty \overset{(3.8),(4.4)}{\le} c\sqrt{\log N}\, \|G(V_N F - [G\langle V_N F\rangle]_0)\|_\infty \overset{(3.12),(4.4)}{\le} c\,\|F\|_\infty \,. \tag{4.28}$$

Collecting (4.25) - (4.28), the claim (4.24) now follows. □

Before proving (4.12), (4.13) we still need the following lemma, which shows that the kernel of the linear operator $F \to \langle V_N F\rangle$, is almost invariant under $(GV_N)^2$. We will later see, cf.(4.61), that the function $F = 1_B$, which belongs to this kernel, is in an appropriate sense, close to being an eigenvector of $(GV_N)^2$.

Lemma 4.5. *When F is a function on B, one has*

$$\|\langle V_N(GV_N)^2F\rangle\|_\infty \le c_2\|\langle V_N F\rangle\|_\infty + \frac{c_3}{(\log N)^{\frac{3}{2}}}\|F\|_\infty. \tag{4.29}$$

Proof. We use (4.25) and write

$$\langle V_N(GV_N)^2F\rangle = \langle V_N A_1\rangle + \langle V_N A_2\rangle + \langle V_N A_3\rangle. \tag{4.30}$$

With the help of Lemma 3.2 we find that

$$\|\langle V_N A_1\rangle\|_\infty \overset{(3.18)}{\le} \frac{c}{\sqrt{\log N}}\, \|V_N G(V_N F - \langle V_N F\rangle)\|_\infty \overset{(3.10)}{\le} \frac{c}{(\log N)^{\frac{3}{2}}}\, \|F\|_\infty, \tag{4.31}$$

$$\|\langle V_N A_2\rangle\|_\infty \overset{(3.18)}{\le} \frac{c}{\sqrt{\log N}}\, \|V_N[G\langle V_N F\rangle]_0\|_\infty \overset{(3.8)}{\le} c\,\|\langle V_N F\rangle\|_\infty, \tag{4.32}$$

and that

$$\begin{aligned}\|\langle V_N A_3\rangle\|_\infty &\overset{(3.18)}{\le} \frac{c}{\sqrt{\log N}}\, \|V_N(G\langle V_N F\rangle - [G\langle V_N F\rangle]_0)\|_\infty\\ &\overset{(3.12)}{\le} \frac{c}{(\log N)^{\frac{3}{2}}}\, \|F\|_\infty.\end{aligned} \tag{4.33}$$

Collecting (4.30) - (4.33), we obtain (4.29). □

We now prove (4.12), (4.13). As a result of (4.24), (4.29) we see that for $k \ge 1$ and F a function on B, one has

$$(4.34)\qquad \|\langle V_N(GV_N)^{2k}F\rangle\|_\infty \overset{(4.29),(4.24)}{\le} c_2\|\langle V_N(GV_N)^{2(k-1)}F\rangle\|_\infty + \frac{c_3}{(\log N)^{\frac{3}{2}}}\, c_1^{k-1}\|F\|_\infty$$

and by induction

$$\le c_2^k\|\langle V_N F\rangle\|_\infty + \frac{c_3}{(\log N)^{\frac{3}{2}}}\,(c_1^{k-1} + c_2\, c_1^{k-2} + \cdots + c_2^{k-1})\,\|F\|_\infty$$

$$\le c_2^k\|\langle V_N F\rangle\|_\infty + \frac{c^k}{(\log N)^{\frac{3}{2}}}\,\|F\|_\infty.$$

Keeping in mind that $\langle V_N\rangle = 0$, we thus see that for $k \ge 1$,
(4.35)

$$\begin{aligned} |a_N(2k+1)| &\overset{(4.1),(4.8)}{=} \frac{\alpha\log N}{N}\,|(V_N,(GV_N)^{2k}1)| \le \alpha\log N|\Lambda|\,\|\langle V_N(GV_N)^{2k}1\rangle\|_\infty \\ &\overset{(4.34)}{\le} \frac{\alpha c^k|\Lambda|}{\sqrt{\log N}}. \end{aligned}$$

Together with (4.9), this proves (4.13) as well as (4.12) for odd n. When $n = 2k$, with $k \ge 1$, we note that

$$(4.36)\qquad \begin{aligned} |a_N(2k)| &= \frac{\alpha\log N}{N}\,|(V_N,(GV_N)(GV_N)^{2(k-1)}1)| \\ &= \frac{\alpha\log N}{N}\,|(GV_N, V_N(GV_N)^{2(k-1)}1)| \quad \text{(by symmetry of } G\text{)}, \\ &\le \alpha\log N|\Lambda|\,\|GV_N\|_\infty\,\|V_N(GV_N)^{2(k-1)}1\|_\infty \overset{(3.10),(4.24)}{\le} \alpha c^k. \end{aligned}$$

The proof of (4.12) is now complete.

There remains to prove (4.14), i.e. to analyze the large N behavior of the even coefficients $a_N(2k)$. To motivate the next lemma we recall that

$$(4.37)\qquad a_N(2) \overset{(4.8)}{=} \frac{\alpha\log N}{N}\,(V_N, GV_N) \overset{(3.2)}{=} \frac{\alpha\log N}{N}\,|\Lambda|\sum_{z\in J}\langle V_N\, GV_N\rangle_z.$$

Lemma 4.6. *There exists a function $\Gamma(\cdot)$ on $\mathbb{N}$ tending to 0 at infinity such that*

$$(4.38)\qquad \log N\langle V_N\, GV_N\rangle_z = \frac{1}{2}\,\frac{\mathcal{E}(W)}{|\Lambda|} + f_N(z),\ \text{for } z \in J,$$

with $|f_N(z)| \le \Gamma(d(z,J^c))$.

Moreover if one defines

$$(4.39)\qquad \tau_N = \frac{1}{2\log N}\sum_{|z|\le N} g((0,z)),\ \text{so that } \lim_N \tau_N = \frac{3}{2\pi}\ \text{by (1.9)},$$

one has the identity on B

$$(4.40)\qquad (GV_N)^2 1_B - \tau_N\, \mathcal{E}(W)\, 1_B = \frac{1}{\log N}\, G(f_N \circ \pi_{\mathbb{Z}}) + k_N,$$

where for $x = (y,z) \in B$,

$$(4.41)\qquad |k_N|(x) \le \frac{c}{\log N} \quad \log(N/d(z,J^c)).$$

Proof. We begin with the proof of (4.38). We note that for $z \in J$, $x = (y,z)$ in B and $x' = (y',z')$ in B, one has

$$(4.42)\qquad \begin{aligned} \log N \langle V_N G V_N\rangle_z &= \frac{1}{|\Lambda|} \sum_{y\in\Lambda, x'\in B} W(y)\, g(x,x')\, W(y') \\ &\overset{(4.3)\text{ii)}}{=} \frac{1}{|\Lambda|} \sum_{y,y'\in\Lambda, z'\in J} W(y)\, (g(x,x') - g(x,(y,z'))\, W(y') \\ &= \frac{1}{|\Lambda|} \sum_{y,y'\in\Lambda} W(y)\, W(y') \sum_{z'\in J} \big(g(x'-x) - g((0,z'-z))\big) \\ &\overset{(1.10)}{=} \frac{1}{|\Lambda|} \sum_{y,y'\in\Lambda} W(y)\, W(y')\big(b_N(y'-y,z) - \tfrac{3}{2}\, a(y'-y)\big) \\ &\overset{(4.15)}{=} \tfrac{1}{2}\, \frac{\mathcal{E}(W)}{|\Lambda|} + f_N(z), \end{aligned}$$

where we have set

$$f_N(z) = \frac{1}{|\Lambda|} \sum_{y,y'\in\Lambda} W(y)\, W(y')\, b_N(y'-y,z), \text{ for } z \in J.$$

The estimate in the second line of (4.38) is now an immediate consequence of (1.11). This completes the proof of (4.38).

We then turn to the proof of (4.40), (4.41). We write

$$(4.43)\qquad (GV_N)^2\, 1_B = G\langle V_N\, GV_N\rangle + G(V_N\, GV_N - \langle V_N\, GV_N\rangle) \overset{\text{def}}{=} a_1 + a_2.$$

We know that

$$(4.44)\qquad \|a_2\|_\infty \overset{(3.10)}{\le} c\, \|V_N\, GV_N\|_\infty \overset{(3.10)}{\le} c'/\log N,$$

and by (4.38) we find that

$$(4.45)\qquad \begin{aligned} a_1 &= \tfrac{1}{2}\, \frac{\mathcal{E}(W)}{|\Lambda|}\, \frac{G1_B}{\log N} + \frac{1}{\log N}\, G(f_N \circ \pi_{\mathbb{Z}}) \\ &= \tau_N\, \mathcal{E}(W)\, 1_B + \frac{1}{\log N}\, G(f_N\circ \pi_{\mathbb{Z}}) + r_N, \end{aligned}$$

where for $x = (y,z) \in B$ we have set

$$(4.46)\qquad r_N(x) = \tfrac{1}{2}\, \frac{\mathcal{E}(W)}{|\Lambda|}\, \frac{G1_B}{\log N} - \tfrac{1}{2}\, \frac{\mathcal{E}(W)}{\log N} \sum_{|z'-z|\le N} g((0,z'-z)).$$

We thus see that, (recall $J_y = \{y\} \times J$):

$$
(4.47)\qquad \begin{aligned}
|r_N(x)| \le \frac{c}{\log N}\, \Big\|G\Big(\frac{1_B}{|\Lambda|} - 1_{J_0}\Big)\Big\|_\infty + \frac{c}{\log N}\, \|G1_{J_0} - [G1_{J_0}]_0\|_\infty \\
+ \frac{c}{\log N} \sum_{|z'-z|\le N, z'\notin J} g((0,z'-z)) \overset{(3.10),(3.12),(1.3)}{\le} \frac{c}{\log N}\, \log(N/d(z,J^c))\,.
\end{aligned}
$$

Collecting (4.43) - (4.47) we have completed the proof of (4.40), (4.41). □

As a result of (4.37), (4.38) we see that

$$
(4.48)\qquad a_N(2) = \frac{\alpha}{2}\, \mathcal{E}(W) + \frac{\alpha|\Lambda|}{N} \sum_{z\in J} f_N(z) \underset{N}{\longrightarrow} \frac{\alpha}{2}\, \mathcal{E}(W).
$$

This proves (4.14) in the case $k = 1$. To handle the case $k > 1$, we will need to control the propagation of boundary effects corresponding to terms with $\mathbb{Z}$-component close to the complement of J, when proving the convergence of $a_N(2k) = \alpha \frac{\log N}{N}\, (V_N, (GV_N)^{2k-1}1)$ for $N \to \infty$. The next lemma will be useful for this purpose. We first introduce some notation.

We consider non-empty sub-intervals of J

$$
(4.49)\qquad I_0 \subseteq I_1 \subseteq I_2 \subsetneq J,
$$

and define

$$
(4.50)\qquad L_0 = d(I_0, J\backslash I_1) \ge 1,\; L_1 = d(I_1, J\backslash I_2) \ge 1,
$$

as well as

$$
(4.51)\qquad C_0 = \Lambda \times I_0 \subseteq C_1 = \Lambda \times I_1 = C_2 = \Lambda \times I_2 \subseteq B.
$$

We have the following variation on Lemma 4.4.

Lemma 4.7. *For F a function on B, one has*

$$
(4.52)\qquad \|1_{C_0}(GV_N)^2F\|_\infty \le c_4\|1_{C_2}F\|_\infty + c_5\Big(\frac{1}{L_0} + \frac{1}{L_1} + \frac{1}{\log N}\, \log\Big(\frac{(N+1)^2}{L_0L_1}\Big)\Big)\, \|F\|_\infty.
$$

Proof. By Lemma 4.4 we can assume that F vanishes on C_2. We use the decomposition (4.25) of $(GV_N)^2F$. We find that

$$
(4.53)\qquad \begin{aligned}
&\|1_{C_0}A_1\|_\infty = \|1_{C_0}GV_N\, G(V_NF - \langle V_NF\rangle)\|_\infty \\
&\le \|1_{C_0}GV_N\, 1_{C_1}G(V_NF - \langle V_NF\rangle)\|_\infty + \|1_{C_0}GV_N\, 1_{B\backslash C_1}G(V_NF - \langle V_NF\rangle)\|_\infty \\
&\overset{(3.8),(3.9)}{\le} c\sqrt{\log N}\, \|1_{C_1}G(V_NF - \langle V_NF\rangle)\|_\infty \\
&+ \frac{c}{\sqrt{\log N}}\, \log\Big(\frac{N+1}{L_0}\Big)\, \|G(V_NF - \langle V_NF\rangle)\|_\infty \\
&\overset{(3.11),(3.10)}{\le} \frac{c}{L_1}\, \|F\|_\infty + \frac{c}{\log N}\, \Big(\frac{N+1}{L_0}\Big)\, \|F\|_\infty,
\end{aligned}
$$

using the fact that F vanishes on C_2 in the last step. In a similar fashion we find that

$$
\begin{aligned}
&\|1_{C_0} A_2\|_\infty = \|1_{C_0} G V_N \, [G\langle V_N F\rangle]_0\|_\infty \overset{(3.10),(3.11)}{\leq} \\
&\frac{c}{\sqrt{\log N}} \ \|1_{C_1}[G\langle V_N F\rangle]_0\|_\infty + \frac{c}{\sqrt{\log N}} \ \frac{1}{L_0} \ \|1_{B\backslash C_1}[G\langle V_N F\rangle]_0\|_\infty \overset{(3.9),(3.8)}{\leq} \\
&\frac{c}{\log N} \ \log\Big(\frac{N+1}{L_1}\Big) \ \|F\|_\infty + \frac{c}{L_0} \ \|F\|_\infty,
\end{aligned}
\tag{4.54}
$$

where once again we have used that F vanishes on C_2 in the last step. Finally we have

$$
\begin{aligned}
&\|1_{C_0} A_3\|_\infty = \|1_{C_0} G V_N \, (G\langle V_N F\rangle - [G\langle V_N F\rangle]_0)\|_\infty \overset{(3.8),(3.9)}{\leq} \\
&c\sqrt{\log N} \ \|1_{C_1}(G\langle V_N F\rangle - [G\langle V_N F\rangle]_0)\|_\infty \\
&+ \frac{c}{\sqrt{\log N}} \ \log\Big(\frac{N+1}{L_0}\Big) \|1_{B\backslash C_1}(G\langle V_N F\rangle - [G\langle V_N F\rangle]_0)\|_\infty \\
&\overset{(3.13),(3.12)}{\leq} \ \frac{c}{L_1} \ \|F\|_\infty + \frac{c}{\log N} \ \log\Big(\frac{N+1}{L_0}\Big) \ \|F\|_\infty,
\end{aligned}
\tag{4.55}
$$

using that F vanishes on C_2 in the last step.

Collecting (4.53) - (4.55), we obtain (4.52). □

We now introduce the following sequence of possibly empty sub-intervals of J:

$$
J_k = \{i \geq 1; 1 + 4k[Ne^{-\sqrt{\log N}}] \leq i \leq N - 4k[Ne^{-\sqrt{\log N}}]\}, \text{ for } k \geq 0. \tag{4.56}
$$

There is some freedom in the above definition. The proof below would work with minor changes if one replaces $Ne^{-\sqrt{\log N}}$ by $N^{1-\varepsilon_N}$, with $\varepsilon_N \to 0$, and $\varepsilon_N \log N \to \infty$.

Setting $B_k = \Lambda \times J_k$, we find that for any $k \geq 1$, when $N \geq c(k)$,

$$
\emptyset \neq B_k \subsetneq B_{k-1} \subsetneq \cdots \subsetneq B_1 \subsetneq B,
$$

and that with the notation (4.39)

$$
\begin{aligned}
&\|1_{B_k}((GV_N)^{2k} 1_B - (\tau_N \, \mathcal{E}(W))^k 1_B)\|_\infty \leq \\
&\sum_{m=0}^{k-1} \|1_{B_k}((GV_N)^{2(m+1)}(\tau_N \, \mathcal{E}(W))^{k-(m+1)} 1_B - (GV_N)^{2m}(\tau_N \mathcal{E}(W))^{k-m} 1_B)\|_\infty \leq \\
&\sum_{m=0}^{k-1} (\tau_N \, \mathcal{E}(W))^{k-(m+1)} \|1_{B_{m+1}}(GV)^{2m}((GV_N)^2 1_B - \tau_N \, \mathcal{E}(W) 1_B)\|_\infty.
\end{aligned}
\tag{4.57}
$$

We set $F_N = (GV_N)^2 1_B - \tau_N \, \mathcal{E}(W) 1_B$, and now want to bound $\|1_{B_{m+1}}(GV_N)^{2m} \ F_N\|_\infty$ with the help of (4.52), when $0 < m < k$. To this end we introduce $\widehat{J}_m$ with a similar definition as in (4.56), simply replacing $4k$ by $2m+2$, so that $J_{m+1} \subseteq \widehat{J}_m \subseteq J_m$ play the role of $I_0 \subseteq I_1 \subseteq I_2$ in (4.49). We note that $d(J_{m+1}, J\backslash \widehat{J}_m)$ and $d(\widehat{J}_m, J\backslash J_m)$ are bigger than $c\,Ne^{-\sqrt{\log N}}$. As a result the expression inside the parenthesis after c_5 in (4.52) is smaller than

$$
c\, N^{-1} \, e^{\sqrt{\log N}} + \frac{1}{\log N} \ \log\Big(\frac{(N+1)^2}{N^2} \ e^{2\sqrt{\log N}}\Big) \leq \frac{c}{\sqrt{\log N}}.
$$

It thus follows that for $0 < m < k$:

$$\|1_{B_{m+1}}(GV_N)^{2m}F_N\|_\infty \overset{(4.52),(4.24)}{\le} c_4\|1_{B_m}(GV_N)^{2(m-1)}F_N\|_\infty + \frac{c_1^{m-1}}{\sqrt{\log N}}\, c_6\|F_N\|_\infty \tag{4.58}$$

and by induction

$$\begin{aligned} &\le c_4^m\|1_{B_1}F_N\|_\infty + \frac{c_6}{\sqrt{\log N}}\,(c_1^{m-1} + c_4\,c_1^{m-2} + \dots + c_4^{m-1})\|F_N\|_\infty \\ &\le c_4^m\|1_{B_1}F_N\|_\infty + \frac{c^m}{\sqrt{\log N}}\,\|F_N\|_\infty. \end{aligned}$$

Coming back to the last line of (4.57), we see that for $k \ge 1$, $N \ge c(k)$, each term under the sum, thanks to the above bound and (4.39), is smaller than $c(k)(\|1_{B_1}F_N\|_\infty + \frac{1}{\sqrt{\log N}}\|F_N\|_\infty)$. Hence we see that for $k \ge 1$ and $N \ge c(k)$,

$$\|1_{B_k}((GV_N)^{2k}1_B - (\tau_N\,\mathcal{E}(W))^k 1_B)\|_\infty \le c(k)\Big(\|1_{B_1}F_N\|_\infty + \frac{1}{\sqrt{\log N}}\Big), \tag{4.59}$$

where we have used the bound $\|F_N\|_\infty \le c$, which follows from (4.24) and (4.39). Note that the form of the correction term $(\log N)^{-\frac{1}{2}}$ in (4.59) mainly reflects our choice for the intervals J_k in (4.56). In view of Lemma 4.6, we also find that

$$\|1_{B_1}F_N\|_\infty \overset{(4.40)}{\le} \Big\|1_{B_1}\,\frac{G(f_N\circ\pi_{\mathbb{Z}})}{\log N}\Big\|_\infty + \|1_{B_1}\,k_N\|_\infty \longrightarrow 0, \text{ as } N \to \infty, \tag{4.60}$$

where we have used the bounds in the second line of (4.38) and (4.41) to conclude in the last step. Since τ_N converges to $\frac{3}{2\pi}$, see (4.39), we can infer from (4.59), (4.60) that

$$\Delta_{k,N} \overset{\text{def}}{=} \Big\|1_{B_k}\Big((GV_N)^{2k}1_B - \Big(\frac{3}{2\pi}\,\mathcal{E}(W)\Big)^k 1_B\Big)\Big\|_\infty \underset{N}{\longrightarrow} 0, \text{ for each } k \ge 1. \tag{4.61}$$

We can now use this estimate to study the asymptotic behavior of $a_N(2(k+1))$ as N goes to infinity. Indeed one has

$$\begin{aligned} &\Big|a_N(2(k+1)) - \Big(\frac{3}{2\pi}\,\mathcal{E}(W)\Big)^k a_N(2)\Big| = \\ &\alpha\,\frac{\log N}{N}\,\Big|\Big(V_N, GV_N\Big((GV_N)^{2k}1_B - \Big(\frac{3}{2\pi}\,\mathcal{E}(W)\Big)^k 1_B\Big)\Big)\Big| = I_1 + I_2, \end{aligned} \tag{4.62}$$

where in the last step, using the symmetry of G, we have set:

$$\begin{aligned} I_1 &= \alpha\,\frac{\log N}{N}\,\Big|\Big(GV_N, V_N\,1_{B_k}\Big((GV_N)^{2k}1_B - \Big(\frac{3}{2\pi}\,\mathcal{E}(W)\Big)^k 1_B\Big)\Big)\Big|, \\ I_2 &= \alpha\,\frac{\log N}{N}\,\Big|\Big(GV_N, V_N\,1_{B\setminus B_k}\Big((GV_N)^{2k}1_B - \Big(\frac{3}{2\pi}\,\mathcal{E}(W)\Big)^k 1_B\Big)\Big)\Big|. \end{aligned}$$

We then observe that

$$I_1 \le c\alpha\log N\,\|GV_N\|_\infty\,\|V_N\|_\infty\,\Delta_{k,N} \overset{(3.10)}{\le} c'\alpha\,\Delta_{k,N} \underset{N}{\overset{(4.61)}{\longrightarrow}} 0,$$

and that

$$I_2 \overset{(4.24)}{\le} \alpha \log N \, \|GV_N\|_\infty \, \|V_N\|_\infty \times c(k) \times \frac{|B \backslash B_k|}{N}$$

$$\overset{(3.10)}{\le} c'(k)\alpha \, \frac{|J \backslash J_k|}{|J|} \;\; \underset{N}{\overset{(4.56)}{\longrightarrow}} 0.$$

We have thus shown that

$$\lim_N \Big| a_N(2(k+1)) - \Big(\frac{3}{2\pi}\,\mathcal{E}(W)\Big)^k a_N(2)\Big| = 0, \text{ for any } k \ge 1. \tag{4.63}$$

Combined with (4.48) this completes the proof of (4.14) and hence of Theorem 4.1. □

Our next objective is to study the convergence in distribution of the random fields $(\frac{\mathcal{L}_{y,N}}{\log N})_{y \in \mathbb{Z}^2}$ and $(\frac{\mathcal{L}'_{y,N}}{N u'_N})_{y \in \mathbb{Z}^2}$, as N goes to infinity, where we recall the notation from (4.1), (4.2). The task is simplified by the fact that we have already proved Theorem 4.2: we only need to investigate the convergence in distribution of these random fields at the origin. We now focus on the case where $\Lambda = \{0\}$, and J_0 plays the role of B. We further define

$$\widetilde{V}_N(x) = \frac{1}{\log N}\, 1_{J_0}(x), \; \widetilde{V}'_N(x) = \frac{1}{\gamma_N}\, \widetilde{V}_N(x), \text{ for } x \in \mathbb{Z}^3, \tag{4.64}$$

and set

$$\widetilde{\mathcal{L}}_N = \frac{1}{\log N}\, \mathcal{L}_{0,N}, \; \widetilde{\mathcal{L}}'_N = \frac{1}{N u'_N}\, \mathcal{L}'_{0,N}\,. \tag{4.65}$$

Just as in (4.7), (4.10), we know by Theorem 2.1 that

$$\mathbb{E}[\exp\{z \widetilde{\mathcal{L}}_N\}] = \exp\Big\{\sum_{n \ge 1} \widetilde{a}_N(n)\, z^n\Big\}, \text{ for } |z| < \widetilde{r}_N \text{ in } \mathbb{C}, \tag{4.66}$$

with $r_N > 0$ and where we have set

$$\widetilde{a}_N(n) = u_N(\widetilde{V}_N, (G\widetilde{V}_N)^{n-1} 1), \text{ for } n \ge 1, \tag{4.67}$$

and that

$$\mathbb{E}[\exp\{z \widetilde{\mathcal{L}}'_N\}] = \exp\Big\{\sum_{n \ge 1} \widetilde{a}'_N(n)\, z^n\Big\}, \text{ for } |z| < \widetilde{r}'_N \text{ in } \mathbb{C}, \tag{4.68}$$

with $r'_N > 0$ and

$$\widetilde{a}'_N(n) = u'_N(\widetilde{V}'_N, (G\widetilde{V}'_N)^{n-1} 1) = \frac{1}{\alpha}\, \gamma_N^{1-n}\, \widetilde{a}_N(n), \text{ for } n \ge 1. \tag{4.69}$$

The heart of the matter for the proof of Theorem 4.9 below lies in the control of the large N behavior of the sequence $\widetilde{a}_N(n)$, $n \ge 1$.

Theorem 4.8.

$$0 \le \widetilde{a}_N(n) \le \alpha c_7^n, \; \textit{for } n \ge 1, N > 1, \tag{4.70}$$

$$\textit{for any } n \ge 1, \; \lim_N \widetilde{a}_N(n) = \alpha \Big(\frac{3}{\pi}\Big)^{n-1}. \tag{4.71}$$

We first explain how Theorems 4.2 and 4.8 enable us to infer the convergence in law of the appropriately scaled fields $\mathcal{L}_{y,N}$, $y \in \mathbb{Z}^2$, and $\mathcal{L}'_{y,N}$, $y \in \mathbb{Z}^2$.

Theorem 4.9.

(4.72) *As N goes to infinity, $(\frac{\mathcal{L}_{y,N}}{\log N})_{y\in\mathbb{Z}^2}$ converges in distribution to a flat random field with constant valued distributed as R^2 with R as in* (4.18).

(4.73) *As N goes to infinity, $(\frac{\mathcal{L}'_{y,N}}{Nu'_N})_{y\in\mathbb{Z}^2}$ converges in distribution to a flat random field with value* 1.

Proof of Theorem 4.9 (assuming Theorem 4.8): A repetition of the arguments used in the proof of Theorem 4.2 shows that $\widetilde{\mathcal{L}}_N$ converges in distribution to a non-negative random variable $\widetilde{\mathcal{L}}$ with Laplace transform

$$E[\exp\{-\lambda\widetilde{\mathcal{L}}\}] = \exp\Big\{-\frac{\alpha\lambda}{1+\frac{3}{\pi}\lambda}\Big\}, \text{ for } \lambda \ge 0, \tag{4.74}$$

so that by (1.30), $\widetilde{\mathcal{L}}$ is $BESQ^0(\alpha, \frac{3}{2\pi})$-distributed, i.e. has same distribution as V^2 in the notation of (4.18).

Moreover we know from Theorem 4.2 that for any $y \in \mathbb{Z}^2$, when N goes to infinity, $\frac{1}{\log N}(\mathcal{L}_{y,N} - \mathcal{L}_{0,N})$ converges to zero in distribution, and (4.72) follows.

In the case of (4.73) we note instead that the arguments used in the proof of Theorem 4.2 now show that $\widetilde{\mathcal{L}}'_N$ converges in distribution to a non-negative random variable with Laplace transform $e^{-\lambda}$, $\lambda \ge 0$, i.e. to the constant 1. Since by Theorem 4.2, $\frac{1}{Nu'_N}(\mathcal{L}'_{y,N} - \mathcal{L}'_{0,N})$ converges in distribution to zero for any $y \in \mathbb{Z}^2$, we obtain (4.73).

Proof of Theorem 4.8: We now write $\|\cdot\|_\infty$ for the supremum norm on $\widetilde{B} = J_0$ and the linear operators we consider are restricted to functions that vanish outside $\widetilde{B}$. The fact that $\widetilde{a}_N(n)$ is non-negative is plain, see (4.64), (4.67). Moreover the right-hand inequality in (4.70) is a direct consequence of (3.8). This proves (4.70).

We now turn to the proof of (4.71). For $k \ge 1$, we introduce $\widetilde{B}_k = \{0\} \times J_k$, with J_k as in (4.56), so that for any $k \ge 1$, and $N \ge c(k)$, $\phi \ne \widetilde{B}_k \subsetneq \widetilde{B}_{k-1} \subsetneq \cdots \subsetneq \widetilde{B}_1 \subsetneq \widetilde{B}$. In a much simpler fashion than (4.59) we now find that, (see (4.39) for notation),

$$\|1_{\widetilde{B}_k}((G\widetilde{V}_N)^k 1_{\widetilde{B}} - (2\tau_N)^k 1_{\widetilde{B}})\|_\infty \le c(k)\Big(\|1_{\widetilde{B}_1}\widetilde{F}_N\|_\infty + \frac{1}{\sqrt{\log N}}\Big), \tag{4.75}$$

where we have set

$$\widetilde{F}_N = (G\widetilde{V}_N)1_{\widetilde{B}} - 2\tau_N\, 1_{\widetilde{B}} = G\widetilde{V}_N - 2\tau_N\, 1_{\widetilde{B}}.$$

It already follows from the definitions of $\widetilde{V}_N$ and τ_N in (4.64), (4.39) that

$$\lim_N \|1_{\widetilde{B}_1}\widetilde{F}_N\|_\infty = 0. \tag{4.76}$$

We thus see that for $k \geq 1$,

$$\lim_N \widetilde{a}_N(k) = \lim_N \alpha \, \frac{\log N}{N} \, (\widetilde{V}_N, (G\widetilde{V}_N)^{k-1} 1) = \alpha (2 \lim_N \tau_N)^{k-1} = \alpha \Big(\frac{3}{\pi}\Big)^{k-1}, \tag{4.77}$$

and this proves (4.71). □

Remark 4.10.

1) There is an important connection between random interlacements at level u and the structure left by random walk on a large torus $(\mathbb{Z}/N\mathbb{Z})^d$,(here $d = 3$), at a microscopic scale of order 1, see [23], or even at a mesoscopic scale of order $N^{1-\varepsilon}$, with $0 < \varepsilon < 1$, see [22], when the walk runs for times of order uN^d. This naturally raises the question whether the above limiting results might also be relevant for the field of occupation times left close to the origin, by continuous time simple random walk with uniform starting point on a large two-dimensional torus $(\mathbb{Z}/N\mathbb{Z})^2$, at times of order $\alpha N^2 \log N$ $(= u_N \, N^3)$, or at much larger times $u'_N \, N^3$. Let us incidentally point out that the time scale $\alpha N^2 \log N$ is much smaller than the cover time of the torus which has order $\frac{4}{\pi} \, N^2 (\log N)^2$, see [4].

2) We can consider the discrete occupation times $\ell_{x,u}$, $x \in \mathbb{Z}^d$, $u \geq 0$, see Remark 2.4 5), and define $\mathfrak{L}_{y,N}$ and $\mathfrak{L}'_{y,N}$, for $y \in \mathbb{Z}^2$, $N > 1$, as in (4.2), simply replacing $L_{x,u}$ by $\ell_{x,u}$.

Theorems 4.2 and 4.9 enable us to show that $\big(\frac{\mathfrak{L}'_{y,N} - \mathfrak{L}'_{0,N}}{\sqrt{N u'_N}}\big)_{y \in \mathbb{Z}^2}$ converges in distribution to a centered Gaussian field which vanishes at the origin. However this limiting field is different from $(\psi_y)_{y \in \mathbb{Z}^2}$ in (4.20).

The heart of the matter lies in the fact that for any $W(\cdot)$ as in (4.3), when N tends to infinity,

(4.78) $\sum_{y \in \mathbb{Z}^2} W(y) \, \frac{\mathfrak{L}'_{y,N}}{\sqrt{N u'_N}}$ converges in distribution to a Gaussian variable with variance $\mathcal{E}(W) - \sum_{y \in \mathbb{Z}^2} W(y)^2$,

(in the case of $\mathcal{L}'_{y,N}$ the limiting variance instead equals $\mathcal{E}(W)$).

Indeed it follows from (2.27) that for real z and $N > 1$,

$$\mathbb{E}\Big[\exp\Big\{z \sum_{y \in \mathbb{Z}^2} \frac{W(y)}{\sqrt{N u'_N}} \, \mathfrak{L}'_{y,N}\Big\}\Big] = \mathbb{E}\Big[\exp\Big\{\sum_{y \in \mathbb{Z}^2} \big(1 - e^{-z \frac{W(y)}{\sqrt{N u'_N}}}\big) \, \mathcal{L}'_{y,N}\Big\}\Big], \tag{4.79}$$

and for $|z| < r$ and $N \geq c$, we can use Taylor's expansion and write

$$1 - e^{-z \frac{W(y)}{\sqrt{N u'_N}}} = z \, \frac{W(y)}{\sqrt{N u'_N}} - \frac{1}{2} \, z^2 \, \frac{W(y)^2}{N u'_N} \, (1 + \varepsilon_y(z, N)),$$

where $|\varepsilon_y(z,N)| \leq \frac{1}{2}$ and $\lim_N \varepsilon_y(z,N) = 0$, for each $y \in \mathbb{Z}^2$, $|z| < r$.

Inserting the above identity in (4.79) shows that for $|z| < r$ and $N \geq c$, the left-hand side of (4.79) equals

$$\mathbb{E}\Big[\exp\Big\{z \sum_{y \in \mathbb{Z}^2} \frac{W(y)}{\sqrt{N u'_N}} \, \mathcal{L}'_{y,N} - \frac{1}{2} \, z^2 \sum_{y \in \mathbb{Z}^2} \frac{W(y)^2}{N u'_N} \, (1 + \varepsilon_y(z,N)) \, \mathcal{L}'_{y,N}\Big\}\Big]. \tag{4.80}$$

By the end of the proof of Theorem 4.2 we know that for $|z| < c$,

$$\lim_N \mathbb{E}\Big[\exp\Big\{z \sum_{y\in\mathbb{Z}^2} \frac{W(y)}{\sqrt{Nu'_N}}\, \mathcal{L}'_{y,N}\Big\}\Big] = \exp\Big\{\frac{z^2}{2}\, \mathcal{E}(W)\Big\}.$$

A straightforward uniform integrability argument combined with (4.73) and (4.80) shows that for real z with $|z| < c$,

$$\lim_N \mathbb{E}\Big[\exp\Big\{z \sum_{y\in\mathbb{Z}^2} \frac{W(y)}{\sqrt{Nu'_N}}\, \mathfrak{L}'_{y,N}\Big\}\Big] = \exp\Big\{\frac{z^2}{2}\, \mathcal{E}(W) - \frac{z^2}{2} \sum_{y\in\mathbb{Z}^2} W(y)^2\Big\}. \tag{4.81}$$

Similar arguments as in the proof of Theorem 4.2 now yield (4.78).

Note incidentally that Theorems 4.2 and 4.9 are not quite sufficient to study the limit in law of $(\frac{\mathfrak{L}_{y,N}-\mathfrak{L}_{0,N}}{\sqrt{\log N}})_{y\in\mathbb{Z}^2}$. As shown by the above proof, see in particular (4.80), to handle this case we would in essence need a limiting result for the joint law of the two random fields that appear in (4.17) and (4.72).

□

5 Occupation times at high level u

In this section we relate occupation times at a high level u of the random interlacements with the d-dimensional Gaussian free field. The limit $u \to \infty$ bypasses the obstructions present when one considers a fixed level u, see Remarks 2.4 2). Our main result appears in Theorem 5.1. It has a similar flavor to (4.20) of Theorem 4.2 and (4.73) of Theorem 4.9. Moreover it can rather straightforwardly be extended to the case of random interlacements on transient weighted graphs, see Remark 5.2. However we keep the set-up of $\mathbb{Z}^d$, $d \geq 3$ for the main body of this section, not to overburden notation.

We consider on an auxiliary probability space

(5.1) $\varphi_x, x \in \mathbb{Z}^d$, a centered Gaussian field with covariance function $E[\varphi_x \varphi_{x'}] = g(x'-x) + g(0) - g(x) - g(x')$, for $x, x' \in \mathbb{Z}^d$.

This field has the same distribution as the field $(\gamma_x - \gamma_0)_{x\in\mathbb{Z}^d}$ of increments at the origin of the d-dimensional Gaussian free field, $(\gamma_x)_{x\in\mathbb{Z}^d}$, i.e. the centered Gaussian field with covariance function $E[\gamma_x\gamma_{x'}] = g(x,x')$, for $x, x' \in \mathbb{Z}^d$.

We can now state the main result of this section.

Theorem 5.1. *As $u \to \infty$,*

(5.2) $\Big(\dfrac{L_{x,u} - L_{x,0}}{\sqrt{2u}}\Big)_{x\in\mathbb{Z}^d}$ *converges in distribution to the Gaussian random field $(\varphi_x)_{x\in\mathbb{Z}^d}$ in* (5.1),

and

(5.3) $\big(\frac{1}{u}\, L_{x,u}\big)_{x\in\mathbb{Z}^d}$ *converges in distribution to the constant field equal to* 1.

Proof. We follow the same strategy as in the previous section, the problem is however much simpler now. It clearly suffices to prove (5.2) and (5.3) with u replaced by a sequence u_N such that

$$u_N \geq 1, \text{ for } N \geq 1, \text{ and } \lim_N u_N = \infty. \tag{5.4}$$

We thus consider a function V on $\mathbb{Z}^d$ such that

$$V \text{ is finitely supported and } \sum_{x \in \mathbb{Z}^d} V(x) = 0. \tag{5.5}$$

We define

$$L_N = \sum_{x \in \mathbb{Z}^d} \frac{1}{\sqrt{2u_N}} V(x) \, L_{x,u_N}, \text{ for } N \geq 1. \tag{5.6}$$

It follows from Theorem 2.1 and Remark 2.2 that for some fixed $r > 0$,

$$\mathbb{E}[\exp\{zL_N\}] = \exp\Big\{\sum_{n \geq 1} c_N(n) z^n\Big\}, \text{ for } |z| < r \text{ in } \mathbb{C}, \tag{5.7}$$

where we have set for $n, N \geq 1$,

$$c_N(n) = 2^{-\frac{n}{2}} \, u_N^{1-\frac{n}{2}} (V, (GV)^{n-1} 1). \tag{5.8}$$

In view of (5.5) we have

$$c_N(1) = 0. \tag{5.9}$$

By (5.4), (5.8) it is also plain that for $n \geq 2$,

$$|c_N(n)| \leq c(V)^n, \tag{5.10}$$

with $c(V)$ a positive constant depending on V and d, by our convention. Moreover we find that

$$\lim_N c_N(n) = 0, \text{ for } n > 2, \text{ and} \tag{5.11}$$

$$c_N(2) = \frac{1}{2} \, (V, GV) \overset{(5.1),(5.5)}{=} \frac{1}{2} \, E\Big[\Big(\sum_x V(x) \varphi_x\Big)^2\Big]. \tag{5.12}$$

The same arguments as in the proof of Theorem 4.2 show that

$$\begin{aligned} &L_N \text{ converges in distribution to a centered Gaussian variable with variance} \\ &E\Big[\Big(\sum_x V(x)\varphi_x\Big)^2\Big]. \end{aligned} \tag{5.13}$$

Since V in (5.5) and u_N in (5.4) are arbitrary, the claim (5.2) follows. We then turn to the proof of (5.3). It follows by (2.21) that for $\lambda \geq 0$,

$$\mathbb{E}\Big[\exp\Big\{-\frac{\lambda}{u_N} \, L_{0,u_N}\Big\}\Big] = \exp\Big\{-\frac{\lambda}{1 + g(0)\frac{\lambda}{u_N}}\Big\} \longrightarrow e^{-\lambda}, \text{ as } N \to \infty. \tag{5.14}$$

This shows that $\frac{1}{u_N} L_{0,u_N}$ converges in distribution to the constant 1 as N goes to infinity. Since due to (5.2), for any $x \in \mathbb{Z}^d$, $\frac{1}{u_N}(L_{x,u_N} - L_{0,u_N})$ tends to zero in distribution, as N goes to infinity, our claim follows. □

Remark 5.2.

1) The results of the present section can straightforwardly be extended to the set-up of continuous time random interlacements on a transient weighted graph E, as we now explain. We keep the same notation and assumptions as in Remark 2.4 4). We introduce a base point $x_0 \in E$. In place of (5.1) we consider

$$\begin{aligned} &\varphi_x, x \in E, \text{ a centered Gaussian field with covariance function} \\ &E[\varphi_x \varphi_{x'}] = g(x,x') + g(x_0,x_0) - g(x_0,x) - g(x_0,x'),\ x,x' \in E, \end{aligned} \tag{5.15}$$

where $g(\cdot,\cdot)$ now stands for the Green density.

This field has the same distribution as the field of increments $(\gamma_x - \gamma_{x_0})_{x\in E}$, of the Gaussian free field $(\gamma_x)_{x\in E}$ attached to the transient weighted graph, i.e. the centered Gaussian field with covariance function $E[\gamma_x \gamma_{x'}] = g(x,x')$, $x,x' \in E$.

With the help of (2.25), (2.26), the arguments employed in the proof of Theorem 5.1 now show that as $u \to \infty$,

$$\begin{aligned} &\Big(\frac{L_{x,u} - L_{x,0}}{\sqrt{2u}}\Big)_{x\in E} \text{ converges in distribution to the Gaussian} \\ &\text{random field } (\varphi_x)_{x\in E}, \end{aligned} \tag{5.16}$$

and that

$$\Big(\frac{1}{u}\, L_{x,u}\Big)_{x\in E} \text{ converges in distribution to the constant field equal to 1.} \tag{5.17}$$

2) In the case of the discrete occupation times $\ell_{x,u}$, $x \in \mathbb{Z}^d$, $u \ge 0$, the same arguments used in Remark 4.10 2) show that when $u \to \infty$,

$$\Big(\frac{\ell_{x,u} - \ell_{x,0}}{\sqrt{2u}}\Big)_{x\in \mathbb{Z}^d} \text{ converges in distribution to } (\nu_x)_{x\in\mathbb{Z}^d}, \tag{5.18}$$

where $(\nu_x)_{x\in\mathbb{Z}^d}$ is the centered Gaussian field vanishing at the origin such that for any V as in (5.5), one has in the notation of (5.1):

$$E\Big[\Big(\sum_{x\in\mathbb{Z}^d} V(x)\,\nu_x\Big)^2\Big] + \frac{1}{2}\sum_{x\in\mathbb{Z}^d} V(x)^2 = E\Big[\Big(\sum_{x\in\mathbb{Z}^d} V(x)\,\varphi_x\Big)^2\Big]. \tag{5.19}$$

Moreover looking at the Laplace functional, one sees with the help of (2.27) and (5.3) that for $u \to \infty$,

$$\Big(\frac{1}{u}\, \ell_{x,u}\Big)_{x\in\mathbb{Z}^d} \text{ converges in distribution to the constant field equal to 1.} \tag{5.20}$$

□

References

[1] P. Billingsley. *Convergence of probability measures.* Wiley, New York, 1968.

[2] E. Bolthausen, J.-D. Deuschel, and G. Giacomin. Entropic repulsion and the maximum of the two-dimensional harmonic crystal. *Ann. Probab.*, 29(4):16701692, 2001.

[3] D. Brydges, J. Fröhlich, and T. Spencer. The random walk representation of classical spin systems and correlation inequalities. *Comm. Math. Phys.*, 83(1):123–150, 1982.

[4] A. Dembo, Y. Peres, J. Rosen, and O. Zeitouni. Cover times for Brownian motion and random walks in two dimensions. *Ann. Math.*, 160(2):433–464, 2004.

[5] E.B. Dynkin. Markov processes as a tool in field theory. *J. of Funct. Anal.*, 50(1):167–187, 1983.

[6] N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes.* North-Holland; Amsterdam, Kodansha, Ltd., Tokyo, 2nd edition, 1989.

[7] G.F. Lawler. *Intersections of random walks.* Birkhäuser, Basel, 1991.

[8] G.F. Lawler and V. Limic. *Random walk: a modern introduction.* Cambridge University Press, 2010.

[9] G.F. Lawler and W. Werner. The brownian loop soup. *Probab. Theory Relat. Fields*, 128:565–588, 2004.

[10] Y. Le Jan. Markov loops and renormalization. *Ann. Probab.*, 38(3):1280–1319, 2010.

[11] Y. Le Jan. *Markov paths, loops and fields.* Lecture Notes in Mathematics. Springer, Berlin, 2011.

[12] E. Lukacs. *Characteristic functions.* Griffin, London, 1970.

[13] M.B. Marcus and J. Rosen. *Markov processes, Gaussian processes, and local times.* Cambridge University Press, 2006.

[14] D. Revuz and M. Yor. *Continuous martingales and Brownian motion.* Springer, Berlin, 3rd edition, 1998.

[15] V. Sidoravicius and A.S. Sznitman. Percolation for the vacant set of random interlacements. *Comm. Pure Appl. Math.*, 62(6):831–858, 2009.

[16] V. Sidoravicius and A.S. Sznitman. Connectivity bounds for the vacant set of random interlacements. *Ann. Inst. Henri Poincaré, Probabilités et Statistiques*, 46(4):976–990, 2010.

[17] F. Spitzer. *Principles of random walk.* Springer, Berlin, second edition, 2001.

[18] K. Symanzik. Euclidean quantum field theory. *In:* Scuola internazionale die Fisica “Enrico Fermi”, XLV Corso, 152-223*, Academic Press*, New York, 1969.

[19] A.S. Sznitman. Vacant set of random interlacements and percolation. *Ann. Math.*, 171(3):2039–2087, 2010.

[20] A.S. Sznitman. Decoupling inequalities and interlacement percolation on GxZ. *preprint,* also available at arXiv:1010.1490.

[21] A. Teixeira. Interlacement percolation on transient weighted graphs. *Electron. J. Probab.*, 14:1604–1627, 2009.

[22] A. Teixeira and D. Windisch. On the fragmentation of a torus by random walk. *preprint,* also available at arXiv:1007.0902.

[23] D. Windisch. Random walk on a discrete torus and random interlacements. *Electron. Comm. Probab.*, 13:140–150, 2008.