AN ISOMORPHISM THEOREM FOR RANDOM INTERLACEMENTS

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Abstract

We consider continuous-time random interlacements on a transient weighted graph. We prove an identity in law relating the field of occupation times of random interlacements at level u to the Gaussian free field on the weighted graph. This identity is closely linked to the generalized second Ray-Knight theorem of [2], [4], and uniquely determines the law of occupation times of random interlacements at level u.

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0 Introduction

In this note we consider continuous-time random interlacements on a transient weighted graph E. We prove an identity in law, which relates the field of occupation times of random interlacements at level u to the Gaussian free field on E. The identity can be viewed as a kind of generalized second Ray-Knight theorem, see [2], [4], and characterizes the law of the field of occupation times of random interlacements at level u.

We now describe our results and refer to Section 1 for details. We consider a countable, locally finite, connected graph, with vertex set E, endowed with non-negative symmetric weights $c_{x,y} = c_{y,x}$, $x, y \in E$, which are positive exactly when x, y are distinct and $\{x, y\}$ is an edge of the graph. We assume that the induced discrete-time random walk on E is transient. Its transition probability is defined by

$$p_{x,y} = \frac{c_{x,y}}{\lambda_x}, \text{ where } \lambda_x = \sum_{z \in E} c_{x,z}, \text{ for } x, y \in E. \tag{0.1}$$

In essence, continuous-time random interlacements consist of a Poisson point process on a certain space of doubly infinite E-valued trajectories marked by their duration at each step, modulo time-shift. A non-negative parameter u plays the role of a multiplicative factor of the intensity of this Poisson point process, which is defined on a suitable canonical space $(\Omega, \mathcal{A}, \mathbb{P})$. The field of occupation times of random interlacements at level u is then defined for $x \in E$, $u \geq 0$, $\omega \in \Omega$, by (see (1.8) for the precise expression)

$$\begin{aligned} L_{x,u}(\omega) = \lambda_x^{-1} \times & \text{ the total duration spent at } x \text{ by the trajectories modulo} \\ & \text{time-shift with label at most } u \text{ in the cloud } \omega \end{aligned} \tag{0.2}$$

(informally, the durations of the successive steps of a trajectory are described by independent exponential variables of parameter 1, but occupation times at x get rescaled by a factor λ_x^{-1}).

The Gaussian free field on E is the other ingredient of our isomorphism theorem. Its canonical law P^G on $\mathbb{R}^E$ is such that

$$\begin{aligned} & \text{under } P^G, \text{ the canonical field } \varphi_x,\ x \in E, \text{ is a centered Gaussian field with} \\ & \text{covariance } E^{P^G}[\varphi_x \varphi_y] = g(x,y), \text{ for } x, y \in E, \end{aligned} \tag{0.3}$$

where $g(\cdot,\cdot)$ stands for the Green function attached to the walk on E, see (1.3). The main result of this note is the next theorem:

Theorem 0.1. *For each $u \geq 0$,*

$$\begin{aligned} & \big(L_{x,u} + \tfrac{1}{2}\, \varphi_x^2\big)_{x \in E} \textit{ under } \mathbb{P} \otimes P^G, \textit{ has the same law as} \\ & \big(\tfrac{1}{2}\, (\varphi_x + \sqrt{2u})^2\big)_{x \in E} \textit{ under } P^G. \end{aligned} \tag{0.4}$$

This theorem provides for each u an identity in law very much in the spirit of the so-called generalized second Ray-Knight theorems, see Theorem 1.1 of [2] or Theorem 8.2.2 of [4]. Remarkably, although we are in a transient set-up, (0.4) corresponds to the recurrent case in the context of generalized Ray-Knight theorems. Let us underline that

(0.4) uniquely determines the law of $(L_{x,u})_{x \in E}$ under $\mathbb{P}$, as the consideration of Laplace transforms readily shows. We also refer to Remark 3.1 for a variation of (0.4).

The proof of Theorem 0.1 involves an approximation argument of the law of $(L_{x,u})_{x \in E}$ stated in Theorem 2.1, which is of independent interest. This approximation has a similar flavor to what appears at the end of Section 4.5 of [7], when giving a precise interpretation of random interlacements as "loops going through infinity", see also [3], p. 85. The combination of Theorem 2.1 and the generalized second Ray-Knight theorem readily yields Theorem 0.1. As an application of Theorem 0.1 we give a new proof of Theorem 5.1 of [6] concerning the large u behavior of $(L_{x,u})_{x \in E}$, see Theorem 4.1.

We now explain how this note is organized.

In Section 1, we provide precise definitions and recall useful facts. Section 2 develops the approximation procedure for $(L_{x,u})_{x \in E}$. We give two proofs of the main Theorem 2.1, and an extension appears in Remark 2.2. The short Section 3 contains the proof of Theorem 0.1, and a variation of (0.4) in Remark 3.1. In Section 4, we present an application to the study of the large u behavior of $(L_{x,u})_{x \in E}$, see Theorem 4.1.

1 Notation and useful results

In this section we provide additional notation and recall some definitions and useful facts related to random walks, potential theory, and continuous-time interlacements.

We consider the spaces $\widehat{W}_+$ and $\widehat{W}$ of infinite, and doubly infinite, $E \times (0, \infty)$-valued sequences, such that the E-valued sequences form an infinite, respectively doubly-infinite, nearest-neighbor trajectory spending finite time in any finite subset of E, and such that the $(0, \infty)$-valued components have an infinite sum in the case of $\widehat{W}_+$, and infinite "forward" and "backward" sums, when restricted to positive and negative indices, in the case of $\widehat{W}$.

We write Z_n, σ_n, with $n \geq 0$, or $n \in \mathbb{Z}$, for the respective E- and $(0, \infty)$-valued coordinates on $\widehat{W}_+$ and $\widehat{W}$. We denote by P_x, $x \in E$, the law on $\widehat{W}_+$, endowed with its canonical σ-algebra, under which $Z_n, n \geq 0$, is distributed as simple random walk starting at x, and σ_n, $n \geq 0$, are i.i.d. exponential variables with parameter 1, independent from the Z_n, $n \geq 0$. We denote by E_x the corresponding expectation. Further, when ρ is a measure on E, we write P_ρ for the measure $\sum_{x \in E} \rho(x) P_x$, and E_ρ for the corresponding expectation.

We denote by X_t, $t \geq 0$, the continuous-time random walk on E, with constant jump rate 1, defined for $t \geq 0$, $\widehat{w} \in \widehat{W}_+$, by

$$X_t(\widehat{w}) = Z_k(\widehat{w}), \text{ when } \sigma_0(\widehat{w}) + \dots + \sigma_{k-1}(\widehat{w}) \leq t < \sigma_0(\widehat{w}) + \dots + \sigma_k(\widehat{w}) \tag{1.1}$$

(by convention the term bounding t from below vanishes when $k = 0$).

Given $U \subseteq E$, we write $H_U = \inf\{t \geq 0; X_t \in U\}$, $\widetilde{H}_U = \inf\{t > 0; X_t \in U$, and for some $s \in (0, t)$, $X_s \neq X_0\}$, and $T_U = \inf\{t \geq 0; X_t \notin U\}$, for the entrance time in U, the hitting time of U, and the exit time from U. We denote by $g_U(\cdot,\cdot)$ the Green function of the walk killed when exiting U

$$g_U(x,y) = \frac{1}{\lambda_y}\, E_x\Big[\int_0^{T_U} 1\{X_s = y\}ds\Big], \text{ for } x,y \in E. \tag{1.2}$$

The function $g_U(\cdot,\cdot)$ is known to be symmetric and finite (due to the transience assumption we have made). When $U = E$, no killing takes place (i.e. $T_U = \infty$), and we simply write

$$g(x,y) = g_{U=E}(x,y), \text{ for } x,y \in E, \tag{1.3}$$

for the Green function.

Given a finite subset K of U, the equilibrium measure and capacity of K relative to U are defined by

$$e_{K,U}(x) = P_x[\widetilde{H}_K > T_U]\, \lambda_x\, 1_K(x), \text{ for } x \in E, \tag{1.4}$$

$$\text{cap}_U(K) = \sum_{x\in E} e_{K,U}(x). \tag{1.5}$$

When $U = E$, we simply drop U from the notation, and refer to e_K and $\text{cap}(K)$, as the equilibrium measure and the capacity of K. Further, the probability to enter K before exiting U can be expressed as

$$P_x[H_K < T_U] = \sum_{y\in E} g_U(x,y)\, e_{K,U}(y), \text{ for } x \in E. \tag{1.6}$$

We now turn to the description of continuous-time random interlacements on the transient weighted graph E. We write $\widehat{W}^*$ for the space $\widehat{W}$ (introduced at the beginning of this section), modulo time-shift, i.e. $\widehat{W}^* = W/\sim$, where for $\widehat{w}$, $\widehat{w}' \in \widehat{W}$, $\widehat{w} \sim \widehat{w}'$ means that $\widehat{w}(\cdot) = \widehat{w}'(\cdot + k)$ for some $k \in \mathbb{Z}$. We denote by π^*: $\widehat{W} \to \widehat{W}^*$ the canonical map, and endow $\widehat{W}^*$ with the σ-algebra consisting of sets with inverse image under π^* belonging to the canonical σ-algebra of $\widehat{W}$.

The continuous-time interlacement point process is a Poisson point process on the space $\widehat{W}^* \times \mathbb{R}_+$. Its intensity measure has the form $\nu(d\widehat{w}^*)du$, where $\widehat{\nu}$ is the σ-finite measure on $\widehat{W}^*$ such that for any finite subset K of E, the restriction of $\widehat{\nu}$ to the subset of $\widehat{W}^*$ consisting of those $\widehat{w}^*$ for which the E-valued trajectory modulo time-shift enters K, is equal to $\pi^* \circ \widehat{Q}_K$, the image of $\widehat{Q}_K$ under π^*, where $\widehat{Q}_K$ is the finite measure on $\widehat{W}$ specified by

(1.7)

i) $\widehat{Q}_K(Z_0 = x) = e_K(x)$, for $x \in E$,

ii) when $e_K(x) > 0$, conditionally on $Z_0 = x$, $(Z_n)_{n\geq 0}$, $(Z_{-n})_{n\geq 0}$, $(\sigma_n)_{n\in\mathbb{Z}}$ are independent, respectively distributed as simple random walk starting at x, as simple random walk starting at x conditioned never to return to K, and as a doubly infinite sequence of i.i.d. exponential variables with parameter 1.

As in [6], the canonical continuous-time random interlacement point process is then constructed similarly to (1.16) of [5], or (2.10) of [8], on a space $(\Omega, \mathcal{A}, \mathbb{P})$, with $\omega = \sum_{i \geq 0} \delta_{(\widehat{w}_i^*, u_i)}$ denoting a generic element of Ω. A central object of interest in this note is the random field of occupation times of random interlacements at level $u \geq 0$:

$$
(1.8) \qquad \begin{aligned} L_{x,u}(\omega) = \frac{1}{\lambda_x} \sum_{i \geq 0} \sum_{n \in \mathbb{Z}} \sigma_n(\widehat{w}_i) \, 1\{Z_n(\widehat{w}_i) = x, u_i \leq u\}, \text{ for } x \in E, \, \omega \in \Omega, \\ \text{where } \omega = \sum_{i \geq 0} \delta_{(\widehat{w}_i^*, u_i)} \text{ and } \pi^*(\widehat{w}_i) = \widehat{w}_i^*, \text{ for each } i \geq 0. \end{aligned}
$$

The Laplace transform of $(L_{x,u})_{x \in E}$ has been computed in [6]. More precisely, given a function f: $E \to \mathbb{R}$, such that $\sum_{y \in E} g(x,y) |f(y)| < \infty$, for $x \in E$, one sets

$$
(1.9) \qquad Gf(x) = \sum_{y \in E} g(x,y) \, f(y), \text{ for } x \in E.
$$

One knows from Theorem 2.1 and Remark 2.4 4) of [6], that when V: $E \to \mathbb{R}_+$ has finite support and

$$
(1.10) \qquad \sup_{x \in E} GV(x) < 1,
$$

one has the identity

$$
(1.11) \qquad \mathbb{E}\Big[\exp\Big\{ - \sum_{x \in E} V(x) \, L_{x,u} \Big\}\Big] = \exp\{-u \langle V, (I + GV)^{-1} \, 1_E \rangle\}, \text{ for } u \geq 0,
$$

where the notation $\langle f, g \rangle$ stands for $\sum_{x \in E} f(x) \, g(x)$, when f, g are functions on E such that the previous sum converges absolutely, and 1_E denotes the constant function identically equal to 1 on E.

2 An approximation scheme for random interlacements

In this section we develop an approximation scheme for $(L_{x,u})_{x \in E}$ in terms of the fields of local times of certain finite state space Markov chains. The main result is Theorem 2.1, but Remark 2.2 states a by-product of the approximation scheme concerning the random interlacement at level u. This has a similar flavor to Theorem 4.17 of [7], where one gives one of several possible meanings to random interlacements viewed as "Markovian loops going through infinity", see also Le Jan [3], p. 85.

We consider a non-decreasing sequence $U_n, n \geq 1$, of finite connected subsets of E, increasing to E, as well as x_* some fixed point not belonging to E. We introduce the sets $E_n = U_n \cup \{x_*\}$, for $n \geq 1$, and endow E_n with the weights $c^n_{x,y}$, $x, y \in E_n$, obtained by "collapsing U_n^c on x_*", that is, for any $n \geq 1$, and $x, y \in U_n$, we set

$$
(2.1) \qquad \begin{aligned} c^n_{x,y} &= c_{x,y}, \\ c^n_{x_*,y} &= c^n_{y,x_*} = \sum_{z \in E \setminus U_n} c_{z,y}, \end{aligned}
$$

and otherwise set $c^n_{x,y} = 0$ (i.e. $c^n_{x_*,x_*} = 0$). We also write

$$\lambda^n_x = \sum_{y \in E_n} c^n_{x,y}, \text{ for } x \in E_n \text{ (in particular } \lambda^n_x = \lambda_x, \text{ when } x \in U_n). \tag{2.2}$$

We tacitly view U_n as a subset of both E and E_n. We consider the canonical simple random walk in continuous time on E_n, attached to the weights $c^n_{x,y}$, $x, y \in E_n$, with jump rate equal to 1. We write X^n_t, $t \geq 0$, for its canonical process, P^n_x for its canonical law starting from $x \in E_n$, and E^n_x for the corresponding expectation.

The local time of this Markov chain is defined by

$$\ell^{n,x}_t = \frac{1}{\lambda^n_x} \int_0^t 1\{X^n_s = x\}\, ds, \text{ for } x \in E_n \text{ and } t \geq 0. \tag{2.3}$$

The function $t \geq 0 \to \ell^{n,x}_t \geq 0$ is continuous, non-decreasings, starts at 0, and P^n_y-a.s. tends to infinity, as t goes to infinity (the walk on E_n is irreducible and recurrent). By convention, when $x \in E \backslash U_n$, we set $\ell^{n,x}_t = 0$, for all $t \geq 0$. We introduce the right-continuous inverse of $\ell^{n,x_*}_\cdot$

$$\tau^n_u = \inf\{t \geq 0; \ell^{n,x_*}_t > u\}, \text{ for any } u \geq 0. \tag{2.4}$$

We are now ready for the main result of this section. We tacitly endow $\mathbb{R}^E$ with the product topology, and convergence in distribution, as stated below (and in the sequel), corresponds to convergence in law of all finite dimensional marginals.

Theorem 2.1. $(u \geq 0)$

$$(\ell^{n,x}_{\tau^n_u})_{x \in E} \textit{ under } P^n_{x_*} \textit{ converges in distribution to } (L_{x,u})_{x \in E} \textit{ under } \mathbb{P}. \tag{2.5}$$

Proof. We give two proofs.

First proof: We denote by $\mathcal{T}$ the set of piecewise-constant, right-continuous, $E \cup \{x_*\}$-valued trajectories, which at a finite time reach x_*, and from that time onwards remain equal to x_*. We endow $\mathcal{T}$ with its canonical σ-algebra.

Under $P^n_{x_*}$, one has almost surely two infinite sequences $R_\ell, \ell \geq 1$ and $D_\ell, \ell \geq 1$,

$$R_1 = 0 < D_1 < R_2 < \dots < R_\ell < D_\ell < \dots \tag{2.6}$$

of successive returns R_ℓ of $X^n_\cdot$ to x_*, and departures D_ℓ from x_*, which tend to infinity. One introduces the random point measure on $\mathcal{T}$

$$\Gamma^n_u = \sum_{\ell \geq 1} 1\{D_\ell < \tau^n_u\}\, \delta_{(X^n_{D_\ell + \cdot})_{0 \leq \cdot \leq R_{\ell+1} - D_\ell}}, \ u \geq 0, \tag{2.7}$$

which collects the successive excursions of $X^n_\cdot$ (out of x_* until first return to x_*) that start before τ^n_u. By classical Markov chain excursion theory we know that

$$\begin{array}{l} \Gamma^n_u \text{ is a Poisson point measure on } \mathcal{T} \text{ with intensity measure} \\ \gamma^n_u(\cdot) = u\, P^n_{\kappa_n}[(X^n_{s \wedge T_{U_n}})_{s \geq 0} \in \cdot] \text{ on } \mathcal{T}, \end{array} \tag{2.8}$$

where T_{U_n} stands for the exit time of $X^n_\cdot$ from U_n and κ_n for the measure on U_n

$$\kappa_n(y) = \lambda^n_{x_*} \frac{c^n_{x_*,y}}{\lambda^n_{x_*}} = c^n_{x_*,y} \overset{(2.1)}{=} \sum_{x \in E \setminus U_n} c_{x,y}, \text{ for } y \in U_n. \tag{2.9}$$

When starting in U_n, the Markov chains X on E, and X^n on E_n, have the same evolution strictly before the exit time of U_n. Denoting by $(X_\cdot)_{0 \le \cdot < T_{U_n}}$ the random element of $\mathcal{T}$, which equals X_s, for $0 \le s < T_{U_n}$, and x_* for $s \ge T_{U_n}$, we see that

$$\gamma^n_u(\cdot) = u\, P_{\kappa_n}[(X_\cdot)_{0 \le \cdot < T_{U_n}} \in \cdot], \text{ for all } n \ge 1,\, u \ge 0. \tag{2.10}$$

Let K be a finite subset of E, and assume n large enough so that $K \subseteq U_n$. We introduce the point measure on $\mathcal{T}$ obtained by selecting the excursions in the support of Γ^n_u that enter K, and only keeping track of their trajectory after they enter K, that is

$$\mu^n_{K,u} = \theta_{H_K} \circ (1\{H_K < \infty\}\, \Gamma^n_u), \tag{2.11}$$

where θ_t, $t \ge 0$, stands for the canonical shift on $\mathcal{T}$, and we use similar notation on $\mathcal{T}$ as below (1.1). By (2.8), (2.10) it follows that

$$\begin{array}{l} \mu^n_{K,u} \text{ is a Poisson point measure on } \mathcal{T} \text{ with intensity measure} \\ \gamma^n_{K,u}(\cdot) = u\, P_{\rho^n_K}[(X_\cdot)_{0 \le \cdot < T_{U_n}} \in \cdot] \text{ on } \mathcal{T}, \end{array} \tag{2.12}$$

where ρ^n_K is the measure supported by K such that

$$\rho^n_K(x) = P_{\kappa_n}[H_K < T_{U_n}, X_{H_K} = x] = e_{K,U_n}(x), \text{ for } x \in K, \tag{2.13}$$

where the last equality follows from (1.60) in Proposition 1.8 of [7]. Note that e_{K,U_n} and e_K are concentrated on K, and for $x \in K$,

$$e_{K,U_n}(x) \overset{(1.4)}{=} P_x[\widetilde{H}_K > T_{U_n}]\, \lambda_x \underset{n \to \infty}{\longrightarrow} P_x[\widetilde{H}_K = \infty]\, \lambda_x = e_K(x). \tag{2.14}$$

Consider $V\colon E \to \mathbb{R}_+$ supported in K, and $\Phi\colon \mathcal{T} \to \mathbb{R}_+$, the map

$$\Phi(w) = \sum_{x \in E} V(x)\, \frac{1}{\lambda_x} \int_0^\infty 1\{w(s) = x\} ds, \text{ for } w \in \mathcal{T}.$$

The measure $\mu^n_{K,u}$ contains in its support the pieces of the trajectory $X^n_\cdot$ up to time τ^n_u, where $X^n_\cdot$ visits K, see (2.11), and we have

$$\begin{aligned} &E^n_{x_*}\Big[\exp\Big\{-\sum_{x \in E} V(x)\, \ell^{n,x}_{\tau^n_u}\Big\}\Big] = E^n_{x_*}\Big[\exp\Big\{-\langle \mu^n_{K,u}, \Phi\rangle\Big\}\Big] \overset{(2.12)}{=} \\ &\exp\Big\{\int_{\mathcal{T}} (e^{-\Phi} - 1)\, d\gamma^n_{K,u}\Big\} \overset{(2.12),(2.13)}{=} \exp\Big\{u\, E_{e_{K,U_n}}\Big[e^{-\int_0^{T_{U_n}} \frac{V}{\lambda}(X_s) ds} - 1\Big]\Big\} \\ &\underset{n \to \infty}{\longrightarrow} \exp\Big\{u\, E_{e_K}\big[e^{-\int_0^\infty \frac{V}{\lambda}(X_s) ds} - 1\big]\Big\} = \mathbb{E}\Big[\exp\Big\{-\sum_{x \in E} V(x)\, L_{x,u}\Big\}\Big], \end{aligned} \tag{2.15}$$

where we used (2.14) and the fact that $T_{U_n} \uparrow \infty$, P_x-a.s., for x in E, for the limit in the last line, and a similar calculation as in (2.5) of [6] for the last equality. Since K and the function $V\colon E \to \mathbb{R}_+$, supported in K, are arbitrary, the claim (2.5) follows.

Second Proof: We will now make direct use of (1.11). The argument is more computational, but also of interest. We consider K and V as above, as well as a positive number λ. We assume n large enough so that $K \subseteq U_n$. We further make a smallness assumption on the non-negative function V (supported in K):

$$\sup_{x \in E} (GV)(x) + \lambda^{-1} \sum_{x \in K} V(x) < 1. \tag{2.16}$$

We define the operator G_n on $\mathbb{R}^{E_n}$ attached to the kernel $g_n(\cdot,\cdot)$ in a similar fashion to (1.9), where we use the notation

$$g_n(x,y) = g_{U_n}(x,y) + \lambda^{-1}, \text{ for } x, y \in E_n, \tag{2.17}$$

and we have set $g_{U_n}(x_*, \cdot) = g_{U_n}(\cdot, x_*) = 0$, by convention, to define $g_{U_n}(\cdot,\cdot)$ on $E_n \times E_n$.

Since $g_{U_n}(\cdot,\cdot) \le g(\cdot,\cdot)$ on $E \times E$, it follows from (2.16) that $\sup_{x \in E_n} (G_n V)(x) < 1$, where we have set $V(x_*) = 0$, by convention, so that the operator $I + G_n V$ is invertible.

We introduce the positive number

$$a_n = \int_0^\infty \lambda e^{-\lambda u} E^n_{x_*} \Big[e^{-\sum_{x \in E} V(x) \ell^{n,x}_{\tau^n_u}} \Big] \, du, \tag{2.18}$$

where we recall that $\ell^{n,x}_t = 0$, when $x \in E \backslash U_n$. Using (2.93), (2.41), (2.71) of [7], or by (8.44) and Remark 3.10.3 of Marcus-Rosen [4], we know that

$$a_n = (I + G_n V)^{-1} 1_{E_n}(x_*). \tag{2.19}$$

We then define the function h_n on E_n and the real number b_n:

$$h_n = (I + G_n V)^{-1} \, 1_{E_n} \text{ and } b_n = \sum_{x \in K} V(x) \, h_n(x). \tag{2.20}$$

We let $G^*_{U_n}$ be the operator on $\mathbb{R}^{E_n}$ attached to the kernel $g_{U_n}(\cdot,\cdot)$ (on $E_n \times E_n$), in a similar fashion to (1.9). By (2.17) and (2.20), we have

$$h_n + G^*_{U_n} V h_n + \lambda^{-1} b_n \, 1_{E_n} = 1_{E_n}, \text{ so that} \tag{2.21}$$

$$h_n = \Big(1 - \frac{b_n}{\lambda}\Big) (1 + G^*_{U_n} V)^{-1} 1_{E_n},$$

noting that the above inverse is well defined by the same argument used below (2.17). By the second equality in (2.20) it follows that

$$b_n = \Big(1 - \frac{b_n}{\lambda}\Big) \sum_{x \in K} V(x) (I + G^*_{U_n} V)^{-1}(x) = \big(1 - \frac{b_n}{\lambda}\big) \langle V, (I + G_{U_n} V)^{-1} 1_E \rangle, \tag{2.22}$$

where we refer to below (1.11) for notation, G_{U_n} is the operator on $\mathbb{R}^E$ attached to the kernel $g_{U_n}(\cdot,\cdot)$ on $E \times E$, and the last equality follows by writing the Neumann series for $(I + G^*_{U_n} V)^{-1}$ and $(I + G_{U_n} V)^{-1}$ (note that $V \ge 0$ and (2.16) straightforwardly imply the convergence of these series in the respective operator norms induced by $L^\infty(E_n)$ and $L^\infty(E)$).

We can now solve for b_n. Noting that $a_n = h_n(x_*) = 1 - \frac{b_n}{\lambda}$, by (2.21), we find

$$a_n = (1 + \lambda^{-1}\langle V, (I + G_{U_n}V)^{-1}1_E\rangle)^{-1}. \tag{2.23}$$

Using the Neumann series for $(I+G_{U_n}V)^{-1}$, and applying dominated convergence together with the fact that $g_{U_n}(\cdot,\cdot) \uparrow g(\cdot,\cdot)$ on $E \times E$, we see that

$$a_n \underset{n\to\infty}{\longrightarrow} (1 + \lambda^{-1}\langle V, (I + GV)^{-1}1_E\rangle)^{-1}. \tag{2.24}$$

Taking the identity (1.11) into account, we have shown that under (2.16),

$$\lim_n \int_0^\infty \lambda e^{-\lambda u} E^n_{x_*}\big[e^{-\sum\limits_{x\in E} V(x)\ell^{n,x}_{\tau^n_u}}\big]\, du = \int_0^\infty \lambda e^{-\lambda u} \mathbb{E}\big[e^{-\sum\limits_{x\in E} V(x) L_{x,u}}\big]\, du. \tag{2.25}$$

Note that when $V\colon E \to \mathbb{R}_+$ is supported in K and $\sup_{x\in E} GV(x) < 1$, then (2.16) holds for λ large (depending on V). The expectation under the integral in the left-hand side of (2.25) is non-increasing in u, whereas the expectation under the integral in the right-hand side of (2.25) is continuous in u by (1.11). It then follows from [1], p. 193-194, that for V as above,

$$\lim_n E^n_{x_*}\big[e^{-\sum\limits_{x\in E} V(x)\ell^{n,x}_{\tau^n_u}}\big] = \mathbb{E}\big[e^{-\sum\limits_{x\in E} V(x) L_{x,u}}\big], \text{ for } u \ge 0. \tag{2.26}$$

This readily implies the tightness of the laws of $(\ell^{n,x}_{\tau^n_u})_{x\in K}$ under $P^n_{x_*}$, and uniquely determines the Laplace transform of their possible limit points, see Theorem 6.6.5 of [1]. Letting K vary, the claim (2.5) follows. □

Remark 2.2. The approximation scheme introduced in this section can also be used to approximate the random interlacement at level u, as we now explain. We let $\mathcal{I}^n_u$ stand for the trace left on U_n by the walk on E_n up to time τ^n_u:

$$\mathcal{I}^n_u = \{x \in U_n;\ \ell^{n,x}_{\tau^n_u} > 0\}. \tag{2.27}$$

By (2.12), (2.14), it follows that for any finite subset K of E and $u \ge 0$,

$$P^n_{x_*}[\mathcal{I}^n_u \cap K = \phi] = P^n_{x_*}[\mu^n_{K,u} = 0] = e^{-u\,\mathrm{cap}_{U_n}(K)} \underset{n}{\overset{(1.4),(1.5)}{\longrightarrow}} e^{-u\,\mathrm{cap}(K)} = \mathbb{P}[\mathcal{I}^u \cap K = \phi], \tag{2.28}$$

where $\mathcal{I}^u$ stands for the random interlacement at level u, that is, the trace on E of doubly infinite trajectories modulo time-shift in the Poisson cloud ω with label at most u. By an inclusion-exclusion argument, see for instance Remark 4.15 of [7] or Remark 2.2 of [5], it follows that, as $n \to \infty$,

$$\mathcal{I}^n_u \text{ under } P^n_{x_*}, \text{ converges in distribution to } \mathcal{I}^u \text{ under } \mathbb{P}, \text{ for any } u \ge 0, \tag{2.29}$$

where the above distributions are viewed as laws on $\{0,1\}^E$ endowed with the product topology. □

3 Proof of the isomorphism theorem

In this short section we combine Theorem 2.1 and the generalized second Ray-Knight theorem of [2] to prove Theorem 0.1. We also state a variation of (0.4) in Remark 3.1.

Proof of Theorem 0.1: For $U \subseteq G$ we denote by $P^{G,U}$ the law on $\mathbb{R}^E$ of the centered Gaussian field with covariance $E^{G,U}[\varphi_x \varphi_y] = g_U(x,y)$, $x, y \in E$ (in particular $\varphi_x = 0$, $P^{G,U}$-a.s., when $x \in E \backslash U$). It follows from the generalized second Ray-Knight theorem, see Theorem 8.2.2 of [4], or Theorem 2.17 of [7], that for $n \geq 1$, $u \geq 0$, in the notation of Section 2,

$$\text{(3.1)} \qquad \begin{array}{l} \big(\ell^{n,x}_{\tau^n_u} + \frac{1}{2}\,\varphi^2_x\big)_{x \in U_n} \text{ under } P^n_{x_*} \otimes P^{G,U_n}, \text{ has the same law as} \\ \big(\frac{1}{2}\,(\varphi_x + \sqrt{2u})^2\big)_{x \in U_n} \text{ under } P^{G,U_n}. \end{array}$$

Since $g_{U_n}(\cdot,\cdot) \uparrow g(\cdot,\cdot)$, we see that P^{G,U_n} converges weakly to P^G (looking for instance at characteristic functions of finite dimensional marginals). Taking Theorem 2.1 into account we thus see letting n tend to infinity that

$$\text{(3.2)} \qquad \begin{array}{l} \big(L_{x,u} + \frac{1}{2}\,\varphi^2_x\big)_{x \in E} \text{ under } \mathbb{P} \otimes P^G, \text{ has the same law as} \\ \big(\frac{1}{2}\,(\varphi_x + \sqrt{2u})^2\big)_{x \in E} \text{ under } P^G, \end{array}$$

and Theorem 0.1 is proved. □

Remark 3.1. Let us mention a variation on (0.4) of Theorem 0.1. By Theorem 1.1 of [2], one knows that for $u \geq 0$, $a \in \mathbb{R}$, $n \geq 1$,

$$\text{(3.3)} \qquad \begin{array}{l} \big(\ell^{n,x}_{\tau^n_u} + \frac{1}{2}\,(\varphi_x + a)^2\big)_{x \in U_n} \text{ under } P^n_{x_*} \otimes P^{G,U_n}, \text{ has the same law as} \\ \big(\frac{1}{2}\,\big(\varphi_x + \sqrt{2u + a^2}\big)^2\big)_{x \in U_n} \text{ under } P^{G,U_n}. \end{array}$$

Letting n tend to infinity, the same argument as above shows that for $u \geq 0$, and $a \in \mathbb{R}$,

$$\text{(3.4)} \qquad \begin{array}{l} \big(L_{x,u} + \frac{1}{2}\,(\varphi_x + a)^2\big)_{x \in E} \text{ under } \mathbb{P} \otimes P^G, \text{ has the same law as} \\ \big(\frac{1}{2}\,\big(\varphi_x + \sqrt{2u + a^2}\big)^2\big)_{x \in E} \text{ under } P^G. \end{array}$$

□

4 An application

We illustrate the use of Theorem 0.1 and show how one can study the large u asymptotics of $(L_{x,u})_{x \in E}$ and in particular recover Theorem 5.1 of [6], see also Remark 5.2 of [6]. We denote by x_0 some fixed point of E.

Theorem 4.1. *As* $u \to \infty$,

(4.1) $\Big(\frac{1}{u}\, L_{x,u}\Big)_{x \in E}$ *converges in distribution to the constant field equal to* 1,

(4.2) $\Big(\frac{L_{x,u}-u}{\sqrt{2u}}\Big)_{x \in E}$ *converges in distribution to* $(\varphi_x)_{x \in E}$ *under* P^G.

In particular, as $u \to \infty$,

(4.3) $\Big(\frac{L_{x,u}-L_{x_0,u}}{\sqrt{2u}}\Big)_{x \in E}$ *converges in distribution to* $(\varphi_x - \varphi_{x_0})_{x \in E}$ *under* P^G.

Proof. We first prove (4.1). To this end we note that P^G-a.s., for $x \in E$,

$$\frac{1}{2u}\, \varphi_x^2 \to 0 \text{ and } \frac{1}{2u}\, (\varphi_x + \sqrt{2u})^2 \to 1, \text{ as } u \to \infty. \tag{4.4}$$

Thus Theorem 0.1 implies that $\frac{1}{u}\, L_{x,u}$ converges in distribution to the constant 1 as u tends to infinity, and (4.1) follows.

We then observe that (4.3) is a direct consequence of (4.2), and turn to the proof of (4.3). Note that by Theorem 0.1

$$\begin{aligned}&\Big(\frac{L_{x,u}-u}{\sqrt{2u}} + \frac{1}{2\sqrt{2u}}\, \varphi_x^2\Big)_{x \in E} \text{ under } \mathbb{P} \otimes P^G, \text{ has the same law as} \\ &\Big(\frac{1}{2\sqrt{2u}}\, [(\varphi_x + \sqrt{2u})^2 - 2u]\Big)_{x \in E}.\end{aligned} \tag{4.5}$$

Note also that for each $x \in E$, P^G-a.s., as $u \to \infty$,

$$\frac{1}{2\sqrt{2u}}\, \varphi_x^2 \to 0, \text{ and} \tag{4.6}$$

$$\frac{1}{2\sqrt{2u}}\, [(\varphi_x + \sqrt{2u})^2 - 2u] = \frac{1}{2\sqrt{2u}}\, \varphi_x^2 + \varphi_x \to \varphi_x. \tag{4.7}$$

Looking at the characteristic function of finite dimensional marginals of the fields in the first and second line of (4.5), we readily obtain (4.3). □

Remark 4.2. In view of the above illustration of the use of Theorem 0.1, one can naturally wonder about the nature of its scope as a transfer mechanism between random interlacements and the Gaussian free field. □

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