PHASE TRANSITION AND LEVEL-SET PERCOLATION FOR THE GAUSSIAN FREE FIELD

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Abstract

We consider level-set percolation for the Gaussian free field on $\mathbb{Z}^d$, $d \geq 3$, and prove that, as h varies, there is a non-trivial percolation phase transition of the excursion set above level h for all dimensions $d \geq 3$. So far, it was known that the corresponding critical level $h_*(d)$ satisfies $h_*(d) \geq 0$ for all $d \geq 3$ and that $h_*(3)$ is finite, see [2]. We prove here that $h_*(d)$ is finite for all $d \geq 3$. In fact, we introduce a second critical parameter $h_{**} \geq h_*$, show that $h_{**}(d)$ is finite for all $d \geq 3$, and that the connectivity function of the excursion set above level h has stretched exponential decay for all $h > h_{**}$. Finally, we prove that h_* is strictly positive in high dimension. It remains open whether h_* and h_{**} actually coincide and whether $h_* > 0$ for all $d \geq 3$.

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0 Introduction

In the present work, we investigate level-set percolation for the Gaussian free field on $\mathbb{Z}^d$, $d \geq 3$. This problem has already received much attention in the past, see for instance [12], [2], and more recently [5], [15]. The long-range dependence of the model makes this problem particularly interesting, but also harder to analyze. Here, we prove the existence of a non-trivial critical level for all $d \geq 3$, and the positivity of this critical level when d is large enough. Some of our methods are inspired by the recent progress in the study of the percolative properties of random interlacements, where a similar long-range dependence occurs, see for instance [19], [21], [24], [27].

We now describe our results and refer to Section 1 for details. We consider the lattice $\mathbb{Z}^d$, $d \geq 3$, endowed with the usual nearest-neighbor graph structure. Our main object of study is the Gaussian free field on $\mathbb{Z}^d$, with canonical law $\mathbb{P}$ on $\mathbb{R}^{\mathbb{Z}^d}$ such that,

$$\text{under } \mathbb{P}, \text{ the canonical field } \varphi = (\varphi_x)_{x \in \mathbb{Z}^d} \text{ is a centered Gaussian field with covariance } \mathbb{E}[\varphi_x \varphi_y] = g(x,y), \text{ for all } x, y \in \mathbb{Z}^d, \tag{0.1}$$

where $g(\cdot,\cdot)$ denotes the Green function of simple random walk on $\mathbb{Z}^d$, see (1.1). Note in particular the presence of strong correlations, see (1.9). For any *level* $h \in \mathbb{R}$, we introduce the (random) subset of $\mathbb{Z}^d$

$$E_\varphi^{\geq h} = \{x \in \mathbb{Z}^d \; ; \; \varphi_x \geq h\}, \tag{0.2}$$

sometimes called *excursion set* (above level h). We are interested in the event that the origin lies in an infinite cluster of $E_\varphi^{\geq h}$, which we denote by $\{0 \overset{\geq h}{\longleftrightarrow} \infty\}$, and ask for which values of h this event occurs with positive probability. Since

$$\eta(h) \overset{\text{def.}}{=} \mathbb{P}[0 \overset{\geq h}{\longleftrightarrow} \infty] \tag{0.3}$$

is decreasing in h, it is sensible to define the critical point for level-set percolation as

$$h_*(d) = \inf\{h \in \mathbb{R} \; ; \; \eta(h) = 0\} \; \in [-\infty, \infty] \tag{0.4}$$

(with the convention $\inf \emptyset = \infty$). A non-trivial phase transition is then said to occur if h_* is finite. It is known that $h_*(d) \geq 0$ for all $d \geq 3$ and that $h_*(3) < \infty$ (see [2], Corollary 2 and Theorem 3, respectively; see also the concluding Remark 5.1 in [2] to understand why the proof does not easily generalize to all $d \geq 3$). It is also known that when $d \geq 4$, for large h, there is no directed percolation inside $E_\varphi^{\geq h}$, see [5], p. 281 (note that this reference studies the percolative properties of the excursion sets of $|\varphi|$ in place of φ).

It is not intuitively obvious why h_* should be finite, for it seems a priori conceivable that infinite clusters of $E_\varphi^{\geq h}$ could exist for *all* $h > 0$ due to the strong nature of the correlations. We show in Corollary 2.7 that this does not occur and that

$$h_*(d) < \infty, \qquad \text{for all } d \geq 3. \tag{0.5}$$

In fact, we prove a stronger result in Theorem 2.6. We define a second critical parameter

$$h_{**}(d) = \inf\big\{h \in \mathbb{R} \; ; \; \text{for some } \alpha > 0, \; \lim_{L \to \infty} L^\alpha \, \mathbb{P}\big[B(0,L) \overset{\geq h}{\longleftrightarrow} S(0,2L)\big] = 0\big\}, \tag{0.6}$$

where the event $\big\{B(0,L) \overset{\geq h}{\longleftrightarrow} S(0,2L)\big\}$ refers to the existence of a (nearest-neighbor) path in $E_\varphi^{\geq h}$ connecting $B(0,L)$, the ball of radius L around 0 in the ℓ^∞-norm, to $S(0,2L)$, the ℓ^∞-sphere of radius $2L$ around 0. It is an easy matter (see Corollary 2.7 below) to show that

$$h_* \leq h_{**}. \tag{0.7}$$

Now, we prove in Theorem 2.6 the stronger statement

$$h_{**} < \infty, \qquad \text{for all } d \geq 3, \tag{0.8}$$

and then obtain as a by-product that

$$\text{the connectivity function of } E_\varphi^{\geq h} \text{ has stretched exponential decay for all } h > h_{**} \tag{0.9}$$

(see Theorem 2.6 below for a precise statement). This immediately leads to the important question of whether h_* and h_{**} actually coincide. In case they differ, our results imply a marked transition in the decay of the connectivity function of $E_\varphi^{\geq h}$ at $h = h_{**}$, see Remark 2.8 below.

Our second result concerns the critical level h_* in high dimension. We are able to show in Theorem 3.3 that

$$h_* \text{ is strictly positive when } d \text{ is sufficiently large.} \tag{0.10}$$

This is in accordance with recent numerical evidence, see [15], Chapter 4. We actually prove a stronger result than (0.10). Namely, we show that one can find a positive level h_0, such that for large d, the restriction of $E_\varphi^{\geq h_0}$ to a thick two-dimensional slab percolates, see above (0.12). Let us however point out that, by a result of [7] (see p. 1151 therein), the restriction of $E_\varphi^{\geq 0}$ to $\mathbb{Z}^2$ (viewed as a subset of $\mathbb{Z}^d$), and, a fortiori, the restriction of $E_\varphi^{\geq h}$ to $\mathbb{Z}^2$, when h is positive, do only contain finite connected components: excursion sets above any non-negative level do not percolate in planes. We refer to Remark 3.6 1) for more on this.

We now comment on the proofs. We begin with (0.8). The key ingredient is a certain (static) renormalization scheme very similar to the one developed in Section 2 of [21] for the problem of percolation of the vacant set left by random interlacements (for a precise definition of this model, see [23], Section 1; we merely note that the two "corresponding" quantities are $E_\varphi^{\geq h}$ and $\mathcal{V}^u$, the vacant set at level $u \geq 0$). We will be interested in the probability of certain crossing events viewed as functions of $h \in \mathbb{R}$,

$$f_n(h) \text{ “=” } \mathbb{P}\big[\, E_\varphi^{\geq h} \text{ contains a path from a given block of side length } L_n \text{ to the complement of its } L_n\text{-neighborhood}\big]$$

(see (2.9) for the precise definition), where $(L_n)_{n\geq 0}$ is a geometrically increasing sequence of length scales, see (2.1). Note that by (0.2), f_n is decreasing in h. We then explicitly construct an increasing but bounded sequence $(h_n)_{n\geq 0}$, with (finite) limit h_∞, such that

$$\lim_{n\to\infty} f_n(h_n) = 0. \tag{0.11}$$

This readily implies (0.5), since $\eta(h_\infty) \leq f_n(h_\infty) \leq f_n(h_n)$ for all $n \geq 0$, hence $\eta(h_\infty)$ vanishes. By separating combinatorial complexity estimates from probabilistic bounds in $f_n(\cdot)$, see (2.8) and Lemma 2.1, we are led to investigate the quantity

$$p_n(h) \text{ “=” } \mathbb{P}\big[\, E_\varphi^{\geq h} \text{ contains paths connecting each of } 2^n \text{ “well-separated” boxes of side length } L_0 \text{ (within a given box of side length} \sim L_n) \text{ to the complement of their respective } L_0\text{-neighborhoods}\,\big]$$

(see (2.8) for the precise definition), where the 2^n boxes are indexed by the "leaves" of a dyadic tree of depth n. The key to proving (0.11) is to provide a suitable induction step relating $p_{n+1}(h_{n+1})$ to $p_n(h_n)$, for all $n \geq 0$, where the increase in parameter $h_n \to h_{n+1}$ allows to dominate the interactions ("sprinkling"). This appears in Proposition 2.2. One then makes sure that $p_0(h_0)$ is chosen small enough by picking h_0 large, see Theorem 2.6. The resulting estimates are fine enough to imply not only that $h_\infty \geq h_*$, but even the stretched exponential decay of the connectivity function of $E_\varphi^{\geq h_\infty}$, thus yielding (0.8). The proof of (0.9) then only requires a refinement of this argument. Note that the strategy we have just described is precisely the one used in [21] for the proof of a similar theorem in the context of random interlacements. We actually also provide a generalization of Proposition 2.2, which is of independent interest, but goes beyond what is directly needed here, see Proposition 2.2'. It has a similar spirit to the main renormalization step leading to the decoupling inequalities for random interlacements in [24], see Remark 2.3.

We now comment on the proof of (0.10), which has two main ingredients. The first ingredient is a suitable decomposition of the field φ restricted to the subspace $\mathbb{Z}^3$ into the sum of two independent Gaussian fields. The first field has independent components and the second field only acts as a "perturbation" when d becomes large, see Lemmas 3.1 and 3.2 below. The second ingredient combines the fact that the critical value of Bernoulli site percolation on $\mathbb{Z}^3$ is smaller than $\frac{1}{2}$ (see [4]), with static renormalization (see Chapter 7 of [8], [9], [18]), and a Peierls-type argument to control the perturbation created by the second field. This actually enables us to deduce a stronger result than (0.10). Namely, we show in Theorem 3.3 that one can find a level $h_0 > 0$ and a positive integer L_0, such that for large d and all $h \leq h_0$, the excursion set $E_\varphi^{\geq h}$ already percolates in the two-dimensional slab

$$\mathbb{Z}^2 \times [0, 2L_0) \times \{0\}^{d-3} \subset \mathbb{Z}^d. \tag{0.12}$$

As already pointed out, some of our proofs employ strategies similar to those developed in the study of the percolative properties of the vacant set left by random interlacements, see [21], [24]. This is not a mere coincidence, as we now explain. Continuous-time random interlacements on $\mathbb{Z}^d$, $d \geq 3$, correspond to a certain Poisson point process of doubly infinite trajectories modulo time-shift, governed by a probability P, with a non-negative parameter u playing the role of a multiplicative factor of the intensity measure pertaining to this Poisson point process (the bigger u, the more trajectories "fall" on $\mathbb{Z}^d$), see [25], [23]. This Poisson gas of doubly infinite trajectories (modulo time-shift) induces a random field of occupation times $(L_{x,u})_{x\in\mathbb{Z}^d}$ (so that the interlacement at level u coincides with $\{x \in \mathbb{Z}^d; L_{x,u} > 0\}$, whereas the vacant set at level u equals $\{x \in \mathbb{Z}^d; L_{x,u} = 0\}$). This field is closely linked to the Gaussian free field, as the following isomorphism theorem from [25] shows:

$$\big(L_{x,u} + \frac{1}{2}\varphi_x^2\big)_{x\in\mathbb{Z}^d}, \text{ under } P \otimes \mathbb{P}, \text{ has the same law as } \big(\frac{1}{2}(\varphi_x + \sqrt{2u})^2\big)_{x\in\mathbb{Z}^d}, \text{ under } \mathbb{P}. \tag{0.13}$$

It is tempting to use this identity as a transfer mechanism, and we hope to return to this point elsewhere.

We conclude this introduction by describing the organization of this article. In Section 1, we introduce some notation and review some known results concerning simple random walk on $\mathbb{Z}^d$ and the Gaussian free field. Section 2 is devoted to proving that excursion sets at a high level do not percolate. The main results are Theorem 2.6 and Corollary 2.7. The positivity of the critical level in high dimension and the percolation of excursion sets at low positive level in large enough two-dimensional slabs is established in Theorem 3.3 of Section 3.

One final remark concerning our convention regarding constants: we denote by $c, c', \dots$ positive constants with values changing from place to place. Numbered constants $c_0, c_1, \dots$ are defined at the place they first occur within the text and remain fixed from then on until the end of the article. In Sections 1 and 2, constants will implicitly depend on the dimension d. In Section 3 however, constants will be purely numerical (and *independent* of d). Throughout the entire article, dependence of constants on additional parameters will appear in the notation.

1 Notation and some useful facts

In this section, we introduce some notation to be used in the sequel, and review some known results concerning both simple random walk and the Gaussian free field.

We denote by $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of natural numbers, and by $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ the set of integers. We write $\mathbb{R}$ for the set of real numbers, abbreviate $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$ for any two numbers $x, y \in \mathbb{R}$, and denote by $[x]$ the integer part of x, for any $x \geq 0$. We consider the lattice $\mathbb{Z}^d$, and tacitly assume throughout that $d \geq 3$. On $\mathbb{Z}^d$, we respectively denote by $|\cdot|$ and $|\cdot|_\infty$ the Euclidean and ℓ^∞-norms. Moreover, for any $x \in \mathbb{Z}^d$ and $r \geq 0$, we let $B(x, r) = \{y \in \mathbb{Z}^d;\ |y - x|_\infty \leq r\}$ and $S(x, r) = \{y \in \mathbb{Z}^d;\ |y - x|_\infty = r\}$ stand for the the ℓ^∞-ball and ℓ^∞-sphere of radius r centered at x. Given K and U subsets of $\mathbb{Z}^d$, $K^c = \mathbb{Z}^d \setminus K$ stands for the complement of K in $\mathbb{Z}^d$, $|K|$ for the cardinality of K, $K \subset\subset \mathbb{Z}^d$ means that $|K| < \infty$, and $d(K, U) = \inf\{|x - y|_\infty\ ;\ x \in K, y \in U\}$ denotes the ℓ^∞-distance between K and U. If $K = \{x\}$, we simply write $d(x, U)$. Finally, we define the inner boundary of K to be the set $\partial^i K = \{x \in K;\ \exists y \in K^c, |y - x| = 1\}$, and the outer boundary of K as $\partial K = \partial^i(K^c)$. We also introduce the diameter of any subset $K \subset \mathbb{Z}^d$, $\mathrm{diam}(K)$, as its ℓ^∞-diameter, i.e. $\mathrm{diam}(K) = \sup\{|x - y|_\infty\ ;\ x, y \in K\}$.

We endow $\mathbb{Z}^d$ with the nearest-neighbor graph structure, the edge-set consisting of all pairs of sites $\{x, y\}$, $x, y \in \mathbb{Z}^d$, such that $|x - y| = 1$. A (nearest-neighbor) path is any sequence of vertices $\gamma = (x_i)_{0 \leq i \leq n}$, where $n \geq 0$ and $x_i \in \mathbb{Z}^d$ for all $0 \leq i \leq n$, satisfying $|x_i - x_{i-1}| = 1$ for all $1 \leq i \leq n$. Moreover, two lattice sites x, y will be called $*$-nearest neighbors if $|x - y|_\infty = 1$. A $*$-path is defined accordingly. Thus, any site $x \in \mathbb{Z}^d$ has $2d$ nearest neighbors and $3^d - 1$ $*$-nearest neighbors.

We now introduce the (discrete-time) simple random walk on $\mathbb{Z}^d$. To this end, we let W be the space of nearest-neighbor $\mathbb{Z}^d$-valued trajectories defined for non-negative times, and let $\mathcal{W}$, $(X_n)_{n \geq 0}$, stand for the canonical σ-algebra and canonical process on W, respectively. Since $d \geq 3$, the random walk is transient. Furthermore, we write P_x for the canonical law of the walk starting at $x \in \mathbb{Z}^d$ and E_x for the corresponding expectation. We denote by $g(\cdot, \cdot)$ the Green function of the walk, i.e.

$$g(x, y) = \sum_{n \geq 0} P_x[X_n = y], \qquad \text{for } x, y \in \mathbb{Z}^d, \tag{1.1}$$

which is finite (since $d \geq 3$) and symmetric. Moreover, $g(x, y) = g(x - y, 0) \stackrel{\text{def.}}{=} g(x - y)$ due to translation invariance. Given $U \subset \mathbb{Z}^d$, we further denote the entrance time in U by $H_U = \inf\{n \geq 0; X_n \in U\}$, the hitting time of U by $\widetilde{H}_U = \inf\{n \geq 1; X_n \in U\}$, and the exit time from U by $T_U = \inf\{n \geq 0; X_n \notin U\} = H_{U^c}$. This allows us to define the Green function $g_U(\cdot, \cdot)$ killed outside U as

$$g_U(x, y) = \sum_{n \geq 0} P_x[X_n = y,\ n < T_U], \qquad \text{for } x, y \in \mathbb{Z}^d. \tag{1.2}$$

It vanishes if $x \notin U$ or $y \notin U$. The relation between g and g_U for any $U \subset \mathbb{Z}^d$ is the following (we let $K = U^c$):

$$g(x, y) = g_U(x, y) + E_x[H_K < \infty,\ g(X_{H_K}, y)], \qquad \text{for } x, y \in \mathbb{Z}^d. \tag{1.3}$$

The proof of (1.3) is a mere application of the strong Markov property (at time H_K).

We now turn to a few aspects of potential theory associated to simple random walk. For any $K \subset\subset \mathbb{Z}^d$, we write

$$e_K(x) = P_x[\widetilde{H}_K = \infty], \qquad x \in K, \tag{1.4}$$

for the equilibrium measure (or escape probability) of K, and

$$\text{cap}(K) = \sum_{x \in K} e_K(x) \tag{1.5}$$

for its capacity. It immediately follows from (1.4) and (1.5) that the capacity is subadditive, i.e.

$$\text{cap}(K \cup K') \leq \text{cap}(K) + \text{cap}(K'), \qquad \text{for all } K, K' \subset\subset \mathbb{Z}^d. \tag{1.6}$$

Moreover, the entrance probability in K may be expressed in terms of $e_K(\cdot)$ (see for example [22], Theorem 25.1, p. 300) as

$$P_x[H_K < \infty] = \sum_{y \in K} g(x, y) \cdot e_K(y), \tag{1.7}$$

from which, together with classical bounds on the Green function (c.f. (1.9) below), one easily obtains (see [21], Section 1 for a derivation) the following useful bound for the capacity of a box:

$$\text{cap}(B(0, L)) \leq cL^{d-2}, \qquad \text{for all } L \geq 1. \tag{1.8}$$

We next review some useful asymptotics of $g(\cdot)$. Given two functions $f_1, f_2 : \mathbb{Z}^d \longrightarrow \mathbb{R}$, we write $f_1(x) \sim f_2(x)$, as $|x| \to \infty$, if they are asymptotic, i.e. if $\lim_{|x|\to\infty} f_1(x)/f_2(x) = 1$.

Lemma 1.1. ($d \geq 3$)

$$g(x) \sim c|x|^{2-d}, \qquad \textit{as } |x| \to \infty. \tag{1.9}$$

$$g(0) = 1 + \frac{1}{2d} + o(d^{-1}), \qquad \textit{as } d \to \infty. \tag{1.10}$$

$$P_0[\widetilde{H}_{\mathbb{Z}^3} = \infty] = 1 - \frac{7}{2d} + o(d^{-1}), \qquad \textit{as } d \to \infty, \tag{1.11}$$

where $\mathbb{Z}^3$ *is viewed as* $\big(\mathbb{Z}^3 \times \{0\}^{d-3}\big) \subset \mathbb{Z}^d$ *in* (1.11).

Proof. For (1.9), see [11], Theorem 1.5.4, for (1.10), see [16], pp. 246-247. In order to prove (1.11), we assume that $d \geq 6$ and define $\pi : \mathbb{Z}^d \longrightarrow \mathbb{Z}^{d-3} : (x^1, \dots, x^d) \mapsto (x^4, \dots, x^d)$. Then, under P_0,

$$Y_n \overset{\text{def.}}{=} \pi \circ X_n, \qquad \text{for all } n \geq 0,$$

is a "lazy" walk on $\mathbb{Z}^{d-3}$ starting at the origin. Clearly, $\{\widetilde{H}_{\mathbb{Z}^3} = \infty\} = \{\widetilde{H}_0^{(Y)} = \infty\}$, where $\widetilde{H}_0^{(Y)}$ refers to the first return to 0 for the walk Y. Hence,

$$P_0[\widetilde{H}_{\mathbb{Z}^3} = \infty] = \big[g^{(Y)}(0)\big]^{-1} = \Big[\frac{d}{d-3} \cdot g^{(d-3)}(0)\Big]^{-1} \overset{(1.10)}{=} 1 - \frac{7}{2d} + o(d^{-1}),$$

as $d \to \infty$, where $g^{(Y)}(\cdot)$ denotes the Green function of Y and $g^{(d-3)}(\cdot)$ that of simple random walk on $\mathbb{Z}^{d-3}$. □

We now turn to the Gaussian free field on $\mathbb{Z}^d$, as defined in (0.1). Given any subset $K \subset \mathbb{Z}^d$, we frequently write φ_K to denote the family $(\varphi_x)_{x \in K}$. For arbitrary $a \in \mathbb{R}$ and $K \subset\subset \mathbb{Z}^d$, we also use the shorthand $\{\varphi_{|K} > a\}$ for the event $\{\min\{\varphi_x;\ x \in K\} > a\}$ and similarly $\{\varphi_{|K} < a\}$ instead of $\{\max\{\varphi_x;\ x \in K\} < a\}$. Next, we introduce certain crossing events for the Gaussian free field. To this end, we first consider the space $\Omega = \{0, 1\}^{\mathbb{Z}^d}$ endowed with its canonical σ-algebra and define, for arbitrary disjoint subsets $K, K' \subset \mathbb{Z}^d$, the event (subset of Ω)

$$\{K \longleftrightarrow K'\} = \{\text{there exists an open path (i.e. along which the configuration has value 1) connecting } K \text{ and } K'\}. \tag{1.12}$$

For any level $h \in \mathbb{R}$, we write Φ^h for the measurable map from $\mathbb{R}^{\mathbb{Z}^d}$ into Ω which sends $\varphi \in \mathbb{R}^{\mathbb{Z}^d}$ to $\big(1\{\varphi_x \geq h\}\big)_{x\in\mathbb{Z}^d} \in \Omega$, and define

$$\{K \overset{\geq h}{\longleftrightarrow} K'\} = (\Phi^h)^{-1}\big(\{K \longleftrightarrow K'\}\big) \tag{1.13}$$

(a measurable subset of $\mathbb{R}^{\mathbb{Z}^d}$ endowed with its canonical σ-algebra $\mathcal{F}$), which is the event that K and K' are connected by a (nearest-neighbor) path in $E_{\varphi}^{\geq h}$, c.f. (0.2). Denoting by Q^h the image of $\mathbb{P}$ under Φ^h, i.e. the law of $\big(1\{\varphi_x \geq h\}\big)_{x\in\mathbb{Z}^d}$ on Ω, we have that $\mathbb{P}[K \overset{\geq h}{\longleftrightarrow} K'] = Q^h[K \longleftrightarrow K']$. Note that $\{K \overset{\geq h}{\longleftrightarrow} K'\}$ is an increasing event upon introducing on $\mathbb{R}^{\mathbb{Z}^d}$ the natural partial order (i.e. $f \leq f'$ when $f_x \leq f'_x$ for all $x \in \mathbb{Z}^d$).

We proceed with a classical fact concerning conditional distributions for the Gaussian free field on $\mathbb{Z}^d$. We could not find a precise reference in the literature, and include a proof for the Reader's convenience. We first define, for $U \subset \mathbb{Z}^d$, the law $\mathbb{P}^U$ on $\mathbb{R}^{\mathbb{Z}^d}$ of the centered Gaussian field with covariance

$$\mathbb{E}^U[\varphi_x\varphi_y] = g_U(x,y), \qquad \text{for all } x,y \in \mathbb{Z}^d, \tag{1.14}$$

with $g_U(\cdot,\cdot)$ given by (1.2). In particular, $\varphi_x = 0$, $\mathbb{P}^U$-almost-surely, whenever $x \in K = U^c$. We then have

Lemma 1.2.

Let $\emptyset \neq K \subset\subset \mathbb{Z}^d$, $U = K^c$ and define $(\widetilde{\varphi}_x)_{x\in\mathbb{Z}^d}$ by

$$\varphi_x = \widetilde{\varphi}_x + \mu_x, \qquad \textit{for } x \in \mathbb{Z}^d, \tag{1.15}$$

where μ_x is the $\sigma(\varphi_x; x \in K)$-measurable map defined as

$$\mu_x = E_x[H_K < \infty, \varphi_{X_{H_K}}] = \sum_{y\in K} P_x[H_K < \infty, X_{H_K} = y]\cdot \varphi_y, \qquad \textit{for } x \in \mathbb{Z}^d. \tag{1.16}$$

Then, under $\mathbb{P}$,

$$(\widetilde{\varphi}_x)_{x\in\mathbb{Z}^d} \textit{ is independent from } \sigma(\varphi_x; x \in K)\textit{, and distributed as } (\varphi_x)_{x\in\mathbb{Z}^d} \textit{ under } \mathbb{P}^U. \tag{1.17}$$

Proof. Note that for all $x \in K$, $\widetilde{\varphi}_x = 0$ (since $\mu_x = \varphi_x$ for $x \in K$, by (1.16)) and that for all $x \in K$, $\varphi_x = 0$, $\mathbb{P}^U$-almost surely. Hence, it suffices to consider $(\widetilde{\varphi}_x)_{x\in U}$. We first show independence. By (1.16), $(\widetilde{\varphi}_x)_{x\in U}$, $(\varphi_y)_{y\in K}$, are centered and jointly Gaussian. Moreover, they are uncorrelated, since for $x \in U$, $y \in K$,

$$\mathbb{E}[\widetilde{\varphi}_x\varphi_y] = \mathbb{E}[\varphi_x\varphi_y] - \mathbb{E}[\mu_x\varphi_y] \overset{(0.1),(1.16)}{=} g(x,y) - \sum_{z\in K} P_x[H_K < \infty, X_{H_K} = z]g(z,y) \overset{(1.3)}{=} 0.$$

Thus, $(\widetilde{\varphi}_x)_{x\in U}$, $(\varphi_y)_{y\in K}$, are independent. To conclude the proof of Lemma 1.2, it suffices to show that

$$\mathbb{E}\left[1_A\big((\widetilde{\varphi}_x)_{x\in U}\big)\right] = \mathbb{E}^U\left[1_A\big((\varphi_x)_{x\in U}\big)\right], \qquad \text{for all } A \in \mathcal{F}_U, \tag{1.18}$$

where $\mathcal{F}_U$ stands for the canonical σ-algebra on $\mathbb{R}^U$. Furthermore, choosing some ordering $(x_i)_{i\geq 0}$ of U, by Dynkin's Lemma, it suffices to assume that A has the form

$$A = A_{x_0} \times \cdots \times A_{x_n} \times \mathbb{R}^{U\setminus\{x_0,\dots,x_n\}}, \quad \text{for some } n \geq 0 \text{ and } A_{x_i} \in \mathcal{B}(\mathbb{R}),\ i = 0,\dots,n. \tag{1.19}$$

We fix some A of the form (1.19), and consider a subset V such that $K \cup \{x_0,\dots,x_n\} \subseteq V \subset\subset \mathbb{Z}^d$ (we will soon let V increase to $\mathbb{Z}^d$). We let P_x^V, $x \in V$, denote the law of simple random walk on

V starting at x killed when exiting V (its Green function corresponds to $g_V(\cdot,\cdot)$), and define $\widetilde{\varphi}_x^V$ for $x \in V$ as in (1.15) but with P_x^V replacing P_x in the definition (1.16) of μ_x. It then follows from Proposition 2.3 in [26] (an analogue of the present lemma for *finite* graphs) that

$$\mathbb{E}^V\left[1_{A^V}\big((\widetilde{\varphi}_x^V)_{x\in V\setminus K}\big)\right] = \mathbb{E}^{V\setminus K}\left[1_{A^V}\big((\varphi_x)_{x\in V\setminus K}\big)\right], \tag{1.20}$$

with $\mathbb{P}^V$, $\mathbb{P}^{V\setminus K}$ as defined in (1.14) and $A^V = A_{x_0}\times\cdots\times A_{x_n}\times\mathbb{R}^{V\setminus(K\cup\{x_0,\dots,x_n\})}$. Letting $V \nearrow \mathbb{Z}^d$, it follows that $g_V(x,y) \nearrow g(x,y)$, $g_{V\setminus K}(x,y) \nearrow g_U(x,y)$, hence by dominated convergence that both sides of (1.20) converge towards the respective sides of (1.18), thus completing the proof. □

Remark 1.3.
Lemma 1.2 yields a choice of regular conditional distributions for $(\varphi_x)_{x\in\mathbb{Z}^d}$ conditioned on the variables $(\varphi_x)_{x\in K}$, which is tailored to our future purposes. Namely, $\mathbb{P}$-almost surely,

$$\mathbb{P}\big[(\varphi_x)_{x\in\mathbb{Z}^d}\in\cdot\ \big|(\varphi_x)_{x\in K}\big] = \widetilde{\mathbb{P}}\big[(\widetilde{\varphi}_x+\mu_x)_{x\in\mathbb{Z}^d}\in\cdot\ \big], \tag{1.21}$$

where μ_x, $x\in\mathbb{Z}^d$ is given by (1.16), $\widetilde{\mathbb{P}}$ does not act on $(\mu_x)_{x\in\mathbb{Z}^d}$, and $(\widetilde{\varphi}_x)_{x\in\mathbb{Z}^d}$ is a centered Gaussian field under $\widetilde{\mathbb{P}}$, with $\widetilde{\varphi}_x = 0$, $\widetilde{\mathbb{P}}$-almost surely for $x \in K$. Lemma 1.2 also provides the covariance structure of this field (namely $g_U(\cdot,\cdot)$, with $U = K^c$), but its precise form will be of no importance in what follows. Note that conditioning on $(\varphi_x)_{x\in K}$ produces the (random) *shift* μ_x, which is *linear* in the variables φ_y, $y\in K$. □

The explicit form of the conditional distributions in (1.21) readily yields the following result, which can be viewed as a consequence of the FKG-inequality for the free field (see for example [6], Chapter 4).

Lemma 1.4.
Let $\alpha\in\mathbb{R}$, $\emptyset\neq K\subset\subset\mathbb{Z}^d$, and assume $A\in\mathcal{F}$ (the canonical σ-algebra on $\mathbb{R}^{\mathbb{Z}^d}$) is an increasing event. Then

$$\mathbb{P}\big[A\big|\ \varphi_{|K}=\alpha\big]\ \le\ \mathbb{P}\big[A\big|\ \varphi_{|K}\ge\alpha\big], \tag{1.22}$$

where the left-hand side is defined by the version of the conditional expectation in (1.21).

Intuitively, augmenting the field can only favor the occurrence of A, an increasing event.

Proof. On the event $\big\{\varphi_{|K}\ge\alpha\big\}$, we have, for μ_x, $x\in\mathbb{Z}^d$, as defined in (1.16),

$$\mu_x = \sum_{y\in K}\varphi_y P_x[H_K<\infty, X_{H_K}=y] \ge \alpha P_x[H_K<\infty] \overset{\text{def.}}{=} m_x(\alpha), \qquad \text{for all } x\in\mathbb{Z}^d, \tag{1.23}$$

with equality instead on the event $\big\{\varphi_{|K}=\alpha\big\}$. Since A is increasing, this yields, with a slight abuse of notation,

$$\begin{aligned}\mathbb{P}[A|\varphi_{|K}=\alpha]\cdot 1_{\{\varphi_{|K}\ge\alpha\}} &\overset{(1.21)}{=} \widetilde{\mathbb{P}}\big[A\big((\widetilde{\varphi}_x+m_x(\alpha))_{x\in\mathbb{Z}^d}\big)\ \big]\cdot 1_{\{\varphi_{|K}\ge\alpha\}}\\ &\le \widetilde{\mathbb{P}}\big[A\big((\widetilde{\varphi}_x+\mu_x)_{x\in\mathbb{Z}^d}\big)\ \big]\cdot 1_{\{\varphi_{|K}\ge\alpha\}}\\ &\overset{(1.21)}{=} \mathbb{P}\big[A\big|\varphi_{|K}\big]\cdot 1_{\{\varphi_{|K}\ge\alpha\}}.\end{aligned}$$

Integrating both sides with respect to the probability measure $\nu(\cdot)\overset{\text{def.}}{=}\mathbb{P}[\ \cdot\ |\varphi_{|K}\ge\alpha]$, we obtain

$$\mathbb{P}[A|\varphi_{|K}=\alpha]\le E_\nu\big[\mathbb{P}\big[A|\varphi_{|K}\big]\cdot 1_{\{\varphi_{|K}\ge\alpha\}}\big] = \mathbb{P}\big[A\big|\ \varphi_{|K}\ge\alpha\big].$$

This completes the proof of Lemma 1.4. □

We now introduce the canonical shift τ_z on $\mathbb{R}^{\mathbb{Z}^d}$, such that $\tau_z(f)(\cdot) = f(\cdot + z)$, for arbitrary $f \in \mathbb{R}^{\mathbb{Z}^d}$ and $z \in \mathbb{Z}^d$. The measure $\mathbb{P}$ is invariant under τ_z, i.e. $\mathbb{P}[\tau_z^{-1}(A)] = \mathbb{P}[A]$, for all $A \in \mathcal{F}$ (the canonical σ-algebra on $\mathbb{R}^{\mathbb{Z}^d}$), by translation invariance of $g(\cdot,\cdot)$ (see below (1.1)), and has the following mixing property:

$$\lim_{z\to\infty} \mathbb{P}[A \cap \tau_z^{-1}(B)] = \mathbb{P}[A]\,\mathbb{P}[B], \qquad \text{for all } A, B \in \mathcal{F} \tag{1.24}$$

(one first verifies (1.24) for A, B depending on finitely many coordinates with the help of (1.9) and the general case follows by approximation, see [3], pp.157-158). The following lemma gives a 0-1 law for the probability of existence of an infinite cluster in $E_\varphi^{\geq h}$, the excursion set above level $h \in \mathbb{R}$, c.f. (0.2).

Lemma 1.5.
Let $\Psi(h) = \mathbb{P}[E_\varphi^{\geq h}$ contains an infinite cluster $]$, for arbitrary $h \in \mathbb{R}$. One then has the following dichotomy:

$$\Psi(h) = \begin{cases} 0, & \text{if } \eta(h) = 0, \\ 1, & \text{if } \eta(h) > 0, \end{cases} \tag{1.25}$$

where $\eta(h) = \mathbb{P}[0 \xleftrightarrow{\geq h} \infty]$. In particular, recalling the definition (0.4) of h_*, (1.25) implies that $\Psi(h) = 1$ for all $h < h_*$, and $\Psi(h) = 0$ for all $h > h_*$.

Proof. This follows by ergodicity, which is itself a consequence of the mixing property (1.24). □

Remark 1.6.
When $\Psi(h) = 1$, in particular in the supercritical regime $h < h_*$, the infinite cluster in $E_\varphi^{\geq h}$ is $\mathbb{P}$-almost surely unique. This follows by Theorem 12.2 in [10] (Burton-Keane theorem), because the field $\big(1\{\varphi_x \geq h\}\big)_{x\in\mathbb{Z}^d}$ is translation invariant and has the finite energy property (see [10], Definition 12.1). □

2 Non-trivial phase transition

The main goal of this section is the proof of Theorem 2.6 below, which roughly states that $h_{**}(d)$ (and hence $h_*(d)$, c.f. Corollary 2.7) is finite for all $d \geq 3$, and that the connectivity function of $E_\varphi^{\geq h}$, c.f. (0.2), has stretched exponential decay for arbitrary $h > h_{**}$. The proof involves a certain renormalization scheme akin to the one developed in [21] and [24] in the context of random interlacements. This scheme will be used to derive recursive estimates for the probabilities of certain crossing events, c.f. Proposition 2.2, which can subsequently be propagated inductively, c.f. Proposition 2.4. The proper initialization of this induction requires a careful choice of the parameters occurring in the renormalization scheme. The resulting bounds constitute the main tool for the proof of the central Theorem 2.6. In addition, an extension of Proposition 2.2 can be found in Remark 2.3 2).

We begin by defining on the lattice $\mathbb{Z}^d$ a sequence of length scales

$$L_n = l_0^n L_0, \quad \text{for } n \geq 0, \tag{2.1}$$

where $L_0 \geq 1$ and $l_0 \geq 100$ are both assumed to be integers and will be specified below. Hence, L_0 represents the finest scale and $L_1 < L_2 < \dots$ correspond to increasingly coarse scales. We further introduce renormalized lattices

$$\mathbb{L}_n = L_n \mathbb{Z}^d \subset \mathbb{Z}^d, \qquad n \geq 0, \tag{2.2}$$

and note that $\mathbb{L}_k \supseteq \mathbb{L}_n$ for all $0 \le k \le n$. To each $x \in \mathbb{L}_n$, we attach the boxes

$$B_{n,x} \stackrel{\text{def.}}{=} B_x(L_n), \qquad \text{for } n \ge 0,\ x \in \mathbb{L}_n, \tag{2.3}$$

where we define $B_x(L) = x + \big([0,L) \cap \mathbb{Z}\big)^d$, the box of side length L *attached to* x, for any $x \in \mathbb{Z}^d$ and $L \ge 1$ (not to be confused with $B(x,L)$). Moreover, we let

$$\widetilde{B}_{n,x} = \bigcup_{y \in \mathbb{L}_n:\ d(B_{n,y}, B_{n,x}) \le 1} B_{n,y}, \qquad n \ge 0,\ x \in \mathbb{L}_n, \tag{2.4}$$

so that $\{B_{n,x};\ x \in \mathbb{L}_n\}$ defines a partition of $\mathbb{Z}^d$ into boxes of side length L_n for all $n \ge 0$, and $\widetilde{B}_{n,x}$, $x \in \mathbb{L}_n$, is simply the union of $B_{n,x}$ and its $*$-neighboring boxes at level n. Moreover, for $n \ge 1$ and $x \in \mathbb{L}_n$, $B_{n,x}$ is the disjoint union of the l_0^d boxes $\{B_{n-1,y};\ y \in B_{n,x} \cap\ \mathbb{L}_{n-1}\}$ at level $n-1$ it contains. We also introduce the indexing sets

$$\mathcal{I}_n = \{n\} \times \mathbb{L}_n, \qquad n \ge 0, \tag{2.5}$$

and given $(n,x) \in \mathcal{I}_n$, $n \ge 1$, we consider the sets of labels

$$\begin{aligned} \mathcal{H}_1(n,x) &= \big\{(n-1,y) \in \mathcal{I}_{n-1};\ B_{n-1,y} \subset B_{n,x} \text{ and } B_{n-1,y} \cap\ \partial^i B_{n,x} \neq \emptyset\big\}, \\ \mathcal{H}_2(n,x) &= \big\{(n-1,y) \in \mathcal{I}_{n-1};\ B_{n-1,y} \cap \big\{z \in \mathbb{Z}^d;\ d(z, B_{n,x}) = [L_n/2]\big\} \neq \emptyset\big\}. \end{aligned} \tag{2.6}$$

Note that for any two indices $(n-1, y_i) \in \mathcal{H}_i(n,x)$, $i = 1,2$, we have $\widetilde{B}_{n-1,y_1} \cap \widetilde{B}_{n-1,y_2} = \emptyset$ and $\widetilde{B}_{n-1,y_1} \cup \widetilde{B}_{n-1,y_2} \subset \widetilde{B}_{n,x}$. Finally, given $x \in \mathbb{L}_n$, $n \ge 0$, we introduce $\Lambda_{n,x}$, a family of subsets $\mathcal{T}$ of $\bigcup_{0 \le k \le n} \mathcal{I}_k$ (soon to be thought of as binary trees) defined as

$$\begin{aligned} \Lambda_{n,x} = \Big\{ \mathcal{T} \subset \bigcup_{k=0}^{n} \mathcal{I}_k\ ;\ & \mathcal{T} \cap \mathcal{I}_n = (n,x) \text{ and every } (k,y) \in \mathcal{T} \cap \mathcal{I}_k,\ 0 < k \le n, \text{ has} \\ & \text{two `descendants' } (k-1, y_i(k,y)) \in \mathcal{H}_i(k,y),\ i = 1,2, \text{ such} \\ & \text{that } \mathcal{T} \cap \mathcal{I}_{k-1} = \bigcup_{(k,y) \in \mathcal{T} \cap \mathcal{I}_k} \{(k-1, y_1(k,y)),\ (k-1, y_2(k,y))\} \Big\}. \end{aligned} \tag{2.7}$$

Hence, any $\mathcal{T} \in \Lambda_{n,x}$ can naturally be identified as a binary tree having root $(n,x) \in \mathcal{I}_n$ and depth n. Moreover, the following bound on the cardinality of $\Lambda_{n,x}$ is easily obtained,

$$|\Lambda_{n,x}| \le (c l_0^{d-1})^2 \cdot (c l_0^{d-1})^{2^2} \cdots (c l_0^{d-1})^{2^n} = (c l_0^{d-1})^{2(2^n - 1)} \le (c_0 l_0^{2(d-1)})^{2^n}, \tag{2.8}$$

where $c_0 \ge 1$ is a suitable constant.

We now consider the Gaussian free field $\varphi = (\varphi_x)_{x \in \mathbb{Z}^d}$ on $\mathbb{Z}^d$ defined in (0.1) and introduce the crossing events (c.f. (1.13) for the notation)

$$A^h_{n,x} = \{B_{n,x} \stackrel{\ge h}{\longleftrightarrow} \partial^i \widetilde{B}_{n,x}\}, \qquad \text{for } h \in \mathbb{R},\ n \ge 0, \text{ and } x \in \mathbb{L}_n\,. \tag{2.9}$$

Three properties of the events $A^h_{n,x}$ will play a crucial role in what follows. Denoting by $\sigma\big(\varphi_y\ ;\ y \in \widetilde{B}_{n,x}\big)$ the σ-algebra on $\mathbb{R}^{\mathbb{Z}^d}$ generated by the variables φ_y, $y \in \widetilde{B}_{n,x}$, we have

$$A^h_{n,x} \in \sigma\big(\varphi_y\ ;\ y \in \widetilde{B}_{n,x}\big), \tag{2.10}$$

$$A^h_{n,x} \text{ is increasing (in } \varphi) \qquad \text{(see the discussion below (1.13))}, \tag{2.11}$$

$$A^h_{n,x} \supseteq A^{h'}_{n,x}, \quad \text{for all } h, h' \in \mathbb{R} \text{ with } h \le h'. \tag{2.12}$$

Indeed, the property (2.12) that $A^h_{n,x}$ decreases with h follows since $E^{\geq h}_{\varphi} \supseteq E^{\geq h'}_{\varphi}$ for all $h \leq h'$ by definition, c.f. (0.2). Next, we provide a lemma which separates the combinatorial complexity of the number of crossings in $A^h_{n,x}$ from probabilistic estimates, using $\Lambda_{n,x}$ as introduced in (2.7). This separation will be key in obtaining estimates fine enough to yield the desired stretched exponential decay. Albeit being completely analogous to Lemma 2.1 in [23], we repeat its proof, for it comprises an essential geometric observation concerning the events $A^h_{n,x}$.

Lemma 2.1. *($n \geq 0$, $(n,x) \in \mathcal{I}_n$, $h \in \mathbb{R}$)*

$$\mathbb{P}[A^h_{n,x}] \leq |\Lambda_{n,x}| \sup_{\mathcal{T} \in \Lambda_{n,x}} \mathbb{P}[A^h_{\mathcal{T}}], \qquad \textit{where} \qquad A^h_{\mathcal{T}} = \bigcap_{(0,y) \in \mathcal{T} \cap \mathcal{I}_0} A^h_{0,y}. \tag{2.13}$$

Proof. We use induction on n to show that

$$A^h_{n,x} \subseteq \bigcup_{\mathcal{T} \in \Lambda_{n,x}} A^h_{\mathcal{T}}, \tag{2.14}$$

for all $(n,x) \in \mathcal{I}_n$, from which (2.13) immediately follows. When $n = 0$, (2.14) is trivial. Assume it holds for all $(n-1,y) \in \mathcal{I}_{n-1}$. For any $(n,x) \in \mathcal{I}_n$, a path in $E^{\geq h}_{\varphi}$ starting in $B_{n,x}$ and ending in $\partial^i \widetilde{B}_{n,x}$ must first cross the box B_{n-1,y_1} for some $(n-1,y_1) \in \mathcal{H}_1(n,x)$, and subsequently B_{n-1,y_2} for some $(n-1,y_2) \in \mathcal{H}_2(n,x)$ before reaching $\partial^i \widetilde{B}_{n,x}$, c.f. Figure 1 below. Thus,

$$A^h_{n,x} \subseteq \bigcup_{\substack{(n-1,y_i) \, \in \, \mathcal{H}_i(n,x) \\ i=1,2}} A^h_{n-1,y_1} \cap A^h_{n-1,y_2}.$$

Upon applying the induction hypothesis to A^h_{n-1,y_1} and A^h_{n-1,y_2} separately, the claim (2.14) follows. □

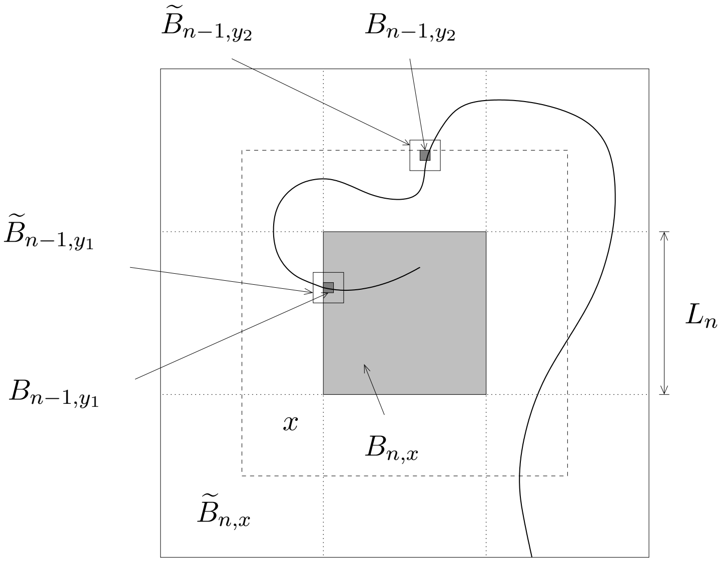

Figure 1: the event $A^h_{n,x}$.

Before proceeding, we remark that the events $A^h_{\mathcal{T}}$, with $h \in \mathbb{R}$ and $\mathcal{T} \in \Lambda_{n,x}$ for some $(n,x) \in \mathcal{I}_n$, $n \geq 0$, defined in (2.13) inherit certain properties from the events $A^h_{0,y}$, $(0,y) \in \mathcal{T} \cap \mathcal{I}_0$. Namely, it follows from (2.11) and (2.12) that

$$A^h_{\mathcal{T}} \text{ is an increasing event (in } \varphi), \tag{2.15}$$

and that, for any two levels $h, h' \in \mathbb{R}$,

$$A_{\mathcal{T}}^{h} \supseteq A_{\mathcal{T}}^{h'} \text{ whenever } h \le h'. \tag{2.16}$$

Further, given any $n \ge 0$, $(n,x) \in \mathcal{I}_n$, and $\mathcal{T} \in \Lambda_{n,x}$, we define the set

$$K_{\mathcal{T}} = \bigcup_{(0,y)\,\in\,\mathcal{T}\cap\mathcal{I}_0} \widetilde{B}_{0,y}. \tag{2.17}$$

Hence, $K_{\mathcal{T}}$ is the disjoint union of 2^n boxes of side length $3L_0$ each, and $K_{\mathcal{T}} \subset \widetilde{B}_{n,x}$. It then immediately follows from the definition of $A_{\mathcal{T}}^{h}$ in (2.13) that (see above (2.10) for the notation $\sigma(\,\cdot\,)$)

$$A_{\mathcal{T}}^{h} \in \sigma\big(\varphi_y \ ; \ y \in K_{\mathcal{T}}\big). \tag{2.18}$$

Finally, upon introducing

$$p_n(h) = \sup_{\mathcal{T}\in\Lambda_{n,x}} \mathbb{P}[A_{\mathcal{T}}^{h}], \qquad \text{for } (n,x) \in \mathcal{I}_n,\ n \ge 0, \tag{2.19}$$

which is well-defined (i.e. independent of $x \in \mathbb{L}_n$) by translation invariance, we obtain $p_n(h) \ge p_n(h')$ whenever $h \le h'$, by (2.16). Note also that

$$p_0(h) = \mathbb{P}\big[B_{0,x=0} \overset{\ge h}{\longleftrightarrow} \partial^i \widetilde{B}_{0,x=0}\big]. \tag{2.20}$$

We now derive the aforementioned "recursive bounds" for the probabilities $p_n(h_n)$, c.f. (2.24) below, along a suitable increasing sequence $(h_n)_{n\ge0}$ (one-step renormalization). These estimates will be key in proving Theorem 2.6 below.

Proposition 2.2. ($L_0 \ge 1$, $l_0 \ge 100$)

There exist positive constants c_1 and c_2 such that, defining

$$M(n, L_0) = c_2\big(\log(2^n(3L_0)^d)\big)^{1/2}, \tag{2.21}$$

then, given any positive sequence $(\beta_n)_{n\ge0}$ satisfying

$$\beta_n \ge (\log 2)^{1/2} + M(n, L_0), \qquad \text{for all } n \ge 0, \tag{2.22}$$

and any increasing, real-valued sequence $(h_n)_{n\ge0}$ satisfying

$$h_{n+1} \ge h_n + c_1\beta_n\big(2l_0^{-(d-2)}\big)^{n+1}, \qquad \text{for all } n \ge 0, \tag{2.23}$$

one has

$$p_{n+1}(h_{n+1}) \le p_n(h_n)^2 + 3e^{-(\beta_n - M(n,L_0))^2}, \qquad \text{for all } n \ge 0. \tag{2.24}$$

The main idea of the proof is to "decouple" the event $A_{\mathcal{T}'}^{h_n} \cap A_{\mathcal{T}''}^{h_n}$, where $\mathcal{T}'$ and $\mathcal{T}''$ are the (binary) subtrees at level n of some given tree $\mathcal{T} \in \Lambda_{n+1,x}$, $x \in \mathbb{L}_{n+1}$, using the increase in parameter $h_n \to h_{n+1}$ to dominate the interactions ("sprinkling").

Proof. We let $n \ge 0$, consider some $m = (n+1, x) \in \mathcal{I}_{n+1}$ and some tree $\mathcal{T} \in \Lambda_m$. We decompose

$$\mathcal{T} = \{m\} \cup \mathcal{T}_{n,y_1(m)} \cup \mathcal{T}_{n,y_2(m)}, \tag{2.25}$$

where $(n, y_i(m))$, $i = 1, 2$ are the two descendants of m in $\mathcal{T}$ and

$$\mathcal{T}_{n,y_i(m)} = \{(k,z) \in \mathcal{T} :\ \widetilde{B}_{k,z} \subseteq \widetilde{B}_{n,y_i(m)}\}, \quad \text{for } i = 1, 2, \tag{2.26}$$

that is $\mathcal{T}_{n,y_i(m)}$ is the (sub-)tree consisting of all descendants of $(n, y_i(m))$ in $\mathcal{T}$. Thus, the union in (2.25) is over disjoint sets. Note in particular that $\mathcal{T}_{n,y_i(m)} \in \Lambda_{n,y_i(m)}$. By construction, the subsets $K_{\mathcal{T}_{n,y_i(m)}}$ $\big(\subset \widetilde{B}_{n,y_i(m)}\big)$, for $i = 1, 2$, see (2.17), satisfy $K_{\mathcal{T}_{n,y_1(m)}} \cap K_{\mathcal{T}_{n,y_2(m)}} = \emptyset$. For sake of clarity, and since m and $\mathcal{T}$ will be fixed throughout the proof, we abbreviate

$$\mathcal{T}_{n,y_i(m)} = \mathcal{T}_i \quad \text{and} \quad K_{\mathcal{T}_{n,y_i(m)}} = K_i, \quad \text{for } i = 1, 2. \tag{2.27}$$

In order to estimate the probability of the event $A^h_{\mathcal{T}} = A^h_{\mathcal{T}_1} \cap A^h_{\mathcal{T}_2}$, $h \in \mathbb{R}$, defined in (2.13), we introduce a parameter $\alpha > 0$ and write

$$\begin{aligned} \mathbb{P}[A^h_{\mathcal{T}}] &\leq \mathbb{P}\big[A^h_{\mathcal{T}_1} \cap A^h_{\mathcal{T}_2} \cap \big\{\max\nolimits_{K_1} \varphi \leq \alpha\big\}\big] \;+\; \mathbb{P}\big[\max\nolimits_{K_1} \varphi > \alpha\big] \\ &= \mathbb{E}\big[1_{A^h_{\mathcal{T}_1}} \cdot 1_{\{\max_{K_1} \varphi \,\leq \alpha\}} \cdot \mathbb{P}[A^h_{\mathcal{T}_2} \,|\varphi_{K_1}]\big] \;+\; \mathbb{P}\big[\max\nolimits_{K_1} \varphi > \alpha\big], \end{aligned} \tag{2.28}$$

where $\max_{K_1} \varphi = \max\{\varphi_x;\ x \in K_1\}$ and the second line follows because $A^h_{\mathcal{T}_1} \cap \big\{\max_{K_1} \varphi \leq \alpha\big\}$ is measurable with respect to $\sigma(\varphi_{K_1})$, c.f. (2.18). We begin by focusing on the conditional probability $\mathbb{P}[A^h_{\mathcal{T}_2} \,|\varphi_{K_1}]$ in (2.28). Using (1.21) and (2.18) applied to $A^h_{\mathcal{T}_2}$, and with a slight abuse of notation, we find

$$\mathbb{P}[A^h_{\mathcal{T}_2} \,|\varphi_{K_1}] = \widetilde{\mathbb{P}}\big[A^h_{\mathcal{T}_2}\big((\widetilde{\varphi}_x + \mu_x)_{x\in K_2}\big)\ \big], \qquad \mathbb{P}\text{-almost surely,} \tag{2.29}$$

where $\mu_x = E_x\big[H_{K_1} < \infty, \varphi_{X_{H_{K_1}}}\big]$. On the event $\big\{\max_{K_1} \varphi \leq \alpha\big\}$, we have, for all $x \in K_2$,

$$\mu_x = \sum_{y\in K_1} \varphi_y P_x[H_{K_1} < \infty, X_{H_{K_1}} = y] \leq \alpha \cdot P_x[H_{K_1} < \infty] \overset{\text{def.}}{=} m_x(\alpha), \tag{2.30}$$

which is deterministic and linear in α. Moreover, we can bound $m_x(\alpha)$ as follows. By virtue of (1.7), $P_x[H_{K_1} < \infty] \leq \text{cap}(K_1) \cdot \sup_{y\in K_1} g(x, y)$ for all $x \in K_2$. Since K_1 consists of 2^n disjoint boxes of side length $3L_0$, c.f. (2.27) and (2.17), its capacity can be bounded, using (1.6) and (1.8), as $\text{cap}(K_1) \leq c2^n L_0^{d-2}$. By (1.9), (2.1) and the observation that $|x-y| \geq c' L_{n+1}$ whenever $x \in K_1$ and $y \in K_2$, it follows that

$$m_x(\alpha) \leq c_1\big(2g(0)\big)^{-1/2} \cdot \alpha \cdot 2^n l_0^{-(n+1)(d-2)} \overset{\text{def.}}{=} \frac{\gamma}{2}, \qquad \text{for } x \in K_2, \tag{2.31}$$

which defines the constant c_1 from (2.23), and the factor $(2g(0))^{-1/2}$ is kept for later convenience.

Returning to the conditional probability $\mathbb{P}[A^h_{\mathcal{T}_2} \,|\varphi_{K_1}]$, we first observe that, on the event $\big\{\max_{K_1} \varphi \leq \alpha\big\}$ and for any $x \in K_2$, the inequality $\widetilde{\varphi}_x + \mu_x \geq h$ implies

$$\widetilde{\varphi}_x - m_x(\alpha) \geq h - \mu_x - m_x(\alpha) \overset{(2.30)}{\geq} h - 2m_x(\alpha) \overset{(2.31)}{\geq} h - \gamma.$$

Hence, on the event $\big\{\max_{K_1} \varphi \leq \alpha\big\}$,

$$\begin{aligned} \mathbb{P}[A^h_{\mathcal{T}_2} \,|\varphi_{K_1}] &\overset{(2.29)}{=} \widetilde{\mathbb{P}}\big[A^h_{\mathcal{T}_2}\big((\widetilde{\varphi}_x + \mu_x)_{x\in K_2}\big)\ \big] \\ &\leq \widetilde{\mathbb{P}}\big[A^{h-\gamma}_{\mathcal{T}_2}\big((\widetilde{\varphi}_x - m_x(\alpha))_{x\in K_2}\big)\ \big] = \mathbb{P}[A^{h-\gamma}_{\mathcal{T}_2} \,|\varphi_{|K_1} = -\alpha], \end{aligned} \tag{2.32}$$

where the last equality follows by (1.21), noting that, on the event $\{\varphi_{|K_1} = -\alpha\}$, we have $\mu_x = m_x(-\alpha) = -m_x(\alpha)$ for all $x \in K_2$, c.f. (2.30). Applying Lemma 1.4 to the right-hand side of (2.32), we immediately obtain that, on the event $\big\{\max_{K_1} \varphi \leq \alpha\big\}$,

$$\mathbb{P}[A^h_{\mathcal{T}_2} \,|\varphi_{K_1}] \leq \mathbb{P}[A^{h-\gamma}_{\mathcal{T}_2} \,|\varphi_{|K_1} \geq -\alpha] \leq \mathbb{P}[A^{h-\gamma}_{\mathcal{T}_2}] \cdot \big(\mathbb{P}[\varphi_{|K_1} \geq -\alpha]\big)^{-1}. \tag{2.33}$$

At last, we insert (2.33) into (2.28), noting that, since φ has the same law as $-\varphi$, we have $\mathbb{P}[\varphi_{|K_1} \geq -\alpha] = 1 - \mathbb{P}[\min_{K_1} \varphi < -\alpha] = 1 - \mathbb{P}[\max_{K_1} \varphi > \alpha]$, to get

$$(2.34) \qquad \mathbb{P}[A^h_{\mathcal{T}}] \le \mathbb{P}\big[A^h_{\mathcal{T}_1}\big] \cdot \mathbb{P}\big[A^{h-\gamma}_{\mathcal{T}_2}\big] \cdot \big(1 - \mathbb{P}[\max_{K_1} \varphi > \alpha]\big)^{-1} + \mathbb{P}\big[\max_{K_1} \varphi > \alpha\big].$$

Next, we turn our attention to the term $\mathbb{P}\big[\max_{K_1} \varphi > \alpha\big]$. By virtue of the BTIS-inequality (c.f [1], Theorem 2.1.1), for arbitrary $\emptyset \neq K \subset\subset \mathbb{Z}^d$, we have

$$(2.35) \qquad \mathbb{P}\big[\max_K \varphi > \alpha\big] \le \exp\Big\{ - \frac{\big(\alpha - \mathbb{E}\left[\max_K \varphi\right]\big)^2}{2g(0)}\Big\}, \qquad \text{if } \alpha > \mathbb{E}\left[\max_K \varphi\right].$$

In order to bound $\mathbb{E}\left[\max_K \varphi\right]$, we write, using Fubini's theorem,

$$(2.36)\ \mathbb{E}\left[\max_K \varphi\right] \le \mathbb{E}\left[\max_K \varphi^+\right] = \int_0^\infty du\ \mathbb{P}\big[\max_K \varphi^+ > u\big] \le A + \int_A^\infty du\ \mathbb{P}\big[\max_K \varphi^+ > u\big],$$

for arbitrary $A \ge 0$. Recalling that $\mathbb{E}[\varphi_x^2] = g(0)$ for all $x \in \mathbb{Z}^d$, c.f. (0.1), and introducing an auxiliary variable $\psi \sim \mathcal{N}(0,1)$, we can bound the integrand as

$$P\big[\max_K \varphi^+ > u\big] \le |K| \cdot \mathbb{P}[\varphi_0 > u] = |K| \cdot \mathbb{P}\big[\psi > g(0)^{-1/2} u\big] \le |K| \cdot e^{-u^2/2g(0)},$$

where we have used in the last step that $\mathbb{P}[\psi > a] \le e^{-a^2/2}$, for $a > 0$, which follows readily from Markov's inequality, since $\mathbb{P}[\psi > a] \le \min_{\lambda > 0} e^{-\lambda a}\, \mathbb{E}[e^{\lambda \psi}] = \min_{\lambda>0} e^{-\lambda a + \lambda^2/2}$, and the minimum is attained at $\lambda = a$. Inserting the bound for $P\big[\max_K \varphi^+ > u\big]$ into (2.36) yields, for arbitrary $A > 0$,

$$(2.37) \qquad \mathbb{E}\left[\max_K \varphi\right] \le A + |K| \int_A^\infty du\ e^{-\frac{u^2}{2g(0)}} \le A + c|K| \cdot e^{-A^2/2g(0)}.$$

We select $A = (2g(0) \log |K|)^{1/2}$ (so that $e^{-A^2/2g(0)} = |K|^{-1}$), by which means (2.37) readily implies that

$$(2.38) \qquad \mathbb{E}\left[\max_K \varphi\right] \le c\sqrt{\log |K|}, \qquad \text{for all} \quad \emptyset \neq K \subset\subset \mathbb{Z}^d.$$

In the relevant case $K = K_1$ with $|K_1| = 2^n (3L_0)^d$, we thus obtain

$$(2.39) \qquad \mathbb{E}\left[\max_{K_1} \varphi\right] \le c_2 \big(2g(0) \log(2^n (3L_0)^d)\big)^{1/2} \overset{(2.21)}{=} \sqrt{2g(0)} \cdot M(n, L_0),$$

where the first inequality defines the constant c_2 from (2.21). We now require

$$(2.40) \qquad \alpha / \sqrt{2g(0)} \ge \sqrt{\log 2} + M(n, L_0),$$

thus (2.35) applies and yields

$$(2.41) \qquad \mathbb{P}\big[\max_{K_1} \varphi > \alpha\big] \le \min\Big\{1/2,\ e^{-\big(\frac{\alpha}{\sqrt{2g(0)}} - M(n,L_0)\big)^2}\Big\}.$$

Returning to (2.34), and using that $(1-x)^{-1} \le 1 + 2x$ for all $0 \le x \le 1/2$ (with $x = \mathbb{P}\big[\max_{K_1} \varphi > \alpha\big]$), we finally obtain, for all α satisfying (2.40) and $h' \ge h$,

$$(2.42) \qquad \begin{aligned} \mathbb{P}[A^{h'}_{\mathcal{T}}] \ \le\ \mathbb{P}[A^h_{\mathcal{T}}] \ &\le\ \mathbb{P}\big[A^h_{\mathcal{T}_1}\big] \cdot \mathbb{P}\big[A^{h-\gamma}_{\mathcal{T}_2}\big] + 3 \cdot \mathbb{P}\big[\max_{K_1} \varphi > \alpha\big] \\ &\overset{(2.41)}{\le}\ \mathbb{P}\big[A^{h-\gamma}_{\mathcal{T}_1}\big] \cdot \mathbb{P}\big[A^{h-\gamma}_{\mathcal{T}_2}\big] + 3 e^{-(\beta - M(n,L_0))^2}, \end{aligned}$$

where we have set $\beta = \alpha / \sqrt{2g(0)}$. The claim (2.24) now readily follows upon taking suprema over all $\mathcal{T} \in \Lambda_{n+1,x}$ on both sides of (2.42), letting $\beta_n \overset{\text{def.}}{=} \beta$, $h_n \overset{\text{def.}}{=} h - \gamma \in \mathbb{R}$ (h was arbitrary), $h_{n+1} \overset{\text{def.}}{=} h'$, so that requiring $h_{n+1} = h' \ge h = h_n + \gamma$, by virtue of (2.31), is nothing but (2.23). Noting condition (2.40) for $\beta_n = \beta$, we precisely recover (2.22). This concludes the proof of Proposition 2.2. □

Remark 2.3.

1) The bound (2.38), which we have derived using an elementary argument, also follows from a more general (and stronger) estimate. One knows that (in fact, this holds for a large class of Gaussian fields, c.f. [1], Theorem 1.3.3),

$$\mathbb{E}\left[\sup\nolimits_K \varphi\right] \le C \int_0^{\frac{1}{2}\sup_{x,y,\in K} d(x,y)} \sqrt{\log\left(N(\varepsilon)\right)}\ \mathrm{d}\varepsilon, \tag{2.43}$$

where $d(x,y) = \left(\mathbb{E}[(\varphi_x - \varphi_y)^2]\right)^{1/2}$, $x,y \in \mathbb{Z}^d$, $K \subset\subset \mathbb{Z}^d$, $N(\varepsilon)$ denotes the smallest number of closed balls of radius ε in this metric covering K, and C is a universal constant. Clearly, $N(\varepsilon) \le |K|$ for all $\varepsilon \ge 0$. Moreover, $\sup_{x,y,\in\mathbb{Z}^d} d(x,y) \le \sqrt{2g(0)}$ by virtue of (0.1). Inserting this into (2.43) immediately yields the bound (2.38).

2) We mention a generalization of Proposition 2.2, which is of independent interest, but will not be needed in what follows. Consider integers $L_0 \ge 1$, $l_0 \ge 100$, and a collection D_x, $x \in \mathbb{L}_0$, of events in Ω $(= \{0,1\}^{\mathbb{Z}^d}$, see above (1.12)), such that

$$D_x \text{ is } \sigma(Y_z\ ;\ z \in \widetilde{B}_{0,x})\text{-measurable for each } x \in \mathbb{L}_0, \tag{2.44}$$

where Y_z, $z \in \mathbb{Z}^d$, stand for the canonical coordinates on Ω.

Given $h \in \mathbb{R}$, $n \ge 0$, $x \in \mathbb{L}_n$, and $\mathcal{T} \in \Lambda_{n,x}$, we replace $A^h_{\mathcal{T}}$ in (2.13) by (see below (1.12) for the notation Φ^h)

$$D^h_{\mathcal{T}} = \bigcap_{(0,y)\in\mathcal{T}\cap\mathcal{I}_0} (\Phi^h)^{-1}(D_y), \tag{2.45}$$

and $p_n(h)$ in (2.8) by

$$q_n(h) = \sup_{x\in\mathbb{L}_n, \mathcal{T}\in\Lambda_{n,x}} \mathbb{P}[D^h_{\mathcal{T}}]. \tag{2.46}$$

One then has the following generalization of Proposition 2.2:

Proposition 2.2'. ($L_0 \ge 1$, $l_0 \ge 100$, (2.44))
Assume that

$$\textit{for each } x \in \mathbb{L}_0,\ D_x \textit{ is increasing,} \tag{2.47}$$

that $(\beta_n)_{n\ge0}$ *is a positive sequence,* $(h_n)_{n\ge0}$ *a real increasing sequence, such that* (2.22), (2.23) *hold. Then,*

$$q_{n+1}(h_{n+1}) \le q_n(h_n)^2 + 3e^{-(\beta_n - M(n,L_0))^2}, \qquad \textit{for all } n \ge 0. \tag{2.48}$$

If instead,

$$\textit{for each } x \in \mathbb{L}_0,\ D_x \textit{ is decreasing,} \tag{2.49}$$

$(\beta_n)_{n\ge0}$ *is a positive sequence,* $(h_n)_{n\ge0}$ *a real decreasing sequence, so that* (2.22) *holds and* (2.23) *holds for* $(-h_n)_{n\ge0}$, *then* (2.48) *holds as well.*

Proof. The arguments employed in the proof of Proposition 2.2 yield the first statement (with D_x, $x \in \mathbb{L}_0$, increasing events). To derive the second statement (when D_x, $x \in \mathbb{L}_0$, are decreasing events), one argues as follows. One introduces the inversion $\iota : \Omega \longrightarrow \Omega$ such that $Y_z \circ \iota = 1 - Y_z$, for all $z \in \mathbb{Z}^d$, and the collection of "flipped" events $\overline{D}_x = \iota^{-1}(D_x) = \iota(D_x)$, $x \in \mathbb{L}_0$. One defines

$\overline{D}^h_{\mathcal{T}}$ as in (2.45) with $\overline{D}_y$, $y \in \mathbb{L}_0$, in place of D_y, $y \in \mathbb{L}_0$. Now observe that $(-\varphi_x)_{x\in\mathbb{Z}^d}$ has the same law as $(\varphi_x)_{x\in\mathbb{Z}^d}$ under $\mathbb{P}$, and that for any $h \in \mathbb{R}$, $\big(1\{\varphi_x < -h\}\big)_{x\in\mathbb{Z}^d}$ has the same law Q^h as $\big(1\{\varphi_x \geq h\}\big)_{x\in\mathbb{Z}^d}$ under $\mathbb{P}$. From this, we infer that for all $h \in \mathbb{R}$, $x \in \mathbb{L}_n$, and $\mathcal{T} \in \Lambda_{n,x}$,

$$\begin{aligned} \mathbb{P}\big[D^h_{\mathcal{T}}\big] &= Q^h\Big[\bigcap_{(0,y)\in\mathcal{T}\cap\mathcal{I}_0} D_y\Big] = \mathbb{P}\Big[\big(1\{\varphi_x < -h\}\big)_{x\in\mathbb{Z}^d} \in \bigcap_{(0,y)\in\mathcal{T}\cap\mathcal{I}_0} D_y\Big] \\ &= Q^{-h}\Big[\bigcap_{(0,y)\in\mathcal{T}\cap\mathcal{I}_0} \overline{D}_y\Big] = \mathbb{P}\big[\,\overline{D}^{-h}_{\mathcal{T}}\big]. \end{aligned} \tag{2.50}$$

When (2.49) holds, the events $\overline{D}_x$, $x \in \mathbb{L}_0$, satisfy (2.47), and thanks to the identity (2.50), the second statement of Proposition 2.2' is reduced to the first statement. This concludes the proof of Proposition 2.2'. □

3) There is an analogy between Proposition 2.2' and the main renormalization step Theorem 2.1 of [24], for the decoupling inequalities of random interlacements (see Theorem 2.6 of [24]). Note however, that unlike condition (2.7) of [24] (see also (2.70) in [24]), (2.22) and (2.23) tie in the finest scale L_0 to the sequence $(h_n)_{n\geq 0}$. This feature has to do with the role of the cut-off level α we introduce in (2.28) and the remainder term it produces. □

We now return to Proposition 2.2 and aim at propagating the estimate (2.24) inductively. To this end, we first define, for all $n \geq 0$,

$$\beta_n = (\log 2)^{1/2} + M(n, L_0) + 2^{(n+1)/2}\big(n^{1/2} + K_0^{1/2}\big), \tag{2.51}$$

where $K_0 > 0$ is a certain parameter to be specified below in Proposition 2.4 and later in (2.59). Note in particular that condition (2.22) holds for this choice of $(\beta_n)_{n\geq 0}$. In the next proposition, we inductively derive bounds for $p_n(h_n)$, $n \geq 0$, given any sequence $(h_n)_{n\geq 0}$ satisfying the assumptions of Proposition 2.2, provided the induction can be initiated, see (2.52) below.

Proposition 2.4.

Assume $h_0 \in \mathbb{R}$ and $K_0 \geq 3(1-e^{-1})^{-1} \stackrel{def.}{=} B$ are such that

$$p_0(h_0) \leq e^{-K_0}, \tag{2.52}$$

and let the sequence $(h_n)_{n\geq 0}$ satisfy (2.23) *with $(\beta_n)_{n\geq 0}$ as defined in* (2.51). *Then,*

$$p_n(h_n) \leq e^{-(K_0 - B)2^n}, \qquad \text{for all } n \geq 0. \tag{2.53}$$

Proof. We define a sequence $(K_n)_{n\geq 0}$ inductively by

$$K_{n+1} = K_n - \log\Big(1 + e^{K_n} \cdot 3^{2^{-(n+1)}} e^{-2^{-(n+1)}(\beta_n - M(n,L_0))^2}\Big), \qquad \text{for all } n \geq 0, \tag{2.54}$$

with β_n given by (2.51) (the factor following e^{K_n} in (2.54) should be viewed as the $2^{(n+1)}$-th root of the remainder term on the right-hand side of (2.24)). Then, (2.54) implies that $K_n \leq K_0$ for all $n \geq 0$. Moreover, as we now see,

$$K_n \geq K_0 - B, \qquad \text{for all } n \geq 0. \tag{2.55}$$

This is clear for $n = 0$. When $n \geq 1$, first note that by virtue of (2.54),

$$K_n = K_0 - \sum_{m=0}^{n-1} \log\Big(1 + e^{K_m} \cdot 3^{2^{-(m+1)}} e^{-2^{-(m+1)}(\beta_m - M(m,L_0))^2}\Big), \qquad \text{for all } n \geq 1. \tag{2.56}$$

Moreover, (2.51) implies

$$(2.57)\qquad (\beta_m - M(m,L_0))^2 \geq \log 2 + 2^{m+1}(m^{1/2} + K_0^{1/2})^2 \geq \log 2 + 2^{m+1}(m + K_0),$$

for all $m \geq 0$, which, inserted into (2.56), yields

$$K_n \geq K_0 - \sum_{m=0}^{\infty} \log\Big(1 + e^{K_m} \cdot 3^{2^{-(m+1)}} e^{-K_0 - m}\Big) \geq K_0 - 3\sum_{m=0}^{\infty} e^{-m} = K_0 - B,$$

where we have used $K_n \leq K_0$ and $\log(1+x) \leq x$ for all $x \geq 0$ in the second inequality. Hence, (2.55) holds. We will now show by induction on n that

$$(2.58)\qquad p_n(h_n) \leq e^{-K_n 2^n}, \qquad \text{for all } n \geq 0,$$

which, together with (2.55), implies (2.53). The inequality (2.58) holds for $n = 0$ by assumption, c.f. (2.52). Assume now it holds for some n. By Proposition 2.2, we find

$$\begin{aligned} p_{n+1}(h_{n+1}) &\overset{(2.24)}{\leq} \big(e^{-K_n 2^n}\big)^2 + 3e^{-(\beta_n - M(n,L_0))^2} \\ &\leq \Big[e^{-K_n}\big(1 + e^{K_n} 3^{2^{-(n+1)}} e^{-2^{-(n+1)}(\beta_n - M(n,L_0))^2}\big)\Big]^{2^{n+1}} \overset{(2.54)}{=} e^{-K_{n+1}2^{n+1}}. \end{aligned}$$

This concludes the proof of (2.58) and thus of Proposition 2.4. □

Remark 2.5.

Although we will not need this fact in what follows, let us point out that a straightforward adaptation of Proposition 2.4 holds in the context of Proposition 2.2'. □

We will now state the main theorem of this section and prove it using Proposition 2.4. To this end, we select K_0 appearing in Proposition 2.4 as follows:

$$(2.59)\qquad K_0 = \log(2c_0 l_0^{2(d-1)}) + B \qquad \text{(see (2.8) for the definition of } c_0\text{)}.$$

Moreover, we will solely consider sequences $(h_n)_{n\geq 0}$ with

$$(2.60)\qquad h_0 > 0, \qquad h_{n+1} - h_n = c_1 \beta_n \big(2 l_0^{-(d-2)}\big)^{n+1}, \qquad \text{for all } n \geq 0,$$

so that condition (2.23) is satisfied. We recall that β_n is given by (2.51), which now reads

$$(2.61)\qquad \beta_n = (\log 2)^{1/2} + c_2\big(\log(2^n(3L_0)^d)\big)^{1/2} + 2^{(n+1)/2}\big(n^{1/2} + (\log(2c_0 l_0^{2(d-1)}) + B)^{1/2}\big)$$

where we have substituted $M(n, L_0)$ from (2.21) and K_0 from (2.59). Note that L_0, l_0 and h_0 are the only parameters which remain to be selected. We finally proceed to the main

Theorem 2.6.

The critical point $h_{**}(d)$ defined in (0.6) satisfies

$$(2.62)\qquad h_{**}(d) < \infty, \qquad \text{for all } d \geq 3.$$

Moreover, for all $d \geq 3$ and $h > h_{**}(d)$, there exist positive constants $c(h)$, $c'(h)$ and $0 < \rho < 1$ (ρ depending on d and h) such that

$$(2.63)\qquad \mathbb{P}\big[B(0,L) \overset{\geq h}{\longleftrightarrow} S(0,2L)\big] \leq c(h) \cdot e^{-c'(h)L^\rho}, \qquad \text{for all } L \geq 1.$$

In particular, the connectivity function $\mathbb{P}\big[0 \overset{\geq h}{\longleftrightarrow} x\big]$ of the excursion set above level h has stretched exponential decay, i.e. there exists $c''(h) > 0$ such that

$$(2.64)\qquad \mathbb{P}\big[0 \overset{\geq h}{\longleftrightarrow} x\big] \leq c(h) \cdot e^{-c''(h)|x|^\rho}, \qquad \text{for all } x \in \mathbb{Z}^d,\ h > h_{**}(d),\ \text{and } d \geq 3.$$

Corollary 2.7.
The excursion set $E_{\varphi}^{\geq h}$ *above level* h *defined in* (0.2) *undergoes a non-trivial percolation phase transition for all* $d \geq 3$, *i.e.*

$$(0 \leq)\ h_*(d) < \infty, \qquad \text{for all } d \geq 3, \tag{2.65}$$

and

$$\mathbb{P}[E_{\varphi}^{\geq h} \text{ contains an infinite cluster }] = \begin{cases} 1, & \text{if } h < h_* \\ 0, & \text{if } h > h_*. \end{cases} \tag{2.66}$$

Proof of Corollary 2.7. The lower bound $h_*(d) \geq 0$ in (2.65) follows from Corollary 2 of [2]. In order to establish the finiteness in (2.65), it suffices to show $h_* \leq h_{**}$ and to invoke the above Theorem 2.6. To this end, we note that by definition (c.f. (0.3)),

$$\eta(h) \leq \mathbb{P}\big[B(0,L) \stackrel{\geq h}{\longleftrightarrow} S(0,2L)\big], \qquad \text{for all } L \geq 1, \tag{2.67}$$

and $h_* \leq h_{**}$ readily follows. As for (2.66), it is an immediate consequence of Lemma 1.5. □

Proof of Theorem 2.6. To prove (2.62), it suffices to construct an explicit level $\bar{h}$ with $0 < \bar{h} < \infty$ such that $\mathbb{P}\big[B(0,L) \stackrel{\geq \bar{h}}{\longleftrightarrow} S(0,2L)\big]$ decays polynomially in L, as $L \to \infty$. In fact, we will even show that $\mathbb{P}\big[B(0,L) \stackrel{\geq \bar{h}}{\longleftrightarrow} S(0,2L)\big]$ has stretched exponential decay.

We begin by observing that the sequence $(h_n)_{n\geq 0}$ defined in (2.60) has a finite limit $h_\infty = \lim_{n\to\infty} h_n$ for every choice of L_0, l_0 and h_0. Indeed, β_n as given by (2.61) satisfies $\beta_n \leq c(L_0, l_0)2^{n+1}$ for all $n \geq 0$, hence

$$h_\infty \stackrel{(2.60)}{=} h_0 + c_1 \sum_{n=0}^{\infty} \beta_n \big(2 l_0^{-(d-2)}\big)^{n+1} \leq h_0 + c'(L_0, l_0) \sum_{n=0}^{\infty} \big(4 l_0^{-(d-2)}\big)^{n+1} < \infty,$$

since we assumed $l_0 \geq 100$. We set

$$L_0 = 10, \qquad l_0 = 100, \tag{2.68}$$

and now show with Proposition 2.4 that there exists $h_0 > 0$ sufficiently large such that, defining

$$\bar{h} = h_\infty = \lim_{n\to\infty} h_n \quad (< \infty), \tag{2.69}$$

we have

$$\mathbb{P}\big[B(0,L) \stackrel{\geq \bar{h}}{\longleftrightarrow} S(0,2L)\big] \leq c \cdot e^{-c' L^\rho}, \qquad \text{for all } L \geq 1, \tag{2.70}$$

for suitable $c, c' > 0$ and $0 < \rho < 1$. To this end, we note that $p_0(h_0)$ defined in (2.20) is bounded by

$$p_0(h_0) \stackrel{(2.20)}{\leq} \mathbb{P}\big[\max_{\widetilde{B}_{0,x=0}} \varphi \geq h_0\big] \stackrel{(2.35)}{\leq} \exp\Big\{ - \frac{\big(h_0 - \mathbb{E}\left[\max_{\widetilde{B}_{0,x=0}} \varphi\right]\big)^2}{2g(0)} \Big\},$$

when $h_0 \geq c$ (e.g. using (2.38) to bound $\mathbb{E}\left[\max_{\widetilde{B}_{0,x=0}} \varphi\right]$). In particular, since K_0 in (2.59) is completely determined by the choices (2.68), we see that $p_0(h_0) \leq e^{-K_0}$ for all $h_0 \geq c$, i.e.

condition (2.52) holds for sufficiently large h_0. By Proposition 2.4, setting $h_0 = c$ and $\bar{h}$ as in (2.69), we obtain

$$(2.71)\qquad p_n(\bar{h}) \overset{\bar{h}>h_n}{\leq} p_n(h_n) \overset{(2.53)}{\leq} e^{-(K_0-B)2^n} \overset{(2.59)}{=} \big(2c_0 l_0^{2(d-1)}\big)^{-2^n}, \qquad \text{for all } n \geq 0.$$

Therefore, we find that for all $n \geq 0$ and $x \in \mathbb{L}_n$,

$$(2.72)\quad \mathbb{P}\big[B_{n,x} \overset{\geq \bar{h}}{\longleftrightarrow} \partial^i \widetilde{B}_{n,x}\big] \overset{(2.13)}{\leq} |\Lambda_{n,x}| \cdot p_n(\bar{h}) \overset{(2.8),(2.71)}{\leq} \big(c_0 l_0^{2(d-1)}\big)^{2^n} \big(2c_0 l_0^{2(d-1)}\big)^{-2^n} = 2^{-2^n}.$$

We now set $\rho = \log 2/\log l_0$, whence $2^n = l_0^{n\rho} = (L_n/L_0)^\rho$. Given $L \geq 1$, we first assume there exists $n \geq 0$ such that $2L_n \leq L < 2L_{n+1}$. Then, since

$$\mathbb{P}\big[B(0,L) \overset{\geq \bar{h}}{\longleftrightarrow} S(0,2L)\big] \leq \mathbb{P}\Bigg[\bigcup_{x \in \mathbb{L}_n : B_{n,x} \cap S(0,L) \neq \emptyset} \big\{B_{n,x} \overset{\geq \bar{h}}{\longleftrightarrow} \partial^i \widetilde{B}_{n,x}\big\}\Bigg],$$

and the number of sets contributing to the union on the right-hand side is bounded by $c l_0^{d-1}$, (2.72) readily implies (2.70), and by adjusting c, c', (2.70) will hold for $L < 2L_0$ as well. It follows that $\bar{h} \geq h_{**}$, which completes the proof of (2.62).

We now turn to the proof of (2.63). Let h be some level with $h_{**} < h < \infty$, and define $h_0 = (h_{**}+h)/2$. Since $h_0 > h_{**}$, we may choose $\varepsilon = \varepsilon(h) > 0$ such that $\lim_{L\to\infty} L^\varepsilon \mathbb{P}\big[B(0,L) \overset{\geq h_0}{\longleftrightarrow} S(0,2L)\big] = 0$, which readily implies

$$(2.73)\qquad \lim_{L_0 \to \infty} L_0^\varepsilon \cdot p_0(h_0) = 0,$$

(see (2.20) for the definition of $p_0(\cdot)$). Moreover, we let

$$(2.74)\qquad l_0 = 100\big(\big[L_0^{\frac{\varepsilon}{3(d-1)}}\big] + 1\big),$$

so that $l_0 \geq 100$ is an integer, as required. From (2.61), it is then easy to see, using (2.74), that $\beta_n \leq c(h)\log(l_0)2^{n+1}$, for all $n \geq 0$. Hence, the limit $h_\infty = \lim_{n\to\infty} h_n$ of the increasing sequence $(h_n)_{n\geq 0}$ defined in (2.60) satisfies

$$(2.75)\qquad \begin{aligned} h_\infty = h_0 + c_1 \sum_{n=0}^\infty \beta_n \big(2 l_0^{-(d-2)}\big)^{n+1} &\leq h_0 + c'(h)\log(l_0) l_0^{-(d-2)} \sum_{n=0}^\infty \big(4 l_0^{-(d-2)}\big)^n \\ &= h_0 + c'(h)\frac{\log(l_0)}{l_0^{d-2}} \cdot \frac{1}{1 - 4 l_0^{-(d-2)}}. \end{aligned}$$

Thus, (2.74) and (2.75) imply that $h_\infty \leq h$ whenever $L_0 \geq c(h)$. Moreover,

$$(2.76)\qquad e^{-K_0} \overset{(2.59)}{=} c l_0^{-2(d-1)} \overset{(2.74)}{\geq} c' L_0^{-\frac{2\varepsilon}{3}} \geq p_0(h_0), \qquad \text{for all } L_0 \geq c(h),$$

where the last inequality follows by (2.73). We thus select $L_0 = c(h)$ so that both (2.76) and $h_\infty \leq h$ hold. Since condition (2.52) is satisfied, Proposition 2.4 yields

$$p_n(h) \overset{h \geq h_n}{\leq} p_n(h_n) \overset{(2.53)}{\leq} e^{-(K_0-B)2^n} \overset{(2.59)}{=} \big(2c_0 l_0^{2(d-1)}\big)^{-2^n}, \qquad \text{for all } n \geq 0,$$

from which point on one may argue in the same manner as for the proof of (2.62) to infer (2.72) (with h in place of $\bar{h}$) and subsequently deduce (2.63). In particular, this involves defining $\rho = \log 2/\log l_0$, which depends on h (and d) through l_0. The stretched exponential bound (2.64) for the connectivity function of $E_\varphi^{\geq h}$ is an immediate corollary of (2.63), since $\mathbb{P}\big[0 \overset{\geq h}{\longleftrightarrow} x\big] \leq \mathbb{P}\big[B(0,L) \overset{\geq h}{\longleftrightarrow} S(0,2L)\big] \leq c(h) e^{-c''(h)|x|^\rho}$ whenever $2L \leq |x|_\infty < 2(L+1)$. □

Remark 2.8.

1) An important open question is whether h_* equals h_{**} or not. In case the two differ, the decay of $\mathbb{P}\big[0 \stackrel{\geq h}{\longleftrightarrow} S(0,L)\big]$ as $L \to \infty$, for $h > h_*$, exhibits a sharp transition. Indeed, first note that by (0.4), for all $h > h_*$, $\mathbb{P}\big[0 \stackrel{\geq h}{\longleftrightarrow} S(0,L)\big] \longrightarrow 0$, as $L \to \infty$. If $h_{**} > h_*$, then by definition of h_{**},

$$\text{for } h \in (h_*, h_{**}) \text{ and any } \alpha > 0, \qquad \limsup_{L\to\infty} L^{d-1+\alpha}\mathbb{P}\big[0 \stackrel{\geq h}{\longleftrightarrow} S(0,L)\big] = \infty.$$

Hence $\mathbb{P}\big[0 \stackrel{\geq h}{\longleftrightarrow} S(0,L)\big]$ decays to zero with L, but with an at most polynomial decay for $h \in (h_*, h_{**})$. However, for $h > h_{**}$, $\mathbb{P}\big[0 \stackrel{\geq h}{\longleftrightarrow} S(0,L)\big]$ has a stretched exponential decay in L, by (2.63).

2) The proof of Theorem 2.6 works just as well if we replace the assumption $h > h_{**}$ by $h > \tilde{h}_{**}$, where $\tilde{h}_{**} (\leq h_{**})$ is defined similarly as h_{**} in (0.6), simply replacing the "lim" by a "liminf" in (0.6), i.e.

$$\tilde{h}_{**} = \inf\big\{h \in \mathbb{R}\,; \text{for some } \alpha > 0,\ \liminf_{L\to\infty} L^{\alpha}\,\mathbb{P}\big[B(0,L) \stackrel{\geq h}{\longleftrightarrow} S(0,2L)\big] = 0\big\}. \tag{2.77}$$

Hence $\mathbb{P}\big[B(0,L) \stackrel{\geq h}{\longleftrightarrow} S(0,2L)\big]$ has stretched exponential decay in L when $h > \tilde{h}_{**}$, and one has in fact the equality

$$h_{**} = \tilde{h}_{**}. \tag{2.78}$$

□

3 Positivity of h_* in high dimension

The main goal of this section is the proof of Theorem 3.3 below, which roughly states that in high dimension, for small but *positive* h, the excursion set $E_\varphi^{\geq h}$ contains an infinite cluster with probability 1. We will prove the stronger statement that percolation already occurs in a two-dimensional slab $\mathbb{Z}^2 \times [0, 2L_0) \times \{0\}^{d-3} \subset \mathbb{Z}^d$ for sufficiently large L_0, see (3.20) below.

The proof essentially relies on two main ingredients. The first ingredient is a suitable decomposition, for large d, of the free field φ restricted to $\mathbb{Z}^3$ (viewed as a subset of $\mathbb{Z}^d$), into the sum of two independent Gaussian fields ψ and ξ (c.f. (3.12) and (3.13) below for their precise definition). The field ψ is i.i.d. and the dominant part, while ξ only acts as a "perturbation." The key step towards this decomposition appears in Lemma 3.1.

The second ingredient is a Peierls-type argument, which comprises several steps: first, the sublattice $\mathbb{Z}^3$ is partitioned into blocks of side length L_0, which are declared "good" if certain events defined separately for ξ and ψ occur simultaneously. Roughly speaking, these events are chosen in a way that suitable excursion sets of the dominant field ψ percolate well and the perturbative part ξ doesn't spoil this percolation (see (3.26), (3.27) and (3.28) for precise definitions). Moreover, $*$-connected components of bad blocks are shown to have small probability, see Lemma 3.5. This ensures that the usual method of Peierls contours is applicable, which in turn allows the conclusion that an infinite cluster of $E_\varphi^{\geq h}$ exists within the above-mentioned slab with positive probability (and with probability 1 by ergodicity). We note that ξ doesn't have finite-range dependence, which renders impractical the use of certain well-known stochastic domination theorems (see for example [14], [18]).

One word on notation: in what follows, we identify $\mathbb{Z}^k$, $k = 2, 3$, with the set of points $(x^1, \dots, x^d) \in \mathbb{Z}^d$ satisfying $x^{k+1} = x^{k+2} = \dots = x^d = 0$. We recall that in this section, constants are numerical unless dependence on additional parameters is explicitly indicated. Moreover, we shall assume throughout this section that

$$d \geq 6. \tag{3.1}$$

We also recall that $g(\cdot,\cdot)$, c.f. (1.1), stands for the Green function on $\mathbb{Z}^d$. Without further ado, we begin with

Lemma 3.1. (*Covariance decomposition*)
Let $K = \mathbb{Z}^3$. There exists a function g' on $K \times K$ such that

$$g(x,y) = \sigma^2(d) \cdot \delta(x,y) \; + \; g'(x,y), \qquad \text{for all } x,y \in K, \tag{3.2}$$

where $1/2 \leq \sigma^2(d) < 1$, $\sigma^2(d) \to 1$ as $d \to \infty$, $\delta(\cdot,\cdot)$ denotes the Kronecker symbol, and g' is the kernel of a translation invariant, bounded, positive operator G' on $\ell^2(K)$, which is the operator of convolution with $g'(\cdot,0)$. Its spectral radius $\rho(G')$ satisfies

$$\rho(G') \leq c_3/d. \tag{3.3}$$

Proof. The operator $Af(x) = \sum_{y \in K} g(x,y)f(y)$, for $x \in K$ and $f \in \ell^2(K)$, is a convolution operator, which is bounded and self-adjoint on $\ell^2(K)$, by (1.9), (3.1), as well as the translation invariance and the symmetry of $g(\cdot,\cdot)$ (letting $h(\cdot) = g(\cdot,0)$, we also use that $\|Af\|_{\ell^2(K)} = \|h * f\|_{\ell^2(K)} \leq \|h\|_{\ell^1(K)} \|f\|_{\ell^2(K)}$, a special case of Young's inequality, see [20], pp. 28-29). Moreover, by [22], P25.2 (b), p. 292, it has an inverse

$$A^{-1} = I - \Pi, \tag{3.4}$$

where Π is the bounded self-adjoint operator on $\ell^2(K)$, $\Pi f(x) = \sum_{y \in K} \pi(x,y) f(y)$, for $x \in K$ and $f \in \ell^2(K)$, with kernel

$$\pi(x,y) = P_x[\widetilde{H}_K < \infty, X_{\widetilde{H}_K} = y] = \pi(0, y-x), \qquad \text{for } x,y \in K. \tag{3.5}$$

Introducing

$$\kappa = P_0[\widetilde{H}_K = \infty] \in (0,1) \qquad \text{(recall (3.1))}, \tag{3.6}$$

we can write

$$A^{-1} = \kappa I + \big((1-\kappa)I - \Pi\big) \stackrel{\text{def.}}{=} \kappa I + \Gamma, \tag{3.7}$$

where Γ is the bounded self-adjoint operator on $\ell^2(K)$ defined by $\Gamma f(x) = \sum_{y \in K} \gamma(x,y) f(y)$, for $x \in K$, $f \in \ell^2(K)$, and

$$\gamma(x,y) = \begin{cases} -\pi(x,y), & \text{if } y \neq x \\ 1 - \kappa - \pi(x,x) \stackrel{(3.5),(3.6)}{=} P_x[\widetilde{H}_K < \infty, X_{\widetilde{H}_K} \neq x] = \sum_{y \neq x} \pi(x,y), & \text{if } y = x. \end{cases} \tag{3.8}$$

Note that by (3.5), (3.8), $\gamma(x,y) = \gamma(0, y-x)$ and Γ is a convolution operator on $\ell^2(K)$. By Young's inequality (see above (3.4)), its operator norm $\|\Gamma\|$ thus satisfies

$$\|\Gamma\| \leq \|\gamma(0,\cdot)\|_{\ell^1(K)} = \sum_{y \neq 0} \pi(0,y) + \sum_{y \neq 0} \pi(0,y) \leq 2P_0\big[\widetilde{H}_K < \infty\big] \stackrel{(3.6)}{=} 2(1-\kappa). \tag{3.9}$$

Observe also that Γ is a positive operator. Indeed, $(\Gamma f, f)_{\ell^2(K)} \overset{(3.8)}{=} \frac{1}{2}\sum_{x,y\in K} \pi(x,y)(f(x) - f(y))^2$, for all $f \in \ell^2(K)$, where $(\cdot,\cdot)_{\ell^2(K)}$ denotes the inner product in $\ell^2(K)$. By (3.4) and (3.7), we can write, for arbitrary, $a \in (0,1)$,

$$\begin{aligned} A = \big(\kappa I + \Gamma\big)^{-1} &= \kappa^{-1}\big(\kappa I + \Gamma - \Gamma\big)\big(\kappa I + \Gamma\big)^{-1} \\ &= \kappa^{-1}\big[I - \Gamma\big(\kappa I + \Gamma\big)^{-1}\big] \\ &= \kappa^{-1}\big[(1-a)I + T_a\big], \end{aligned} \tag{3.10}$$

where T_a is the bounded operator on $\ell^2(K)$,

$$T_a = aI - \Gamma\big(\kappa I + \Gamma\big)^{-1}, \tag{3.11}$$

which is self-adjoint by (3.10), since A is. We will now select $a \in (0,1)$ in such a way that the operator T_a is positive. By self-adjointness, we know that the spectrum $\sigma(T_a)$ of T_a satisfies

$$\sigma(T_a) \subset [m(T_a), \rho(T_a)] \subseteq \mathbb{R},$$

where $m(T_a) = \inf\{(T_a f, f)_{\ell^2(K)}; \|f\|_{\ell^2(K)} = 1\}$ and $\rho(T_a) = \sup\{(T_a f, f)_{\ell^2(K)}; \|f\|_{\ell^2(K)} = 1\}$. By the spectral theorem, and using that the function $x \mapsto x/(\kappa + x)$ is increasing in $x \geq 0$, it follows that

$$m(T_a) \geq a - \frac{\|\Gamma\|}{\kappa + \|\Gamma\|} \overset{(3.9)}{\geq} a - \frac{2(1-\kappa)}{2-\kappa}.$$

Selecting $a_0 = 2(1-\kappa)/(2-\kappa)$, we see that T_{a_0} is a positive operator. Moreover, the application of the spectral theorem and the positivity of Γ also show that $\rho(T_{a_0}) \leq a_0$. If we now define $\sigma^2(d) = \kappa^{-1}(1-a_0) = 1/(2-\kappa) \in \big(\frac{1}{2}, 1\big)$ (by (3.6)), and $G' = \kappa^{-1}T_{a_0}$, so that $G'f(x) = \sum_{y\in K} g'(x,y)f(y)$ for $x \in K$ and $f \in \ell^2(K)$, then (3.10) readily yields (3.2). In addition, G' is translation invariant, and by (1.11), we see that $\sigma^2(d)$ tends to 1 as $d \to \infty$, and the spectral radius of G' satisfies

$$\rho(G') \leq \kappa^{-1}\rho(T_{a_0}) \leq \kappa^{-1}a_0 = \frac{2}{\kappa(2-\kappa)} \cdot (1-\kappa) \overset{(1.11)}{\leq} c_3/d,$$

for a suitable constant $c_3 > 0$. This concludes the proof of Lemma 3.1. □

We now decompose the Gaussian free field according to Lemma 3.1. To this end, we let $\mathbb{P}_\psi$, $\mathbb{P}_\xi$, be probabilities on auxiliary probability spaces Ω_ψ, Ω_ξ, respectively endowed with random fields $(\psi_x)_{x\in\mathbb{Z}^3}$, $(\xi_x)_{x\in\mathbb{Z}^3}$, such that

(3.12) under $\mathbb{P}_\psi$, $(\psi_x)_{x\in\mathbb{Z}^3}$, is a centered Gaussian field with covariance $\mathbb{E}_\psi[\psi_x\psi_y] = \sigma^2(d)\cdot\delta(x,y)$, for all $x, y \in \mathbb{Z}^3$,

and

(3.13) under $\mathbb{P}_\xi$, $(\xi_x)_{x\in\mathbb{Z}^3}$, is a centered Gaussian field with covariance $\mathbb{E}_\xi[\xi_x\xi_y] = g'(x,y)$, for all $x, y \in \mathbb{Z}^3$.

Then, by (3.2) and usual Gaussian field arguments (see [1], p.11),

(3.14) $(\varphi_x)_{x\in\mathbb{Z}^3}$, under $\mathbb{P}$, has the same law as $(\psi_x + \xi_x)_{x\in\mathbb{Z}^3}$, under $\mathbb{P}_\psi \otimes \mathbb{P}_\xi$.

Moreover, given any level $h \in \mathbb{R}$, we define the (random) sets

$$E_\psi^{\geq h} = \{x \in \mathbb{Z}^3 \ ; \ \psi_x \geq h\}, \qquad E_\psi^{<h} = \mathbb{Z}^3 \setminus E_\psi^{\geq h}, \tag{3.15}$$

and $E_\xi^{\geq h}$, $E_\xi^{<h}$ in analogous manner. A crucial point is that the field ξ acts only as a small perturbation when d is large, which is entailed in (3.3) and more quantitatively in the following lemma, the proof of which uses ideas developed in [17] (see in particular Theorem 2.4 therein).

Lemma 3.2.
There exists a decreasing function $v : (c_3, \infty] \longrightarrow (0,1)$, *with* $\lim_{u\to\infty} v(u) = 0$, *such that for all* $h > 0$ *and* $d \geq 6$ *satisfying* $h^2 > c_3 d^{-1}$, *and all* $A \subset\subset \mathbb{Z}^3$,

$$\mathbb{P}_\xi\Big[\bigcap_{x\in A}\{|\xi_x| > h\}\Big] \leq \big[v(h^2 d)\big]^{|A|}. \tag{3.16}$$

Proof. First note that

$$\mathbb{P}_\xi\Big[\bigcap_{x\in A}\{|\xi_x| > h\}\Big] \leq \mathbb{P}_\xi\Big[\sum_{x\in A}\xi_x^2 > h^2|A|\Big], \qquad \text{for all } h > 0. \tag{3.17}$$

Now, assume some ordering of $A \subset\subset \mathbb{Z}^3$ has been specified, and let $G'_A = \big(g'(x,y)\big)_{x,y\in A}$ denote the covariance matrix of the Gaussian vector $\xi_A = (\xi_x)_{x\in A}$, with decreasing eigenvalues $\lambda_i \geq 0$, $1 \leq i \leq |A|$, and write $\rho_A = \rho(G'_A) = \lambda_1$ for its spectral radius. Finally, define the diagonal matrix $\Lambda = \mathrm{diag}(\{\lambda_i\})$. By spectral decomposition, $G'_A = O\Lambda O^T$ for some orthogonal matrix O. Let $\tilde{\xi} = (\tilde{\xi}_i)_{1\leq i\leq |A|}$ be a Gaussian vector whose components are i.i.d. standard Gaussian variables, and $\mathbb{P}_{\tilde{\xi}}$ be its law. Then $O\sqrt{\Lambda}\tilde{\xi} \sim \mathcal{N}(0, O\Lambda O^T)$, i.e. $O\sqrt{\Lambda}\tilde{\xi} \overset{d}{=} \xi_A$, and thus $\sum_{x\in A}\xi_x^2 \overset{d}{=} (O\sqrt{\Lambda}\tilde{\xi})^T O\sqrt{\Lambda}\tilde{\xi} = \sum_{1\leq i\leq |A|}\lambda_i\tilde{\xi}_i^2$. Inserting this into (3.17) yields

$$\mathbb{P}_\xi\Big[\bigcap_{x\in A}\{|\xi_x| > h\}\Big] \leq \mathbb{P}_{\tilde{\xi}}\Big[\sum_{1\leq i\leq |A|}\tilde{\xi}_i^2 > \rho_A^{-1}h^2|A|\Big] \leq \min_{0<a<1} e^{-\frac{ah^2|A|}{2\rho_A}}\mathbb{E}_{\tilde{\xi}}\Big[\prod_{1\leq i\leq |A|} e^{\frac{a}{2}\tilde{\xi}_i^2}\Big],$$

where we have used Markov's inequality in the last step. But $\mathbb{E}_{\tilde{\xi}}[e^{a\tilde{\xi}_i^2/2}] = (1-a)^{-1/2}$ for all $0 < a < 1$ and $1 \leq i \leq |A|$, thus yielding

$$\mathbb{P}_\xi\Big[\bigcap_{x\in A}\{|\xi_x| > h\}\Big] \leq \min_{0<a<1}\Big[\sqrt{1-a}\cdot e^{\frac{ah^2}{2\rho_A}}\Big]^{-|A|}. \tag{3.18}$$

One easily verifies that the function $q(a) = \sqrt{1-a}\cdot e^{ah^2/2\rho_A}$ attains a maximum in the interval $(0,1)$ only if $h^2/\rho_A > 1$, which certainly holds if $h^2 > c_3 d^{-1}$ by Lemma 3.1. In this regime, the maximum is reached for $a = 1 - \rho_A/h^2$. Inserting this into (3.18), we obtain

$$\mathbb{P}_\xi\Big[\bigcap_{x\in A}\{|\xi_x| > h\}\Big] \leq \Big[\big(eh^2\rho_A^{-1}\big)^{1/2}\cdot e^{-\frac{h^2}{2\rho_A}}\Big]^{|A|}.$$

The function $\tilde{v}(u) = (2eu)^{1/2}e^{-u}$ is $(0,1)$-valued and monotonically decreasing on $(1/2,\infty)$. Thus, $h^2 > c_3 d^{-1}$ ensures that $h^2/2\rho_A > 1/2$ and $\tilde{v}(h^2/2\rho_A) \leq \tilde{v}(h^2 d/2c_3) \overset{\text{def.}}{=} v(h^2 d)$. This completes the proof of Lemma 3.2. □

We are now ready to introduce a central quantity before proceeding to the main theorem of this section. Namely, we define, for all $d \geq 6$, $h \in \mathbb{R}$ and positive integers L_0,

$$\Psi^{(slab)}(d,h,L_0) = \mathbb{P}\big[E_\varphi^{\geq h} \cap \big(\mathbb{Z}^2\times[0,2L_0)\times\{0\}^{d-3}\big) \text{ contains an infinite cluster }\big]. \tag{3.19}$$

Theorem 3.3.
There exists $d_0 \geq 6$, *a level* $h_0 > 0$ *and an integer* $L_0 \geq 1$ *such that*

$$\Psi^{(slab)}(d,h_0,L_0) = 1, \qquad \textit{for all } d \geq d_0. \tag{3.20}$$

In particular, the critical level $h_*(d)$ *defined in* (0.4) *satisfies*

$$h_*(d) \geq h_0 > 0, \qquad \textit{for all } d \geq d_0. \tag{3.21}$$

Proof. To begin with, we note that (3.21) immediately follows from (3.20). In order to prove (3.20), we will use the decomposition (3.14) of the Gaussian free field restricted to $\mathbb{Z}^3$ and the bounds obtained in Lemma 3.2 to perform a Peierls-type argument.

We let $\mathbf{P}_p = (p\delta_1 + (1-p)\delta_0)^{\otimes \mathbb{Z}^3}$, for $p \in (0,1)$, and observe that the law of $\big(1\{\psi_x \geq h\}\big)_{x\in\mathbb{Z}^3}$ on $\{0,1\}^{\mathbb{Z}^3}$ (endowed with its canonical σ-algebra) is $\mathbf{P}_{p(h,\sigma(d))}$, where

$$p(h,\sigma) \stackrel{\text{def.}}{=} \frac{1}{\sqrt{2\pi}\sigma}\int_h^\infty e^{-x^2/2\sigma^2}\mathrm{d}x, \qquad \text{for } h,\sigma > 0, \tag{3.22}$$

so that $p(h,\sigma(d)) = \mathbb{P}_\psi[\psi_0 \geq h]$ (see (3.12)). For arbitrary $h > 0$, $L_0 \geq 1$, $d \geq 6$ and $\omega_\xi \in \Omega_\xi$, we define the increasing event (part of $\{0,1\}^{\mathbb{Z}^3}$),

$$A^h_\infty(\omega_\xi) = \big\{\omega \in \{0,1\}^{\mathbb{Z}^3} \ ; \ \{x\in\mathbb{Z}^3; \omega_x = 1\} \cap E_\xi^{\geq -h}(\omega_\xi) \cap (\mathbb{Z}^2 \times [0,2L_0)\big) \text{ has an infinite cluster}\big\},$$

and obtain, for all $p' \leq p(2h,\sigma(d))$, using the decomposition (3.14),

$$\begin{aligned} \Psi^{(slab)}(d,h,L_0) &\geq \mathbb{P}_\psi \otimes \mathbb{P}_\xi\big[E_\psi^{\geq 2h} \cap E_\xi^{\geq -h} \cap \big(\mathbb{Z}^2\times[0,2L_0)\big) \text{ contains an infinite cluster }\big] \\ &= \int_{\Omega_\xi} \mathrm{d}\mathbb{P}_\xi(\omega_\xi)\, \mathbf{P}_{p(2h,\sigma(d))}\,\big[A^h_\infty(\omega_\xi)\big] \\ &\geq \int_{\Omega_\xi} \mathrm{d}\mathbb{P}_\xi(\omega_\xi)\, \mathbf{P}_{p'}\,\big[A^h_\infty(\omega_\xi)\big], \end{aligned} \tag{3.23}$$

where we have used that $A^h_\infty(\omega_\xi)$ is increasing in ω for every fixed ω_ξ in the last line. Since $\sigma^2(d) \geq 1/2$ for $d \geq 6$, we have $p(2h,\sigma(d)) \geq p(2h, 1/\sqrt{2}) = p'$ for all $h > 0$ and $d \geq 6$. Thus, (3.23) yields

$$\Psi^{(slab)}(d,h,L_0) \geq \mathbb{P}_{\psi^0} \otimes \mathbb{P}_\xi\big[E_{\psi^0}^{\geq 2h} \cap E_\xi^{\geq -h} \cap \big(\mathbb{Z}^2\times[0,2L_0)\big) \text{ contains an infinite cluster }\big], \tag{3.24}$$

for all $d \geq 6$, $h > 0$ and $L_0 \geq 1$, where $\psi^0 = (\psi^0_x)_{x\in\mathbb{Z}^3}$ is a field of independent centered Gaussian variables with variance $1/2$, i.e. as in (3.12) but with $1/2$ in place of $\sigma^2(d)$, and $\mathbb{P}_{\psi^0}$ denotes the probability on $(\Omega_0, \mathcal{A}_0)$ governing ψ^0. We will prove that the probability on the right-hand-side of (3.24) is equal to one for all $d \geq d_0$, $0 < h \leq h_0$, with suitable d_0, h_0 and L_0. The claim (3.20) will immediately follow from this.

We first construct certain families of "good" events for the Gaussian fields ψ^0 and ξ. To this end, we define boxes $B^{(3)}_x(L) = x + \big([0,L)\cap\mathbb{Z}\big)^3$ for any $x \in \mathbb{Z}^3$ and positive integer L, and introduce a renormalized lattice

$$\mathscr{L} = L_0\mathbb{Z}^2 \ \big(\subset \mathbb{Z}^3\big). \tag{3.25}$$

We begin with ψ^0 and define $\mathcal{C}_x(\omega)$, for any $x \in \mathscr{L}$ and $\omega \in \{0,1\}^{\mathbb{Z}^3}$, to be the (possibly empty) open (i.e. value 1) cluster in $B^{(3)}_x(2L_0)$ containing the most vertices. If several such clusters exist, we choose one according to some given prescription (say using lexicographic order). For arbitrary $x \in \mathscr{L}$, we introduce the event F_x (in the canonical σ-algebra of $\{0,1\}^{\mathbb{Z}^3}$) as follows: given some $\omega \in \{0,1\}^{\mathbb{Z}^3}$, we let $\omega \in F_x$ if and only if

i) $\mathcal{C}_x(\omega)$ is a *crossing* cluster for $B^{(3)}_x(2L_0)$ in the first two axes-directions, i.e. for $i = 1,\ 2$, there exists an open path γ_i in $\mathcal{C}_x(\omega)$ with endvertices $y_{(i)}, z_{(i)}$ satisfying $y^i_{(i)} = x^i$ and $z^i_{(i)} = x^i + 2L_0 - 1$.

ii) $\mathcal{C}_x(\omega)$ is the *only* open cluster $\mathcal{C}$ of $B^{(3)}_x(2L_0)$ having the property $\mathrm{diam}(\mathcal{C}) \geq L_0 - 1$.

Having introduced F_x for all $n \geq 0$ and $x \in \mathscr{L}$, we define, for $h > 0$, the measurable map $\Phi_0^h : \Omega_0 \longrightarrow \{0,1\}^{\mathbb{Z}^3}$, $\omega_0 \mapsto \big(1\{\omega_{0_x} \geq h\}\big)_{x\in\mathbb{Z}^3}$, and the events (in $\mathcal{A}_0$)

$$F_x^h = (\Phi_0^h)^{-1}(F_x), \qquad \text{for all } h > 0 \text{ and } x \in \mathscr{L}\,. \tag{3.26}$$

We now turn to the ("good") events for ξ, set

$$G_x^{-h} = \bigcap_{y \in B_x^{(3)}(2L_0)} \{\xi_y \geq -h\}, \qquad \text{for all } x \in \mathscr{L},\ h > 0, \tag{3.27}$$

and note that $\mathbb{P}_\xi[G_x^{-h}] = \mathbb{P}_\xi[G_0^{-h}]$, for all $x \in \mathscr{L}$, by translation invariance of $g'(\cdot,\cdot)$, see (3.2).

With "good" events for ψ^0 and ξ at hand (see (3.26) and (3.27)), for any $h > 0$, we define a vertex $x \in \mathscr{L}$ to be *h-good* if the event

$$F_x^{2h} \times G_x^{-h} \tag{3.28}$$

occurs (under $\mathbb{P}_{\psi^0} \otimes \mathbb{P}_\xi$), and *h-bad* otherwise. A sequence $\gamma = (x_i)_{0\leq i \leq m}$, $m \in \mathbb{N} \cup \{\infty\}$, in $\mathscr{L}$ such that $|x_{i+1} - x_i| = L_0$ for all $0 \leq i < m$ will be called a *nearest-neighbor path in $\mathscr{L}$* (we will refer to m as the length of the path). Similarly, a *$*$-nearest-neighbor path in $\mathscr{L}$* is any sequence in $\mathscr{L}$ subject to the weaker condition $|x_{i+1} - x_i|_\infty = L_0$ for all $0 \leq i < m$.

A crucial property is that percolation of h-good sites in $\mathscr{L}$ implies percolation of $E_{\psi^0}^{\geq 2h} \cap E_\xi^{\geq -h}$ (in the slab $\mathbb{Z}^2 \times [0, 2L_0)$), which we state as a

Lemma 3.4.
Let $h > 0$, $\gamma = (x_i)_{0\leq i\leq m}$, $m \in \mathbb{N} \cup \{\infty\}$, be a nearest-neighbor path in $\mathscr{L}$ and assume that all vertices x_i, $0 \leq i \leq m$, are h-good. Then, the corresponding clusters $\mathcal{C}_{x_i}(\psi^0)$ (pertaining to the events $F_{x_i}^{2h}$) are subsets of $E_{\psi^0}^{\geq 2h} \cap E_\xi^{\geq -h}$, which are all connected within the set $\bigcup_{i=0}^m B_{x_i}^{(3)}(2L_0)$.

In particular, Lemma 3.4 implies that if $(x_i)_{i\geq 0}$ is an infinite nearest-neighbor path of h-good vertices in $\mathscr{L}$ which is unbounded, then the set $\bigcup_{i=0}^\infty B_{x_i}^{(3)}(2L_0) \cap E_{\psi^0}^{\geq 2h} \cap E_\xi^{\geq -h}$ (a subset of $\mathbb{Z}^2 \times [0, 2L_0)$) contains an infinite cluster.

Proof of Lemma 3.4. It suffices to consider the case $m = 2$. The general case then follows by induction on m. Thus, let $x_1, x_2 \in \mathscr{L}$, $|x_1 - x_2| = L_0$, be both h-good. The following holds for $i = 1, 2$: by definition of $F_{x_i}^{2h}$, the set $E_{\psi^0}^{\geq 2h} \cap B_{x_i}^{(3)}(2L_0)$ contains a cluster $\mathcal{C}_{x_i}$ $\big(= \mathcal{C}_{x_i}(\psi^0)\,\big)$ which is crossing in the first two axes-directions. Moreover, since $G_{x_i}^{-h}$ occurs, $\xi_y \geq -h$ for all $y \in \mathcal{C}_{x_i}$, i.e. $\mathcal{C}_{x_i} \subset E_\xi^{\geq -h}$.

It remains to show the clusters $\mathcal{C}_{x_1}$ and $\mathcal{C}_{x_2}$ are connected within $B_{x_1}^{(3)}(2L_0) \cup B_{x_2}^{(3)}(2L_0)$. Let $k \in \{1, 2\}$ be such that $|x_1^k - x_2^k| = L_0$. Since $\mathcal{C}_{x_1}$ is crossing for $B_{x_1}^{(3)}(2L_0)$ in the k-th direction, $\mathrm{diam}\big(\mathcal{C}_{x_1} \cap B_{x_2}^{(3)}(2L_0)\big) \geq L_0 - 1$. Hence $\mathcal{C}_{x_1}$ and $\mathcal{C}_{x_2}$ are connected within $B_{x_2}^{(3)}(2L_0)$ by condition ii) in the above definition of the events F_x. This concludes the proof of Lemma 3.4. □

We now carry on with the proof of Theorem 3.3, and select the parameters h_0, L_0 and d_0. First note that $p_c^{\mathrm{site}}(\mathbb{Z}^3)$, the critical level for Bernoulli site percolation on $\mathbb{Z}^3$, satisfies $p_c^{\mathrm{site}}(\mathbb{Z}^3) < 1/2$ (see [4], Theorem 4.1). We may thus choose $h_0 > 0$ such that

$$\mathbb{P}_{\psi^0}[\psi^0_{x=0} \geq 2h_0] \overset{(3.22)}{=} p\Big(2h_0, \frac{1}{\sqrt{2}}\Big) = \frac{1}{2}\Big(\frac{1}{2} + p_c^{\mathrm{site}}(\mathbb{Z}^3)\Big), \tag{3.29}$$

which means that the Bernoulli site percolation model on $\mathbb{Z}^3$ associated to choosing sites $x \in \mathbb{Z}^3$ where $\psi^0_x \geq 2h_0$, is supercritical. By the site-percolation version of Theorem 7.61 in [8], we thus obtain that

$$\lim_{L_0 \to \infty} \mathbb{P}_{\psi^0}\big[F_x^{2h_0}\big] = 1, \qquad \text{for all } x \in \mathscr{L}\,. \tag{3.30}$$

Moreover, the collection $\big(1\{F^{2h_0}_{L_0 y}\}\big)_{y\in\mathbb{Z}^2}$ is 2-dependent and, (see [8], Theorem 7.65, or [14], Theorem 0.0), there exists a non-decreasing function $\pi : [0,1] \longrightarrow [0,1]$ with $\lim_{\delta\to 1}\pi(\delta) = 1$ such that if $\mathbb{P}_{\psi^0}[F_0^{2h_0}] \geq \delta$, then

$$\begin{array}{l} \big(1\{F_x^{2h_0}\}\big)_{x\in\mathscr{L}} \quad \text{stochastically dominates a family of independent} \\ \text{Bernoulli random variables indexed by } \mathscr{L}\text{, with success parameter } \pi(\delta). \end{array} \tag{3.31}$$

Using (3.30), we then choose L_0 the smallest positive integer such that

$$\pi\big(\mathbb{P}_{\psi^0}[F_0^{2h_0}]\big) \geq 1 - 1/40. \tag{3.32}$$

Having fixed h_0 and L_0, we choose, in the notation of Lemma 3.2, a constant $c_4 > c_3$ such that

$$(2L_0)^3 \cdot v(u)^{1/4} \leq 1/40, \qquad \text{for all } u \geq c_4, \tag{3.33}$$

(recall that $v(\cdot)$ is monotonically decreasing) and define

$$d_0 = [h_0^{-2} c_4] + 1, \tag{3.34}$$

so that (3.33) and (3.34) yield

$$(2L_0)^3 \cdot v(h_0^2 d)^{1/4} \leq (2L_0)^3 \cdot v(h_0^2 d_0)^{1/4} \leq 1/40, \qquad \text{for all } d \geq d_0. \tag{3.35}$$

We now proceed to the last step of the proof, which mainly encompasses a Peierls argument. To this end, we introduce, for $x \in \mathscr{L}$ and N a multiple of L_0, the event $H^{h_0}(x, N)$ that x is connected to $\{y \in \mathscr{L}; |y - x|_\infty = N\}$, the restriction to $\mathscr{L}$ of the ℓ^∞-sphere of radius N centered around x, by a $*$-path of h_0-bad vertices in $\mathscr{L}$. In the following lemma, we show that this event has small probability.

Lemma 3.5.
For all $d \geq d_0$, $n \geq 1$, and $x \in \mathscr{L}$,

$$\mathbb{P}_{\psi^0} \otimes \mathbb{P}_\xi\big[H^{h_0}(x, nL_0)\big] \leq 2^{-n}. \tag{3.36}$$

Proof of Lemma 3.5. For $x \in \mathscr{L}$ and $n \geq 1$, we denote by $\Gamma^*_{x,n}$ the set of self-avoiding $*$-paths in $\mathscr{L}$ starting at x of length n. For $H^{h_0}(x, nL_0)$ to occur, there must be a self-avoiding $*$-path $\gamma = (x_i)_{0\leq i\leq n}$ of h_0-bad vertices in $\mathscr{L}$ starting at x, hence

$$\begin{aligned} \mathbb{P}_{\psi^0} \otimes \mathbb{P}_\xi\big[H^{h_0}(x, nL_0)\big] &\leq \mathbb{P}_{\psi^0} \otimes \mathbb{P}_\xi\Big[\bigcup_{\gamma\in\Gamma^*_{x,n}} \{\gamma \text{ is } h_0\text{-bad}\}\Big] \\ &\leq |\Gamma^*_{x,n}| \sup_{\gamma=(x_i)_{0\leq i\leq n}\in\Gamma^*_{x,n}} \mathbb{P}_{\psi^0} \otimes \mathbb{P}_\xi\Big[\bigcap_{i=1}^n \Big((F^{2h_0}_{x_i})^c \cup (G^{-h_0}_{x_i})^c\Big)\Big], \end{aligned} \tag{3.37}$$

where we have identified $(F^{2h_0}_{x_i})^c$ with $(F^{2h_0}_{x_i})^c \times \Omega_\xi$ in the last line (and similarly for $(G^{-h_0}_{x_i})^c$). For arbitrary $\gamma = (x_i)_{0\le i\le n} \in \Gamma^*_{x,n}$, the probability on the right-hand side of (3.37) is equal to (setting $[[n]] = \{1,\dots,n\}$)

$$\begin{aligned}(3.38)\quad &\mathbb{P}_{\psi^0}\otimes\mathbb{P}_\xi\Big[\bigcup_{k=0}^{n}\ \bigcup_{\{i_1,\dots,i_k\}\subset[[n]]}\ \bigcap_{j\in[[n]]\setminus\{i_1,\dots,i_k\}}(F^{2h_0}_{x_j})^c\ \bigcap_{i\in\{i_1,\dots,i_k\}}(G^{-h_0}_{x_i})^c\Big]\\ &\le \sum_{k=0}^{n}\binom{n}{k}\sup_{\{i_1,\dots,i_k\}\subset[[n]]}\mathbb{P}_{\psi^0}\Big[\bigcap_{j\in[[n]]\setminus\{i_1,\dots,i_k\}}(F^{2h_0}_{x_j})^c\Big]\cdot\mathbb{P}_\xi\Big[\bigcap_{i\in\{i_1,\dots,i_k\}}(G^{-h_0}_{x_i})^c\Big].\end{aligned}$$

By stochastic domination and our choice of L_0, c.f. (3.31) and (3.32), we have

$$(3.39)\qquad \mathbb{P}_{\psi^0}\Big[\bigcap_{j\in[[n]]\setminus\{i_1,\dots,i_k\}}(F^{2h_0}_{x_j})^c\Big]\le 40^{-(n-k)}.$$

When $(G^{-h_0}_{x_i})^c$, $i=i_1,\dots,i_k$, simultaneously occur, we can choose k sites z_j, $1\le j\le k$, in the respective boxes $B^{(3)}_{x_{i_j}}(2L_0)$, $1\le j\le k$, such that $\xi_{z_j} < -h_0$ (c.f. definition (3.27)). Any such z_j belongs to exactly four boxes $B^{(3)}_x(2L_0)$, $x\in\mathscr{L}$. Since $(x_i)_{0\le i\le n}$ is self-avoiding in $\mathscr{L}$, we thus have $|\{z_j\ ;\ 1\le j\le k\}|\ge k/4$. As a result, Lemma 3.2 yields

$$(3.40)\qquad \mathbb{P}_\xi\Big[\bigcap_{i\in\{i_1,\dots,i_k\}}(G^{-h_0}_{x_i})^c\Big]\le (2L_0)^{3k}\cdot\big(v(h_0^2d_0)\big)^{\frac{k}{4}}\overset{(3.35)}{\le} 40^{-k}.$$

Note that $|\Gamma^*_{x,n}|\le(3^2-1)^n = 8^n$. Putting (3.38), (3.39) and (3.40) together, and substituting the resulting bound into (3.37), we finally obtain,

$$\mathbb{P}_{\psi^0}\otimes\mathbb{P}_\xi\big[H^{h_0}(x,nL_0)\big]\le 8^n\sum_{k=0}^{n}\binom{n}{k}40^{-k}40^{-(n-k)} = (2/5)^n < 2^{-n}.$$

This completes the proof of Lemma 3.5. □

We can now conclude the proof of Theorem 3.3. For arbitrary $d\ge d_0$, we consider the event that the set $\mathscr{L}\cap[0,2L_0)^2$ is surrounded by a $*$-circuit (i.e. a self-avoiding $*$-path except for the end point which coincides with the starting point) of h_0-bad vertices in $\mathscr{L}$. Considering a point of this circuit on the first axis with largest coordinate, we see that the probability of this event is bounded by

$$(3.41)\qquad \sum_{n=2}^{\infty}\mathbb{P}_{\psi^0}\otimes\mathbb{P}_\xi[H^{h_0}(0,nL_0)]\overset{(3.36)}{\le}\sum_{n=2}^{\infty}2^{-n}<1.$$

If this event does not occur, then by planar duality (c.f. [8], Section 11.2), there exists an infinite self-avoiding nearest-neighbor path $\gamma=(x_i)_{i\ge0}$ of h_0-good vertices in $\mathscr{L}$. Lemma 3.4 (see in particular the remark following it) then implies that the set

$$\Big(\bigcup_{i=0}^{\infty}B^{(3)}_{x_i}(2L_0)\cap E^{\ge 2h_0}_{\psi^0}\cap E^{\ge -h_0}_{\xi}\Big)\quad\subset\ \mathbb{Z}^2\times[0,2L_0)$$

contains an infinite cluster. By (3.41), this event happens with positive probability, so that

$$\mathbb{P}_{\psi^0}\otimes\mathbb{P}_\xi\big[E^{\ge 2h_0}_{\psi^0}\cap E^{\ge -h_0}_{\xi}\cap\big(\mathbb{Z}^2\times[0,2L_0)\big)\text{ contains an infinite cluster }\big]>0,\ \text{ for all } d\ge d_0.$$

It then follows by (3.24) and ergodicity (see Lemma 1.5) that the value of $\Psi^{(slab)}(d,h_0,L_0)$ is in fact one. This proves (3.20) and thus concludes the proof of Theorem 3.3. □

Remark 3.6.

1) We show in Theorem 3.3 that $E_{\varphi}^{\geq h}$ percolates in a two-dimensional slab for small but positive h when d is sufficiently large. However, it should be underlined that $E_{\varphi}^{\geq h} \cap \mathbb{Z}^2$ does not percolate for any $h \geq 0$, as we now briefly explain. Indeed, the conditions of Theorem 14.3 in [10] (which is itself a variant of [7] when the finite energy condition holds) are met for the law of $\big(1\{\varphi_x \geq 0\}\big)_{x\in\mathbb{Z}^2}$ on $\{0,1\}^{\mathbb{Z}^2}$ under $\mathbb{P}$ (in particular, positive correlations, see Definition 14.1 in [10], follow from the FKG-inequality). Hence $E_{\varphi}^{\geq 0} \cap \mathbb{Z}^2$ and its complement in $\mathbb{Z}^2$ cannot *both* have infinite clusters. Observe that $\big(1\{\varphi_x \geq 0\}\big)_{x\in\mathbb{Z}^2}$ and $\big(1\{\varphi_x < 0\}\big)_{x\in\mathbb{Z}^2}$ have the same law under $\mathbb{P}$, by symmetry. If $E_{\varphi}^{\geq 0} \cap \mathbb{Z}^2$ percolated (with probability one, by ergodicity), the same would hold true as well for $E_{\varphi}^{<0} \cap \mathbb{Z}^2$, leading to a contradiction. Therefore, $E_{\varphi}^{\geq 0} \cap \mathbb{Z}^2$ does not percolate.

2) With Lemma 3.2 at hand, one may immediately apply the criterion of Molchanov and Stepanov (c.f. [17], Theorem 2.1) to infer that $E_{\xi}^{\geq -h}$ (c.f. (3.15) for notation) percolates *strongly* in $\mathbb{Z}^3$ when $h^2 d$ is sufficiently large, i.e. not only does $E_{\xi}^{\geq -h}$ percolate, but in addition $E_{\xi}^{< -h}$ doesn't, $\mathbb{P}_{\xi}$-almost surely. However, we note that $E_{\psi}^{\geq h}$ does not percolate strongly, since for all $h > 0$, we have $p_c = p_c^{\text{site}}(\mathbb{Z}^3) < p(h, \sigma(d)) = \mathbb{P}_{\psi}[\psi_0 \geq h] < 1/2$, hence in particular $p(h,\sigma(d)) \in (p_c, 1-p_c)$, where both $E_{\psi}^{\geq h}$ and its complement possess an infinite cluster with positive probability (in fact with probability one).

3) It remains open whether $h_*(d)$ is actually strictly positive for *all* $d \geq 3$. Recent simulations suggest this is the case when $d = 3$, with an approximate value $\mathbb{P}[\varphi_0 \geq h_*] \simeq 0.16$, see [15], Section 4.1.2, and Figure 4.1 in Appendix 4.4. □

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