ON $(\mathbb{Z}/N\mathbb{Z})^2$-OCCUPATION TIMES, THE GAUSSIAN FREE FIELD, AND RANDOM INTERLACEMENTS

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Abstract

We study the occupation times left by random walk on $(\mathbb{Z}/N\mathbb{Z})^2$ at times either proportional to $N^2 \log N$ or much larger than $N^2 \log N$, and relate these random fields to the Gaussian free field pinned at the origin. Our results answer a question raised in [18] and mirror limit statements in [18] for the occupation times of large rods of size N in $\mathbb{Z}^3$ by random interlacements at a level u_N such that $u_N N^3$ is either proportional to $N^2 \log N$ or much larger than $N^2 \log N$.

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0 Introduction

In this article, we consider simple random walk with unit jump rate, and uniformly distributed starting point, on a large two-dimensional discrete torus $\mathbb{T}_N = (\mathbb{Z}/N\mathbb{Z})^2$. We study the random field of occupation times left by the walk close to the origin, when the walk runs for times either proportional to $N^2 \log N$, or much larger than $N^2 \log N$. We relate this random field, in the large N limit, to the two-dimensional Gaussian free field pinned at the origin, by looking at scaled differences of occupation times. The results we prove answer positively a question raised in [18]: the limit theorems we derive in the present work mirror the statements obtained in [18] concerning occupation times of long rods of size N in $\mathbb{Z}^3$ by random interlacements at a level u_N such that $u_N N^3$ is either proportional to $N^2 \log N$, or much larger than $N^2 \log N$. They signal the presence of a link between random interlacements and random walk on a large two-dimensional torus.

We now discuss the problem treated in this work in more detail, but we refer to Section 1 for the precise set-up. We consider X_t, $t \geq 0$, the canonical continuous time simple random walk on $\mathbb{T}_N$, with jump rate equal to 1, and uniform starting distribution. We write P for its canonical law and P_x for the canonical law, of the walk starting at $x \in \mathbb{T}_N$. The field of occupation times of the walk is defined as:

$$L_t^x = \int_0^t 1\{X_s = x\}\, ds, \text{ for } x \in \mathbb{T}_N, t \geq 0\,. \tag{0.1}$$

We tacitly view L_t^x as a periodic function of x, and still write L_t^y when $y \in \mathbb{Z}^2$, in place of L_t^x with x the canonical projection of y on $\mathbb{T}_N$. We are interested in two types of time scales when N goes to infinity:

$$\begin{aligned} &\text{i)} \;\; t_N = \alpha N^2 \log N, && \text{with } \alpha > 0, \\ &\text{ii)} \;\; t'_N = \alpha_N\, N^2 \log N, && \text{with } \lim_N \alpha_N = \infty. \end{aligned} \tag{0.2}$$

The choice (0.2) i) corresponds to a non-vanishing limit $e^{-\alpha \frac{\pi}{2}}$ of the probability that the occupation time at a given point of $\mathbb{T}_N$, say the origin, vanishes, see Remark 3.3. On the other hand, the choice (0.2) ii) corresponds to a vanishing limit for this probability. We refer to the time scale (0.2) i) as the Poissonian regime, and to (0.2) ii) as the ergodic regime.

The limit theorems we derive bring into play the two-dimensional Gaussian free field pinned at the origin, that is, the centered Gaussian field $(\psi_x)_{x\in\mathbb{Z}^2}$ with covariance $a(x) + a(x') - a(x' - x)$, for $x, x' \in \mathbb{Z}^2$, with $a(\cdot)$ the potential kernel of the two-dimensional simple random walk, see (1.7), (1.36), and an independent random variable R having the law $\mathrm{BES}^0(\sqrt{\alpha}, \frac{1}{\pi})$ of a zero-dimensional Bessel process at time $\frac{1}{\pi}$ starting in $\sqrt{\alpha}$ at time 0, see (1.37). Our main results are the following. In the Poissonian regime (0.2) i), we show in Theorems 3.1 and 3.7 that under P, when N tends to infinity,

$$\Big(\frac{L_{t_N}^x}{\log N}\Big)_{x\in\mathbb{Z}^2} \text{ converges in distribution to the flat field with value } R^2, \tag{0.3}$$

and

$$\Big(\frac{L_{t_N}^x - L_{t_N}^0}{\sqrt{2\log N}}\Big)_{x\in\mathbb{Z}^2} \text{ converges in distribution to the random field } (R\psi_x)_{x\in\mathbb{Z}^2}. \tag{0.4}$$

In the (simpler) case of the ergodic regime (0.2) ii), we instead show in Theorems 4.1 and 4.3 that when N goes to infinity,

$$(0.5) \qquad \Big(\frac{L^x_{t'_N}}{t'_N N^{-2}}\Big)_{x\in\mathbb{Z}^2} \text{ converges in distribution to the flat field with value 1,}$$

and that

$$(0.6) \qquad \Big(\frac{L^x_{t'_N} - L^0_{t'_N}}{\sqrt{2t'_N N^{-2}}}\Big)_{x\in\mathbb{Z}^2} \text{ converges in distribution to } (\psi_x)_{x\in\mathbb{Z}^2}.$$

The above results provide a positive answer to the question raised in Remark 4.10 of [18] concerning the pertinence of the limit theorems derived in that article for the occupation times of long rods in $\mathbb{Z}^3$ by random interlacements, to reflect an analogous behavior of the occupation times of simple random walk on $\mathbb{T}_N$ at time scales such as in (0.2). We refer to Remarks 3.8 1) and 4.4 for more on this topic.

Let us say a few words concerning proofs. An important role is played by the successive returns to 0 and departures from the box with side-length $\frac{N}{2}$ centered at the origin, for the walk on $\mathbb{T}_N$. Our main results (0.3) - (0.6) can be recast in terms of limit statements for suitable additive functionals of the walk, which are well approximated by discrete sums collecting the contributions of the additive functionals along the above mentioned excursions, cf. Proposition 2.2. The analysis of these discrete sums is more involved in the Poissonian regime (0.2) i) than in the simpler ergodic regime (0.2) ii). In the case of (0.4) the heart of the matter appears in Theorem 3.2, which derives the joint limit law of the duration of an excursion and the contribution of the additive functional during the excursion. Interestingly, the proof of Theorem 3.2 contains some ingredients, see in particular Lemma 3.6, which, although simpler, are quite reminiscent of what was done in Theorem 4.1 of [18] for occupation times of random interlacements. We refer to Remark 3.8 2) for potential alternative routes to the result described here, either by means of random interlacements, or generalized Ray-Knight theorems, see chapter 8 of [13].

Let us now describe how this article is organized.

In Section 1 we introduce additional notation and collect several results concerning the random walk on $\mathbb{T}_N$ and the above mentioned system of excursions, cf. Lemmas 1.2 and 1.3. We also recall some facts about discrete potential theory, the Gaussian free field pinned at the origin, and zero-dimensional Bessel processes.

Section 2 contains some preparation. It introduces the various additive functionals entering the proof of (0.3) - (0.6), and relates them to discrete sums collecting their contributions along excursions, cf. Proposition 2.2.

In Section 3 we study the Poissonian regime and derive (0.4), (0.3) in Theorems 3.1 and 3.7. Most of the work goes into the proof of Theorem 3.2, where the limit joint law of the duration of an excursion and the contribution of the relevant additive functional is analyzed. In Remark 3.8 we explain how these results compare with Theorem 4.2 in [18], in the case of occupation times of long rods by random interlacements in $\mathbb{Z}^3$.

The last Section 4 is devoted to the simpler ergodic regime. We prove (0.6) and (0.5) in Theorems 4.1 and 4.3. The comparison with the results obtained in Theorem 4.9 of [18] in the case of occupation times of long rods by random interlacements in $\mathbb{Z}^3$ is discussed in Remark 4.4.

Finally let us state our convention concerning constants. We denote by $c, c', \widetilde{c}, \overline{c}$ positive constants with value changing from place to place. Numbered constants refer to the value corresponding to their first appearance in the text. In Section 1 constants are numerical, but from Section 2 onward they depend on Λ in (2.1) and V in (2.3).

1 Set-up and some useful facts

In this section we introduce additional notation. We recall various facts concerning simple random walk on $\mathbb{T}_N$, discrete potential theory, the two-dimensional Gaussian free field pinned at the origin, and zero-dimensional Bessel processes. We introduce in (1.12) the system of excursions of the walk on $\mathbb{T}_N$, which plays an important role throughout the article, and derive some of its properties in Lemmas 1.2 and 1.3.

We denote by $\mathbb{N} = \{0, 1, \dots\}$ the set of natural numbers. When u is a non-negative real number, we let $[u]$ stand for the integer part of u. Given a finite set A, we write $|A|$ for its cardinality. We denote by $p_N(\cdot)$ the canonical map from $\mathbb{Z}^2$ onto $\mathbb{T}_N$. We write $|\cdot|$ for the Euclidean norm on $\mathbb{R}^2$, and $d(\cdot,\cdot)$ for the distance on $\mathbb{T}_N$ or $\mathbb{Z}^2$ induced by the sup-norm $|\cdot|_\infty$. For A, A' subsets of $\mathbb{T}_N$ or $\mathbb{Z}^2$, we write $d(A, A') = \inf\{d(x, x');$ $x \in A, x' \in A'\}$ for the mutual distance between A and A'. When $A = \{x\}$, we write $d(x, A')$ for simplicity. We denote by $B(x, r)$ the closed ball with center x (in $\mathbb{T}_N$ or $\mathbb{Z}^2$) and radius $r \geq 0$ in the $d(\cdot,\cdot)$-distance. For $U \subseteq \mathbb{Z}^2$, we write $\partial U = \{x \in \mathbb{Z}^2 \backslash U;\ \exists x' \in U,$ $|x - x'| = 1\}$ for the boundary of U, $\partial_{\text{int}} U = \{x \in U;\ \exists x' \in \mathbb{Z}^2 \backslash U,\ |x - x'| = 1\}$ for the interior boundary of U, and $\overline{U} = U \cup \partial U$ for the closure of U. When $U \subseteq \mathbb{T}_N$ we define $\partial U, \partial_{\text{int}} U$, and $\overline{U}$ in a similar manner.

The canonical space for the continuous time simple random walk on $\mathbb{T}_N$ consists of the set of right-continuous trajectories from $\mathbb{R}_+$ into $\mathbb{T}_N$ with finitely many jumps on any finite interval. It is endowed with the canonical σ-algebra. We denote by X_t, $t \geq 0$, the canonical process and by θ_t, $t \geq 0$, the canonical shift. Given $U \subseteq \mathbb{T}_N$, we write $H_U = \inf\{t \geq 0; X_t \in U\}$, $\widetilde{H}_U = \inf\{t > 0;$ for some $s < t$, $X_s \neq X_0$, and $X_t \in U\}$, and $T_U = \inf\{t \geq 0; X_t \notin U\}$, for the entrance time of U, the hitting time of U, and the exit time from U. When $U = \{x\}$, we write H_x or $\widetilde{H}_x$ for simplicity.

We denote by P_x the canonical law of the simple random walk on $\mathbb{T}_N$ with exponential holding times of parameter 1, starting from $x \in \mathbb{T}_N$, and by E_x the corresponding expectation. When ρ is a measure on $\mathbb{T}_N$ we denote by E_ρ the measure $\sum_{x \in \mathbb{T}_N} \rho(x) P_x$ and by E_ρ the corresponding expectation. We write π for the uniform probability on $\mathbb{T}_N$, i.e.

$$\pi(x) = N^{-2}, \text{ for } x \in \mathbb{T}_N. \tag{1.1}$$

It is a reversible measure for the walk on $\mathbb{T}_N$. When $\rho = \pi$, we simply write P and E in place of P_π and E_π.

We consider the closed ball in the $d(\cdot,\cdot)$-distance with radius $\frac{N}{4}$ and center at the origin:

$$B = B\Big(0, \frac{N}{4}\Big) \subseteq \mathbb{T}_N\,. \tag{1.2}$$

The Green function of the walk on $\mathbb{T}_N$ killed when exiting B is defined by

$$g_B(x, y) = E_x\Big[\int_0^{T_B} 1\{X_s = y\} ds\Big], \text{ for } x, y \in \mathbb{T}_N\,. \tag{1.3}$$

It is a symmetric function of its arguments x and y. When $K \subset B$, the equilibrium measure of K relative to B is the measure $e_{K,B}$ concentrated on $\partial_{\text{int}} K$ defined by

$$e_{K,B}(x) = P_x[\widetilde{H}_K > T_B]\, 1_K(x), \text{ for } x \in \mathbb{T}_N\,. \tag{1.4}$$

Its total mass $\text{cap}_B(K)$ is the capacity of K relative to B. The measure $e_{K,B}$ satisfies the identity:

$$P_x[H_K < T_B] = \sum_{y \in \mathbb{T}_N} g_B(x, y)\, e_{K,B}(y), \text{ for } x \in \mathbb{T}_N. \tag{1.5}$$

It is also known that when $K' \subset K$, the measure $e_{K',B}$ is obtained from $e_{K,B}$ via the sweeping identity:

$$e_{K',B}(x) = P_{e_{K,B}}[H_{K'} < T_B, X_{H_{K'}} = x], \text{ for } x \in \mathbb{T}_N\,. \tag{1.6}$$

Our main interest in this work lies in the large N asymptotics of various quantities, and we will tacitly assume throughout that $N \geq c$ so that $\overline{B} \subseteq \mathbb{T}_N$ with its natural subgraph structure of $\mathbb{T}_N$ can be identified with $p_N^{-1}(\overline{B}) \cap (-\frac{N}{2}, \frac{N}{2})^2$ endowed with its subgraph structure of $\mathbb{Z}^2$ (recall that p_N stands for the canonical map: $\mathbb{Z}^2 \to \mathbb{T}_N$). We can thus view B and $\overline{B}$ as subsets of $\mathbb{Z}^2$, and in particular represent the restriction of $g_B(\cdot,\cdot)$ to $\overline{B} \times \overline{B}$ in terms of the simple random walk on $\mathbb{Z}^2$ killed when exiting B.

We then introduce the potential kernel of simple random walk in $\mathbb{Z}^2$, cf. (1.40), p. 37 of [10], or p. 121, 122, 148 of [15]:

$$a(y) = \lim_n \sum_{j=0}^{n} P_0^{\mathbb{Z}^2}[Y_j = 0] - P_0^{\mathbb{Z}^2}[Y_j = y], \text{ for } y \in \mathbb{Z}^2, \tag{1.7}$$

where $Y_j, j \geq 0$, stands for the canonical discrete time simple random walk on $\mathbb{Z}^2$, and $P_y^{\mathbb{Z}^2}$ for its canonical law starting from $y \in \mathbb{Z}^2$.

It is known that $a(\cdot)$ is a non-negative function on $\mathbb{Z}^2$, vanishing at the origin, which is symmetric and satisfies, cf. Proposition P2, p. 123 of [15]:

$$\lim_{y' \to \infty} a(y' + y) - a(y') = 0, \text{ for any } y \in \mathbb{Z}^2. \tag{1.8}$$

In addition $a(\cdot)$ has the asymptotic behavior, cf. Theorem 1.62, p. 38 of [10]:

$$a(y) \sim \frac{2}{\pi} \log |y|, \text{ as } y \to \infty\,. \tag{1.9}$$

By Proposition 1.6.3, p. 39 of [10], and the above mentioned identification, we can represent $g_B(\cdot,\cdot)$ in terms of $a(\cdot)$, via the formula:

$$g_B(x, x') = E_x[a(X_{T_B} - x')] - a(x - x'), \text{ for } x, x' \in \overline{B}. \tag{1.10}$$

In particular when $y, y' \in \mathbb{Z}^2$ are fixed, setting x, x' to be their canonical images on $\mathbb{T}_N$, whose distance to B^c is equivalent to $\frac{N}{4}$ as $N \to \infty$, we find by (1.8), (1.10) that:

$$\lim_N g_B(x, x) - g_B(x, x') = a(y - y')\,. \tag{1.11}$$

In addition by (1.9), (1.10), we also find that

$$g_B(x, x') \sim \frac{2}{\pi} \log N, \text{ as } N \to \infty\,. \tag{1.12}$$

We now introduce the sequences $R_k, k \geq 1$, and $D_k, k \geq 1$, of successive returns to 0 and departures from B of the walk on $\mathbb{T}_N$. The system of excursions induced by these sequences of stopping times will play an important role throughout the sequel. We define

$$\begin{aligned} &R_1 = H_0,\ D_1 = T_B \circ \theta_{R_1} + R_1, \text{ and for } k \geq 1 \\ &R_{k+1} = R_1 \circ \theta_{D_k} + D_k,\ D_{k+1} = D_1 \circ \theta_{D_k} + D_k, \end{aligned} \tag{1.13}$$

and we set $R_0 = 0$ by convention, so that P-a.s.,

$$0 = R_0 \leq R_1 < D_1 < \cdots < R_k < D_{k+1} < \ldots\,.$$

It follows from the strong Markov property applied at times $R_k, k \ge 1$, that

$$\text{(1.14)} \qquad \begin{array}{l} \text{under } P,\ R_1, R_2 - R_1, \dots, R_{k+1} - R_k, \dots \text{ are independent and the} \\ R_{k+1} - R_k, k \ge 1, \text{ have same distribution as } H_0 \circ \theta_{T_B} + T_B \text{ under } P_0. \end{array}$$

We also introduce the notation

$$\text{(1.15)} \qquad h = E[H_0], h_0 = E[R_2 - R_1] \overset{(1.14)}{=} E_0[H_0 \circ \theta_{T_B} + T_B]\,.$$

The next lemma collects various known controls on the above random times, and the large N behavior of h and h_0.

Lemma 1.1.

$$\text{(1.16)} \qquad h \sim \frac{2}{\pi}\, N^2 \log N, \ \textit{as } N \to \infty.$$

$$\text{(1.17)} \qquad h_0 \sim \frac{2}{\pi}\, N^2 \log N, \ \textit{as } N \to \infty.$$

$$\text{(1.18)} \qquad \sup_{t \ge 0} |P[H_0 > t] - e^{-t/h}| \le c\, \frac{N^2}{h} \le \frac{c'}{\log N}.$$

$$\text{(1.19)} \qquad \sup_{x \in \mathbb{T}_N} E_x\Big[\exp\Big\{c\, \frac{T_B}{N^2}\Big\}\Big] \le 2.$$

$$\text{(1.20)} \qquad \sup_{x \in \mathbb{T}_N} E_x\Big[\exp\Big\{c\, \frac{H_0}{N^2 \log N}\Big\}\Big] \le 2.$$

Proof. Claim (1.16) comes from Proposition 8, p. 21 in chapter 13 of [2]. For (1.17), it follows from a reward renewal argument that

$$\pi(0) = \frac{1}{N^2} = \frac{1}{h_0}\, E_0\Big[\int_0^{H_0 \circ \theta_{T_B} + T_B} 1\{X_s = 0\} ds\Big] = \frac{1}{h_0}\, g_B(0,0),$$

and hence that

$$\text{(1.21)} \qquad h_0 = N^2 g_B(0,0)\,.$$

Claim (1.17) now follows by (1.12). Claim (1.18) follows from Theorem 1 of [1], see also Proposition 20, p. 28 in Chapter 3 of [2], and from p. 34 in Chapter 5 of [2].

The exponential bound (1.19) is a consequence of Khasminskii's lemma, cf. [9] and also [5], p. 71, and of the estimate, see for instance Lemma 1.1 in [16]:

$$\sup_{x \in B} E_x[T_B] \le c N^2\,.$$

As for (1.20) it also follows from Khasminskii's lemma and the estimate, see Proposition 10.13, p. 133 of [11],

$$\text{(1.22)} \qquad E_x[H_{x'}] \le c\, N^2 \log(d(x,x') + 1) \le c\, N^2 \log N, \ \text{for } x, x' \in \mathbb{T}_N.$$

This completes the proof of Lemma 1.1. □

Given a sequence $s_N \geq 0$, we introduce the "number of returns up to time s_N":

$$J_N = \sum_{k\geq 1} 1\{R_k \leq s_N\}, \tag{1.23}$$

so that P-a.s., $R_{J_N} \leq s_N < R_{J_N+1}$ (recall that $R_0 = 0$, by convention). Our main interest lies in the choice $s_N = t_N$ or $s_N = t'_N$, in the notation of (0.2). We then write:

$$K_N = \sum_{k\geq 1} 1\{R_k \leq t_N\} \text{ and } K'_N = \sum_{k\geq 1} 1\{R_k \leq t'_N\}\,. \tag{1.24}$$

The next lemma will be useful for both choices $s_N = t_N$ or $s_N = t'_N$.

Lemma 1.2.

$$\lim_N P[J_N \geq 1, R_{J_N} \leq s_N < D_{J_N}] = 0\,. \tag{1.25}$$

Proof. By reversibility we know that under P the right-continuous modification of $s \in [0, s_N] \to X_{s_N - s} \in \mathbb{T}_N$ has same law as $s \in [0, s_N] \to X_s \in \mathbb{T}_N$. On the event $\{J_N \geq 1, R_{J_N} \leq s_N < D_{J_N}\}$, P-a.s. the right-continuous modification of $(X_{s_N-s})_{0\leq s\leq s_N}$ enters 0 prior to exiting B. Hence the probability in (1.25) is smaller than:

$$P[H_0 < T_B] \leq P[H_0 \leq N^2\sqrt{\log N}] + P[N^2\sqrt{\log N} < T_B] \underset{N}{\longrightarrow} 0,$$

using (1.16), (1.18) for the first term and (1.19) for the second term. □

The next lemma will come to use in Section 4 with the choice $s_N = t'_N$, cf. (4.7).

Lemma 1.3. *There exists a sequence $\delta_N \in (0,1)$, with $\lim_N \delta_N = 0$, such that*

$$\lim_N P\Big[(1-\delta_N)\,\frac{\pi}{2}\,\alpha_N \leq K'_N \leq (1+\delta_N)\,\frac{\pi}{2}\,\alpha_N\Big] = 1, \tag{1.26}$$

(see (0.2) for the definition of α_N).

Proof. By (1.14), (1.15), we see that for $k \geq 1$,

$$E[R_k] = h + (k-1)\,h_0, \tag{1.27}$$

$$\mathrm{var}_P(R_k) = \mathrm{var}_P(R_1) + (k-1)\,\mathrm{var}_P(R_2 - R_1) \overset{(1.19),(1.20)}{\leq} k\,c\,(N^2\log N)^2, \tag{1.28}$$

where var_P stands for the variance under P, and we used (1.14) and the strong Markov at time T_B under P_0 to bound $\mathrm{var}_p(R_2 - R_1)$. By Chebyshev's inequality it follows that for any $\varepsilon \in (0,1)$, $k \geq 1$, $N \geq c$,

$$\begin{aligned} &P[(1-\varepsilon)(h+(k-1)h_0) \leq R_k \leq \\ &(1+\varepsilon)(h+(k-1)h_0)] \geq 1 - \frac{c}{\varepsilon^2}\,\frac{k(N^2\log N)^2}{(h+(k-1)h_0)^2} \geq 1 - \frac{c'}{k\varepsilon^2}\,. \end{aligned} \tag{1.29}$$

We then introduce (with the convention $\sup\phi = 0$):

$$k'_N = \sup\{k \geq 1;\, h + (k-1)\,h_0 \leq t'_N\}, \tag{1.30}$$

and see by (0.2) ii), (1.16), (1.17) that

$$k'_N \sim \frac{\pi}{2}\,\alpha_N, \text{ as } N \to \infty. \tag{1.31}$$

Therefore, when $\delta \in (0, \frac{1}{2})$, choosing $\varepsilon \in (0,1)$ so that $(1-\varepsilon)(1+\delta) > 1$, we have:

$$\begin{aligned}(1.32)\quad & P[K'_N > (1+\delta)k'_N] \le P[R_{[(1+\delta)k'_N]+1} \le h + k'_N\, h_0] \overset{(1.29)}{\le} \\ & \frac{c}{k'_N \varepsilon^2} + P\big[(1-\varepsilon)(h + [(1+\delta)k'_N]h_0) \le R_{[(1+\delta)k'_N]+1} \le h + k'_N\, h_0\big] \underset{N}{\longrightarrow} 0,\end{aligned}$$

since the probability in the last line vanishes for large N by our choice of ε.

Likewise when we instead choose $\varepsilon \in (0,1)$ so that $(1+\varepsilon)(1-\delta) < 1$, we find that

$$\begin{aligned}(1.33)\quad & P[K'_N < (1-\delta)k'_N] \le P[R_{[(1-\delta)k'_N]+1} > h + (k'_N - 1)\, h_0] \overset{(1.29)}{\le} \\ & \frac{c}{k'_N \varepsilon^2} + P\big[h + (k'_N - 1)h_0 < R_{[(1-\delta)k'_N]+1} \le (1+\varepsilon)(h + [(1-\delta)k'_N]h_0)\big] \underset{N}{\longrightarrow} 0,\end{aligned}$$

since the probability in the last line vanishes for large N by our choice of ε.

Collecting (1.31) - (1.33), we see that

$$(1.34)\qquad \gamma_N \overset{\text{def}}{=} E\Big[\Big|\frac{K'_N}{\frac{\pi}{2}\,\alpha_N} - 1\Big| \wedge 1\Big] \underset{N}{\longrightarrow} 0,$$

and hence by Chebyshev's inequality for large N

$$(1.35)\qquad P\Big[\Big|\frac{K'_N}{\frac{\pi}{2}\,\alpha_N} - 1\Big| > \sqrt{\gamma_N}\Big] \le \sqrt{\gamma_N} \underset{N}{\longrightarrow} 0\,.$$

The claim (1.26) readily follows (choosing for instance $\delta_N = \sqrt{\gamma_N}$ for large N). □

We now turn to the definition of the two-dimensional Gaussian free field pinned at the origin. We introduce on some auxiliary space

$$\begin{aligned}(1.36)\quad & \psi_y, y \in \mathbb{Z}^2, \text{ a centered Gaussian field with covariance function} \\ & E[\psi_y \psi_{y'}] = a(y) + a(y') - a(y'-y), \text{ for } y, y' \in \mathbb{Z}^2.\end{aligned}$$

We refer to Lemma 1.2 of [18], where the interpretation of this random field as the limit of the field of increments at the origin of a Gaussian free field with Dirichlet conditions outside $[-N,N]^2$, as $N \to \infty$, or as the limit of the same field with Dirichlet boundary condition outside $[-N,N]^2$, conditioned to take the value 0 at the origin, with $N \to \infty$, is provided. This last limit result explains the terminology of Gaussian free field "pinned at the origin" we use for the random field in (1.36).

As last topic of this section we briefly recall some facts concerning the zero-dimensional Bessel process. We denote by $\text{BES}^0(a,\tau)$ the law at time $\tau \ge 0$ of a zero-dimensional Bessel process starting at $a \ge 0$. If the random variable R is $\text{BES}^0(a,\tau)$-distributed, the Laplace transform of R^2 is given by the formula, see [14], p. 411, or [8], p. 239:

$$(1.37)\qquad E[e^{-\lambda R^2}] = \exp\Big\{-\frac{a^2\lambda}{1+2\tau\lambda}\Big\}, \text{ for } \lambda \ge 0\,.$$

We denote by $\text{BESQ}^0(\alpha^2,\tau)$ the law of R^2, namely the distribution of a zero-dimensional square Bessel process at time τ starting from a^2 at time 0.

2 Further preparation

In this section we introduce the additive functionals that will enter the proof of the main claims (0.3) - (0.6) in the next two sections. We show in Proposition 2.2 that these additive functionals can be replaced by discrete sums collecting their respective contributions between successive times R_k, with $1 \le k \le K_N$, in the case of (0.2) i) and $1 \le k \le K'_N$, in the case of (0.2) ii). Throughout this section and the rest of the article constants depend on the set Λ in (2.1) and the function V in (2.3).

We introduce

$$\Lambda \text{ a finite subset of } \mathbb{Z}^2 \text{ containing } 0\,. \tag{2.1}$$

From now on we assume $N \ge c$ (see the convention concerning constants stated above (2.1)), so that (see (1.2) and below (1.6)):

$$\Lambda \subseteq B\,. \tag{2.2}$$

We will also routinely view Λ as a subset of $\mathbb{T}_N$, keeping in mind the identification discussed below (1.6). We further consider a function

$$V : \mathbb{Z}^2 \to \mathbb{R}, \text{ supported in } \Lambda, \text{ such that } \sum_{x\in\Lambda} V(x) = 0\,. \tag{2.3}$$

In the same fashion we will routinely view V as a function on $\mathbb{T}_N$, vanishing on the complement of Λ in $\mathbb{T}_N$.

The next lemma collects controls, which we will repeatedly use in the sequel.

Lemma 2.1.

$$\sup_{x\in\Lambda} P_x[H_0 > T_B] \le \frac{c}{\log N}\,. \tag{2.4}$$

$$\sup_{x\in\mathbb{T}_N} E_x\Big[\exp\Big\{c\int_0^{H_0} 1\{X_s \in \Lambda\}ds\Big\}\Big] \le 2\,. \tag{2.5}$$

$$\mathrm{cap}_B(\Lambda) \sim \frac{\pi}{2\log N},\ \textit{as } N \to \infty\,. \tag{2.6}$$

Proof. We begin with the proof of (2.4). It follows from the strong Markov property applied at time H_0 that for $x \in \mathbb{T}_N$,

$$g_B(x,0) = P_x[H_0 < T_B]\, g_B(0,0).$$

Using the symmetry of $g_B(\cdot,\cdot)$ we find

$$P_x[H_0 > T_B] = \frac{g_B(0,0) - g_B(0,x)}{g_B(0,0)} \overset{(1.11),(1.12)}{\le} \frac{c}{\log N}\,,\ \text{for } x \in \Lambda, \tag{2.7}$$

and (2.4) follows. We then turn to the proof of (2.5). Using Khasminskii's lemma once again, cf. [5], p. 71, it suffices to show that

$$\sup_{x\in\mathbb{T}_N} E_x\Big[\int_0^{H_0} 1\{X_s \in \Lambda\}ds\Big] \le c\,, \tag{2.8}$$

To this end we note that for $x \in \mathbb{T}_N$,

$$E_x\Big[\int_0^{H_0} 1\{X_s \in \Lambda\} ds\Big] = \sum_{y\in\Lambda} P_x\Big[H_y < H_0, \int_{H_y}^{H_0\circ\theta_{H_y}+H_y} 1\{X_s = y) ds\Big]$$
$$= \sum_{y\in\Lambda\setminus\{0\}} \frac{P_x[H_y < H_0]}{P_y[\widetilde{H}_y > H_0]}$$

using the strong Markov property and the identity $E_y[\int_0^{H_0} 1\{X_s = y\} ds] = P_y[\widetilde{H}_y > H_0]^{-1}$ for $y \neq 0$ in the last step. Now observe that $P_y[\widetilde{H}_y > H_0] \geq c$, for $y \in \Lambda$ and (2.8) follows. This proves (2.5). Finally let us prove (2.6). By (1.6) we find that

$$e_{\{0\},B}(0) = P_{e_{\Lambda,B}}[H_0 < T_B] \overset{(2.4)}{\sim} \mathrm{cap}_B(\Lambda)\ , \text{ as } N \to \infty.$$

On the other hand by (1.4) we have

$$e_{\{0\},B}(0) = P_0[\widetilde{H}_0 > T_B] = \frac{1}{g_B(0,0)} \overset{(1.12)}{\sim} \frac{\pi}{2\log N}\ , \text{ as } N \to \infty\,.$$

The claim (2.6) follows and the proof of Lemma 2.1 is completed. □

We introduce the functions (supported in Λ)

(2.9) $$V_N = \frac{1}{\sqrt{2\log N}}\, V, \quad \widetilde{V}_N = \frac{1}{\log N}\, 1_{\{0\}},$$

and

(2.10) $$V'_N = \frac{1}{\sqrt{\alpha_N}}\, V_N = \frac{1}{\sqrt{2t'_N N^{-2}}}\ V, \quad \widetilde{V}'_N = \frac{1}{\alpha_N}\ \widetilde{V}_N = \frac{1}{t'_N N^{-2}}\ 1_{\{0\}}\,.$$

The respective contributions of these additive functionals between the successive times R_k, $k \geq 0$, is described by the four sequences:

(2.11) $$\xi_k = \int_{R_k}^{R_{k+1}} V_N(X_s)\, ds, \quad \widetilde{\xi}_k = \int_{R_k}^{R_{k+1}} \widetilde{V}_N(X_s)\, ds,\ k \geq 0,$$

and

(2.12) $$\xi'_k = \int_{R_k}^{R_{k+1}} V'_N(X_s)\, ds = \frac{1}{\sqrt{\alpha_n}}\, \xi_k, \quad \widetilde{\xi}'_k = \int_{R_k}^{R_{k+1}} \widetilde{V}'_N(X_s)\, ds = \frac{1}{\alpha_N}\ \widetilde{\xi}_k, \text{ for } k \geq 0.$$

By definition of $R_1 (= H_0)$, see (1.13), and we have

(2.13) $$\widetilde{\xi}_0 = 0 = \widetilde{\xi}'_0\,.$$

The application of the strong Markov property at the successive times R_k yields the following strengthening of (1.14):

(2.14) under P, $(R_1, \xi_0), (R_2 - R_1, \xi_1), \dots, (R_{k+1} - R_k, \xi_k), \dots,$ are independent and the $(R_{k+1} - R_k, \xi_k), k \geq 1$, are identically distributed.

In addition the same holds if one replaces $\xi_k, k \ge 0$, with one of the other sequences that appear in (2.11), (2.12).

We define the partial sums

$$S_k = \xi_1 + \dots + \xi_k, \ \text{for } k \ge 1, \qquad = 0, \ \text{for } k = 0, \tag{2.15}$$

and in a similar fashion $\widetilde{S}_k, k \ge 0$, $S'_k, k \ge 0$, and $\widetilde{S}'_k, k \ge 0$, respectively replacing $\xi_\ell, \ell \ge 0$, by $\widetilde{\xi}_\ell, \ell \ge 0$, $\xi'_\ell, \ell \ge 0$, and $\widetilde{\xi}'_\ell, \ell \ge 0$, in (2.15).

The next proposition enables us in the next two sections to replace the additive functionals based on the functions in (2.9), (2.10), with discrete sums, see (2.15).

Proposition 2.2.

$$\int_0^{t_N} V_N(X_s)\, ds - S_{K_N} \longrightarrow 0, \ \textit{in P-probability, as } N \to \infty, \tag{2.16}$$

and the same holds true with $\widetilde{V}_N$ *and* $\widetilde{S}_k, k \ge 0$, *in place of* V_N *and* $S_k, k \ge 0$.

$$\int_0^{t'_N} V'_N(X_s)\, ds - S'_{K'_N} \longrightarrow 0, \ \textit{in P-probability, as } N \to \infty, \tag{2.17}$$

and the same holds true with $\widetilde{V}'_N$ *and* $\widetilde{S}'_k, k \ge 0$, *in place of* V'_N *and* $\widetilde{S}'_k, k \ge 0$.

Proof. With the notation (1.23), and an arbitrary non-negative sequence s_N we have:

$$\int_0^{s_N} V_N(X_s)\, ds = \xi_0 + S_{J_N} - \int_{s_N}^{R_{J_N+1}} V_N(X_s)\, ds\,. \tag{2.18}$$

By (2.5) of Lemma 2.1 we know that

$$|\xi_0| \le \frac{c}{\sqrt{\log N}} \int_0^{H_0} 1\{X_s \in \Lambda\}\, ds \underset{N}{\longrightarrow} 0, \ \text{in } P\text{-probability}. \tag{2.19}$$

Moreover we have:

$$E\Big[\Big|\int_{s_N}^{R_{J_N+1}} V_N(X_s)\, ds\Big| \wedge 1\Big] \le E\Big[\int_0^{R_1} |V_N(X_s)|\, ds\Big] + E\Big[J_N \ge 1, \Big|\int_{s_N}^{R_{J_N+1}} V_N(X_s)\, ds\Big| \wedge 1\Big].$$

By (2.5) the first term in the right-hand side tends to zero, and by Lemma 1.2 the last term is bounded up to a quantity tending to zero with N by

$$E\Big[J_N \ge 1, D_{J_N} \le s_N, \Big|\int_{s_N}^{R_{J_N+1}} V_N(X_s) ds\Big|\Big] \le E\Big[\int_{s_N}^{H_0 \circ \theta_{s_N} + s_N} |V_N(X_s)|\, ds\Big] \le \frac{c}{\sqrt{\log N}} \sup_{x \in \mathbb{T}_N} E_x\Big[\int_0^{H_0} 1\{X_s \in \Lambda\}\, ds\Big] \underset{N}{\overset{(2.5)}{\longrightarrow}} 0\,.$$

Coming back to (2.18) we have thus shown that

$$\int_0^{s_N} V_N(X_s)\, ds - S_{J_N} \to 0, \ \text{in } P\text{-probability as } N \to \infty. \tag{2.20}$$

A similar and even simpler argument shows that

$$(2.21) \qquad \int_0^{s_N} \widetilde{V}_N(X_s)\, ds - \widetilde{S}_{J_N} \to 0, \text{ in } P\text{-probability as } N \to \infty.$$

Choosing $s_N = t_N$ yields (2.16). Choosing $s_N = t'_N$ and respectively dividing (2.20) and (2.21) by $\frac{1}{\sqrt{\alpha_N}}$ and $\frac{1}{\alpha_N}$ yields (2.17). This completes the proof of Proposition 2.2. □

3 The Poissonian regime

In this section we investigate the field of occupation times left by the random walk on $\mathbb{T}_N$ at times proportional to $N^2 \log N$, close to the origin, as N goes to infinity, and relate this asymptotic behavior to the two-dimensional free field pinned at the origin. The main results are Theorems 3.1 and 3.7, which respectively prove the claims (0.4) and (0.3) from the Introduction. However most of the work goes into the proof of Theorem 3.2. Theorems 3.1 and 3.7 answer positively a question raised in Remark 4.10 1) of [18]. We explain in Remark 3.8 the link with the results obtained in [18] for the occupation field of random interlacements in long rods of $\mathbb{Z}^3$ at a level u_N of order $\frac{\log N}{N}$. We use the convention concerning constants stated at the beginning of Section 2: constants implicitly depend on Λ and V in (2.1), (2.3).

We recall the definition (0.2) i) of the time scale t_N:

$$(3.1) \qquad t_N = \alpha\, N^2 \log N, \text{ with } \alpha > 0\,.$$

It will become clear, cf. Remark 3.3, that this time scale corresponds to a Poissonian regime for the system of excursions of the walk described by the stopping times in (1.13). We tacitly endow $\mathbb{R}^{\mathbb{Z}^2}$ with the product topology, and the convergence stated in Theorems 3.1 and 3.7 below, actually corresponds to the convergence in distribution of all finite dimensional marginals of the relevant random field towards the stated limit. In what follows we routinely identify a random field on $\mathbb{T}_N$ with a periodic random field on $\mathbb{Z}^2$, by means of composition with the canonical map p_N: $\mathbb{Z}^2 \to \mathbb{T}_N$.

Theorem 3.1. *Under P, as N goes to infinity,*

$$(3.2) \qquad \Big(\frac{L^y_{t_N} - L^0_{t_N}}{\sqrt{2 \log N}}\Big)_{y \in \mathbb{Z}^2} \text{ converges in distribution to the random field } (R\psi_y)_{y\in\mathbb{Z}^2},$$

where R and $(\psi_y)_{y\in\mathbb{Z}^2}$ are independent, and

$$(3.3) \qquad R \text{ is } BES^0\Big(\sqrt{\alpha},\, \frac{1}{\pi}\Big)\text{-distributed},$$

$$(3.4) \qquad (\psi_y)_{y\in\mathbb{Z}^2} \text{ is the centered Gaussian field introduced in (1.36).}$$

Theorem 3.1 will follow from Theorem 3.2 stated below. We first need to introduce notation. For V as in (2.3) we define

$$(3.5) \qquad \mathcal{E}(V) = -\sum_{y,y' \in \mathbb{Z}^2} a(y'-y)\, V(y)\, V(y'),$$

and note that $\mathcal{E}(V)$ can be expressed in term of $(\psi_y)_{y\in\mathbb{Z}^2}$ via the identity:

$$(3.6) \qquad \mathcal{E}(V) = E\Big[\Big(\sum_{y \in \mathbb{Z}^2} V(y)\psi_y\Big)^2\Big],$$

(where we made use of the fact that $\sum_y V(y) = 0$). We consider on some auxiliary space two variables τ, ζ such that

(3.7) $\quad \tau$ and ζ are independent,

(3.8) $\quad \tau$ is exponentially distributed with parameter $\left(\frac{\pi}{2}\right)$

(so that τ has expectation $\frac{2}{\pi}$),

(3.9) $\quad \zeta$ has bilateral exponential distribution with parameter $\left(\frac{\pi}{\mathcal{E}(V)}\right)^{1/2}$,

(i.e. when $\mathcal{E}(V) > 0$, ζ has the density $\frac{\rho}{2} e^{-\rho|u|}$ with respect to the Lebesgue measure du on $\mathbb{R}$, with $\rho = (\frac{\pi}{\mathcal{E}(V)})^{\frac{1}{2}}$, and when $\mathcal{E}(V) = 0$, we simply mean that $\zeta = 0$). We also recall the definitions (1.13) and (2.11). The main step in proving Theorem 3.1 is the next

Theorem 3.2. *Under P, as N goes to infinity,*

$$\left(\frac{R_2 - R_1}{N^2 \log N}, \xi_1\right) \text{ converges in law to } (\tau, \zeta), \tag{3.10}$$

$$\frac{R_1}{N^2 \log N} \text{ converges in law to } \tau. \tag{3.11}$$

We first assume Theorem 3.2 and explain how Theorem 3.1 follows.

Proof of Theorem 3.1 *(assuming Theorem* 3.2*):* We first observe that Theorem 3.1 follows once we show that in the notation of (2.9)

$$\int_0^{t_N} V_N(X_s)\, ds \text{ converges in law to } \sigma_J, \text{ as } N \to \infty, \tag{3.12}$$

where $\sigma_k, k \geq 0$, are the partial sums of i.i.d. variables, $\zeta_\ell, \ell \geq 1$, distributed as ζ in (3.9) (i.e. $\sigma_0 = 0$, $\sigma_k = \zeta_1 + \cdots + \zeta_k$, for $k \geq 1$), and J is an independent Poisson variable with parameter $\alpha \frac{\pi}{2}$. Indeed this implies that for $b \in \mathbb{R}$,

$$\begin{aligned} E[\exp\{ib\, \sigma_J\}] &= \exp\left\{\alpha\, \frac{\pi}{2}\, (E[e^{i\zeta}] - 1)\right\} = \exp\left\{-\frac{\alpha}{2}\, \frac{\mathcal{E}(V)\, b^2}{1 + \frac{\mathcal{E}(V)}{\pi}\, b^2}\right\} \\ &\overset{(1.37)}{\underset{(3.6)}{=}} E\Big[\exp\Big\{ibR \sum_{y \in \mathbb{Z}^2} V(y)\, \psi_y\Big\}\Big]. \end{aligned} \tag{3.13}$$

As a result (3.12) implies that for $b \in \mathbb{R}$ and V as in (2.3)

$$\lim_N E\Big[\exp\Big\{ib \sum_{y \in \mathbb{Z}^2} V(y) \frac{L^y_{t_N}}{\sqrt{2 \log N}}\Big\}\Big] = E\Big[\exp\Big\{ib \sum_{y \in \mathbb{Z}^2} V(y)\, R\psi_y\Big\}\Big], \tag{3.14}$$

and Theorem 3.1 follows. We will now prove (3.12) (assuming Theorem 3.2). We consider $b \in \mathbb{R}, K > 1$, and define

$$A_{K,N} = E\Big[R_K > t_N,\, \exp\Big\{ib \int_0^{t_N} V_N(X_s) ds\Big\}\Big]. \tag{3.15}$$

It follows from (2.16) that

$$\lim_N A_{K,N} - E[R_K > t_N,\, e^{ibS_{K_N}}] = 0\,. \tag{3.16}$$

By definition of K_N, see (1.24), we have

$$E[R_K > t_N, e^{ib\,S_{K_N}}] = \sum_{0\le k<K} E[R_k \le t_N < R_{k+1}, e^{ibS_k}]. \tag{3.17}$$

Combining (2.14) and Theorem 3.2, we see that for any $k \ge 1$,

$$\Big(\frac{R_1}{N^2\log N}, \dots, \frac{R_{k+1}}{N^2\log N},\ \xi_1,\dots,\xi_k\Big) \ \underset{N}{\overset{\text{law}}{\longrightarrow}}\ (T_1,\dots,T_{k+1},\zeta_1,\dots,\zeta_k), \tag{3.18}$$

where the $\zeta_i, i\ge 1$, are i.i.d., with same distribution as ζ in (3.9), and the $T_i, i \ge 1$, are independent from the $\zeta_i, i\ge 1$, and distributed as the successive jumps of a Poisson point process of intensity $\frac{\pi}{2}$ on the positive half-line. It nows follows by a routine continuity argument that for each $k \ge 1$,

$$\lim_N E\Big[\frac{R_k}{N^2\log N} \le \alpha < \frac{R_{k+1}}{N^2\log N}\ ,\ e^{ibS_k}\Big] = E[T_k \le \alpha < T_{k+1}, e^{ib\sigma_k}], \tag{3.19}$$

and by (3.11) (recall that $S_0 = 0$ and $\sigma_0 = 0$) that:

$$\lim_N E\Big[\frac{R_0}{N^2\log N} > \alpha,\ e^{ibS_0}\Big] = E[T_1 > \alpha, e^{ib\sigma_0}]\,. \tag{3.20}$$

Setting $J = \sum_{k\ge1} 1\{T_k \le \alpha\}$, we have thus shown that

$$\lim_N A_{K,N} = E[T_K > \alpha, e^{ib\sigma_J}], \tag{3.21}$$

where J is independent of the $\zeta_\ell, \ell\ge 1$, and Poisson $(\alpha\frac{\pi}{2})$-distributed. Hence for $K \ge 1$

$$\begin{aligned}&\limsup_N \big|E\big[e^{ib\int_0^{t_N} V_N(X_s)ds}\big] - E[e^{ib\sigma_J}]\big| \le \\ &\limsup_N P[R_K \le t_N] + P[T_K \le \alpha] \overset{(3.18)}{=} 2P[T_K \le \alpha] \to 0, \text{ as } K\to\infty.\end{aligned} \tag{3.22}$$

The claim (3.12) follows and Theorem 3.1 is thus proved (once Theorem 3.2 is shown). □

Remark 3.3. By (3.19), (3.20), it follows that for any $k \ge 0$,

$$\lim_N P[K_N = k] = P[J = k],$$

and K_N converges in law to a Poisson variable with parameter $\alpha\frac{\pi}{2}$, as N goes to infinity. This explains why we refer to the time scale t_N in (0.2) i) as the Poissonian regime. □

Proof of Theorem 3.2*:* We first note that (3.11) is a straightforward consequence of (1.16), (1.18). We then turn to the proof of (3.10). By (2.5) and the strong Markov property at time D_1, we see that

$$\int_{D_1}^{R_2} V_N(X_s)ds \underset{N}{\longrightarrow} 0 \text{ in } P\text{-probability}\,. \tag{3.23}$$

Similarly by (1.19) and the strong Markov property at time R_1, we see that

$$\frac{D_1 - R_1}{N^2\log N} \underset{N}{\longrightarrow} 0 \text{ in } P\text{-probability}. \tag{3.24}$$

As a result of (3.23), (3.24), we see that (3.10) will follow once we show that

$$\Big(\frac{R_2 - D_1}{N^2\log N}, \int_{R_1}^{D_1} V_N(X_s)\,ds\Big) \overset{\text{law}}{\longrightarrow} (\tau,\zeta), \text{ as } N\to\infty\,. \tag{3.25}$$

To prove (3.25) we will rely on the following two propositions:

Proposition 3.4.

$$\lim_N \sup_{x \in \partial B} \Big| P_x\Big[\frac{H_0}{h} > t\Big] - e^{-t}\Big| = 0, \textit{ for each } t > 0\,. \tag{3.26}$$

Proposition 3.5.

$$\int_0^{T_B} V_N(X_s)\,ds \xrightarrow[N]{\text{law}} \zeta \textit{ under } P_0\,. \tag{3.27}$$

We first explain how (3.25) follows once Propositions 3.4 and 3.5 are proved. The application of the strong Markov property at time D_1 and at time R_1 then shows that for any $t \geq 0$ and $b \in \mathbb{R}$:

$$E\Big[\frac{D_2 - D_1}{h} > t,\ e^{ib\int_{R_1}^{D_1} V_N(X_s)ds}\Big] = E_0\Big[e^{ib\int_0^{T_B} V_N(X_s)ds} P_{X_{T_B}}\Big[\frac{H_0}{h} > t\Big]\Big] \underset{N}{\longrightarrow} E[e^{ib\zeta}]\,e^{-t}.$$

By a tightness and uniqueness of limit points argument, the above convergence implies that under P as $N \to \infty$,

$$\Big(\frac{R_2 - D_1}{h}, \int_{R_1}^{D_1} V_N(X_s)\,ds\Big) \xrightarrow{\text{law}} \Big(\frac{\pi}{2}\,\tau, \zeta\Big).$$

By (1.16) we know that $h \sim \frac{2}{\pi}\,N^2 \log N$ and (3.25) straightforwardly follows.

There remains to prove Propositions 3.4 and 3.5.

Proof of Proposition 3.4*:* We pick $t > 0$ and introduce the shorthand notation

$$T = N^2 \sqrt{\log N}\,. \tag{3.28}$$

We assume by (1.16) that N is large enough so that $t > \frac{T}{h}$. We then write

$$\begin{aligned}
&\sup_{x \in \partial B} \Big| P_x\Big[\frac{H_0}{h} > t\Big] - e^{-t}\Big| \leq a_1 + a_2 + a_3 + a_4, \text{ with} \\
&a_1 = \sup_{x \in \partial B} \Big| P_x\Big[\frac{H_0}{h} > t\Big] - P_x\Big[\frac{H_0 \circ \theta_T + T}{h} > t\Big]\Big|, \\
&a_2 = \sup_{x \in \partial B} \Big| P_x\Big[\frac{H_0 \circ \theta_T + T}{h} > t\Big] - P\Big[\frac{H_0}{h} > t - \frac{T}{h}\Big]\Big| \\
&a_3 = \Big| P\Big[\frac{H_0}{h} > t - \frac{T}{h}\Big] - e^{(t - \frac{T}{h})}\Big|, \\
&a_4 = |e^{-(t - \frac{T}{h})} - e^{-t}|\,.
\end{aligned} \tag{3.29}$$

We now bound a_i, for $i = 4, \dots, 1$. By (1.16), (3.28) we see that

$$a_4 \leq e^{T/h} - 1 \underset{N}{\longrightarrow} 0\,. \tag{3.30}$$

By (1.18) we find that

$$a_3 \underset{N}{\longrightarrow} 0\,. \tag{3.31}$$

Applying the Markov property at time T together with translation invariance and denoting by $p_t(x,y)$ the transition density with respect to π of the walk on $\mathbb{T}_N$, see (1.11), we find

$$a_2 \le \sum_{y\in\mathbb{T}_N} |p_T(0,y)-1|\pi(y) \le \Big(\sum_{y\in\mathbb{T}_N} (p_T(0,y)-1)^2\,\pi(y)\Big)^{1/2} = (p_{2T}(0,0)-1)^{1/2}, \tag{3.32}$$

using reversibility and the Chapman-Kolmogorov equation in the last step.

From (27), p. 13 in Chapter 5 of [2], it follows that

$$p_{2T}(0,0) = N^2 P_0[X_{2T}=0] = \sum_{0\le k_1,k_2<N} e^{-T(1-\cos(2\pi\frac{k_1}{N})+1-\cos(2\pi\frac{k_2}{N}))}. \tag{3.33}$$

Note that the terms k_i and $N-k_i$, for $0<k_i<N$, $i=1,2$, give the same contribution in the above sum. Hence we see that

$$p_{2T}(0,0)-1 \le 2\sum_{k_1+k_2>0} e^{-cT(\frac{k_1^2+k_2^2}{N^2})} \overset{(3.28)}{\le} c\,e^{-c'\sqrt{\log N}} \underset{N}{\longrightarrow} 0\,. \tag{3.34}$$

Coming back to (3.32) we have shown that

$$a_2 \underset{N}{\longrightarrow} 0\,. \tag{3.35}$$

Finally we consider a_1 and note that $H_0 \le H_0\circ\theta_T + T$. Hence we have

$$a_1 = \sup_{x\in\partial B} P_x[H_0\circ\theta_T + T > th \ge H_0] \le \sup_{x\in\partial B} P_x[H_0\le T]\,. \tag{3.36}$$

We introduce the shorthand notation

$$\varphi(y) = P_y[H_0\le T], \text{ for } y\in\mathbb{T}_N, \tag{3.37}$$

and note that

$$\sum_{y\in\mathbb{T}_N}\varphi(y)\,\pi(y) = P[H_0\le T] \overset{(1.18)}{\le} \frac{c}{\log N} + 1 - e^{-\frac{T}{h}} \le \frac{c'}{\sqrt{\log N}}\,.$$

If we now define

$$A = \{y\in\mathbb{T}_N;\ \varphi(y)\le(\log N)^{-\frac14}\}, \tag{3.38}$$

it follows from Chebyshev's inequality that

$$\pi(A^c) \le c'(\log N)^{-\frac14}\,. \tag{3.39}$$

Hence the proportion of sites in A^c inside any ball or radius $N(\log N)^{-\frac{1}{10}}$ in $\mathbb{T}_N$ tends to 0 as N goes to infinity. By Lemma 1.12 of [3] it follows that the probability that the walk starting from any point in $\mathbb{T}_N$ meets A before moving away at distance $c\,N(\log N)^{-\frac{1}{10}}$ is uniformly bounded away from 0. Using the strong Markov property we see that

$$\sup_{x\in\partial B} P_x[H_0 < H_A] \underset{N}{\longrightarrow} 0\,. \tag{3.40}$$

Now for any $x\in\mathbb{T}_N$, we find by the strong Markov property at time H_A

$$\begin{aligned} P_x[H_0\le T] &\le P_x[H_0<H_A] + P_x[H_A\le H_0\le T] \\ &\le P_x[H_0<H_A] + \sup_{y\in A} P_y[H_0\le T]. \end{aligned}$$

Coming back to (3.36) we have shown that

$$a_1 \le \sup_{x \in \partial B} P_x[H_0 < H_A] + \sup_{y \in A} \varphi(y) \underset{N}{\overset{(3.38),(3.40)}{\longrightarrow}} 0\,. \tag{3.41}$$

This completes the proof of (3.26). □

Our next step is the

Proof of Proposition 3.5*:* It will be convenient to replace P_0 by P_ν in (3.27), with

$$\nu = \frac{e_{\Lambda,B}}{\mathrm{cap}_B(\Lambda)}\,, \text{ the normalized equilibrium measure of } \Lambda \text{ relative to } B. \tag{3.42}$$

As we now explain, (3.27) follows once we show that

$$\int_0^{T_B} V_N(X_s)\,ds \underset{N}{\overset{\text{law}}{\longrightarrow}} \zeta, \text{ under } P_\nu\,. \tag{3.43}$$

Indeed for $b \in \mathbb{R}$, by the strong Markov property at time H_0, we have

$$\begin{aligned} E_\nu\big[e^{ib\int_0^{T_B} V_N(X_s)ds}\big] = {} & E_\nu\big[H_0 < T_B, e^{ib\int_0^{H_0} V_N(X_s)ds}\big]\, E_0\big[e^{ib\int_0^{T_B} V_N(X_s)ds}\big] + \\ & E_\nu\big[T_B < H_0, e^{ib\int_0^{T_B} V_N(X_s)ds}\big], \end{aligned} \tag{3.44}$$

so that

$$\begin{aligned} & \big|E_\nu\big[e^{ib\int_0^{T_B} V_N(X_s)ds}\big] - E_0\big[e^{ib\int_0^{T_B} V_N(X_s)ds}\big]\big| \le \\ & E_\nu\big[H_0 < T_B, \,\big|e^{ib\int_0^{H_0} V_N(X_s)ds} - 1\big|\big] + 2P_\nu[H_0 > T_B] \underset{N}{\overset{(2.4),(2.5)}{\longrightarrow}} 0, \end{aligned} \tag{3.45}$$

and the claim follows.

We thus turn to the proof of (3.43). Expanding the exponential function, we find in a classical fashion, see for instance (2.6) of [18] that for $|z| < c_N$,

$$\begin{aligned} E_\nu\big[e^{z\int_0^{T_B} V_N(X_s)ds}\big] & = \sum_{n\ge 0} z^n\, E_\nu\Big[\int_{0<s_1<\dots<s_n<T_B} V_N(X_{s_1})\dots V_N(X_{s_n})\,ds_1\dots ds_n\Big] \\ & = 1 + \sum_{n\ge 1} z^n(\nu, (G_B\, V_N)^n\, 1), \end{aligned} \tag{3.46}$$

where $G_B f(x) = \sum_{y\in\mathbb{T}_N} g_B(x,y)\, f(y)$, for f function on $\mathbb{T}_N$, and $x \in \mathbb{T}_N$, and (f,g) stands for $\sum_{x\in\mathbb{T}_N} f(x)\, g(x)$. We know by (1.5) and (3.42) that

$$\sum_y \nu(y)\, g_B(y,\cdot) = \mathrm{cap}_B(\Lambda)^{-1} \text{ on } \Lambda,$$

and since V_N is supported in Λ, we find that

$$E_\nu\big[e^{z\int_0^{T_B} V_N(X_s)ds}\big] = \sum_{n\ge 0} a_N(n)\, z^n, \text{ for } |z| \le c_N, \text{ where} \tag{3.47}$$

$$a_N(0) = 1, \text{ and } a_N(n) = \frac{1}{\mathrm{cap}_B(\Lambda)}\, (V_N, (G_B V)^{n-1} 1), \text{ for } n \ge 1\,. \tag{3.48}$$

Note that by our assumption (2.3) on V, we have

$$a_N(1) = (V, 1) = 0\,. \tag{3.49}$$

The main step in proving (3.43) lies in the next lemma. Interestingly its proof, although simpler, uses several ingredients of the proof of Theorem 4.1 of [18].

Lemma 3.6. $(N \geq \overline{c})$

$$|a_N(n)| \leq c_0^n, \ \textit{for all } n \geq 1, \tag{3.50}$$

$$\textit{for any } k \geq 0, \ \lim_N a_N(2k+1) = 0, \tag{3.51}$$

$$\textit{for any } k \geq 1, \ \lim_N a_N(2k) = \Big(\frac{1}{\pi}\,\mathcal{E}(V)\Big)^k. \tag{3.52}$$

We first explain how (3.43) and hence Proposition 3.5 follow from Lemma 3.6. By (3.47) the characteristic function of $\int_0^{T_B} V_N(X_s)\,ds$ under P_ν is analytic in the sense of Chapter 7 of [12]. By Theorem 7.11, p. 193 of [12], one finds that (3.47) actually holds for all $z \in \mathbb{C}$ with $c_0|z| < 1$. In particular picking $z = \pm r$ with $0 < r < c_0^{-1}$, we find that

$$\sup_{N \geq \overline{c}} E_\nu\Big[\cosh\Big(r \int_0^{T_B} V_N(X_s)\,ds\Big)\Big] < \infty\,.$$

Hence the laws under P_ν of $\int_0^{T_B} V_N(X_s)\,ds$ are tight and the variables $\exp\{z \int_0^{T_B} V_N(X_s)\,ds\}$, with $|Re\, z| \leq r (< c_0^{-1})$ are uniformly integrable. If the laws of the variables $\int_0^{T_B} V_N(X_s)\,ds$ under P_ν converge along some subsequence N_k, it follows from Theorem 5.4, p. 32 in [4] that for $|z| < c_0^{-1}$,

$$\begin{aligned} \lim_k\ E_\nu\big[e^{z\int_0^{T_B} V_{N_k}(X_s)\,ds}\big] &= \lim_k \sum_{n \geq 0} a_{N_k}(n)\, z^n \\ &\overset{\text{Lemma 3.6}}{=} \Big(1 - \frac{\mathcal{E}(V)}{\pi}\, z^2\Big)^{-1}. \end{aligned} \tag{3.53}$$

Thus the characteristic function of the limit law is analytic in the sense of [12] and coincides in a neighborhood in $\mathbb{C}$ of the origin with the characteristic function of ζ. So the limit law is the distribution of ζ, whence (3.43). Proposition 3.5 now follows.

Proof of Lemma 3.6*:* As pointed out above, although simpler the arguments are similar to the proof of Theorem 4.1 of [18]. We highlight the main steps.

We first need some notation. We tacitly identify functions on Λ with functions vanishing outside Λ. For such a function F: $\Lambda \to \mathbb{R}$, we write $\|F\|_\infty$ in place of $\sup_{x\in\Lambda} |F(x)|$, as well as

$$\langle F \rangle = \Big(\frac{1}{|\Lambda|} \sum_{x \in \Lambda} F(x)\Big)\, 1_\Lambda, \ \text{and } [F]_0 = F(0)\, 1_\Lambda\,, \tag{3.54}$$

so that $\langle F \rangle$ and $[F]_0$ are constant functions on Λ. For F as above we have:

$$\|G_B F\|_\infty \overset{(1.9),(1.10)}{\leq} c \log N\, \|F\|_\infty, \tag{3.55}$$

as well as

$$\|G_B F\|_\infty \leq c\, \|F\|_\infty, \ \text{when } \langle F \rangle = 0, \tag{3.56}$$

indeed

$$\begin{aligned} G_B F(x) &= \frac{1}{|\Lambda|} \sum_{y,y' \in \Lambda} G_B(x,y)\,(F(y) - F(y')) \\ &= \frac{1}{|\Lambda|} \sum_{y,y' \in \Lambda} (G_B(x,y) - G_B(x,y'))\, F(y), \end{aligned}$$

and the claim now follows by (1.11). Note that $G_B F$ is tacitly restricted to Λ in (3.55), (3.56), but F is supported in Λ, and it is a simple fact (which we will not need), that the bounds in (3.55), (3.56) extend to $|G_B F(x)|$ for x outside Λ. In a similar fashion it follows by (1.11) that

$$\|G_B F - [G_B F]_0\|_\infty \le c\,\|F\|_\infty\,. \tag{3.57}$$

By symmetry of $g_B(\cdot,\cdot)$ and (3.56) we also find that for F, H functions on Λ:

$$\|\langle F(G_B H)\rangle\|_\infty \le c\,\|F\|_\infty\,\|H\|_\infty, \text{ if } \langle F\rangle \text{ or } \langle H\rangle = 0. \tag{3.58}$$

The estimates (3.55) - (3.58) correspond in the present context to Lemmas 3.1 and 3.2 of [18]. The next control shows that $(G_B V_N)^2$ operates boundedly on functions on Λ:

$$\|(G_B V_N)^2\|_{L^\infty(\Lambda)\to L^\infty(\Lambda)} \le c_1\,. \tag{3.59}$$

To see this we decompose $(G_B V_N)^2 F$, for $F:\Lambda\to\mathbb{R}$ in the following way:

$$\begin{aligned}
&(G_B V_N)^2 F = A_1 + A_2 + A_3, \text{ where}\\
&A_1 = G_B V_N G_B (V_N F - \langle V_N F\rangle)\\
&A_2 = G_B V_N [G_B\langle V_N F\rangle]_0\\
&A_3 = G_B V_N (G_B\langle V_N F\rangle - [G_B\langle V_N F\rangle]_0),
\end{aligned} \tag{3.60}$$

and bound each term with the help of (3.55) - (3.57), as in Lemma 4.4 of [18].

The next estimate shows that $(G_B V_N)^2$ leaves the kernel of the map $F \to \langle V_N F\rangle$ almost invariant:

$$\|\langle V_N (G_B V_N)^2\, F\rangle\|_\infty \le c_2 \|\langle V_N F\rangle\|_\infty + \frac{c_3}{(\log N)^{3/2}}\;\|F\|_\infty\,. \tag{3.61}$$

The proof is similar to Lemma 4.5 of [18]. One writes for F function on Λ:

$$\langle V_N (G_B V_N)^2\, F\rangle = \langle V_N A_1\rangle + \langle V_N A_2\rangle + \langle V_N A_3\rangle,$$

with A_1, A_2, A_3 as in (3.60), and uses (3.58) and (3.55) - (3.57) to bound the various terms. The term $c_2\|\langle V_N F\rangle\|_\infty$ in the right-hand side of (3.61) actually stems from the bound on $\langle V_N A_2\rangle$.

Iterating (3.61), as in (4.34) of [18], one sees that for F: $\Lambda\to\mathbb{R}$ and $k\ge 1$,

$$\|\langle V_N (G_B V_N)^{2k} F\rangle\|_\infty \le c_2^k\,\|\langle V_N F\rangle\|_\infty + \frac{c^k}{(\log N)^{3/2}}\;\|F\|_\infty\,. \tag{3.62}$$

From this we conclude that for all $k\ge 0$,

$$\begin{aligned}
|a_N(2k+1)| &\overset{(3.48)}{=} \mathrm{cap}_B(\Lambda)^{-1}\,|(V_N, (G_B V_N)^{2k}\,1)|\\
&= |\Lambda|\;\mathrm{cap}_B(\Lambda)^{-1}\|\langle V_N (G_B V_N)^{2k}\,1\rangle\|_\infty\\
&\overset{(2.6),(3.62)}{\le} c\,\log N\frac{c^k}{(\log N)^{3/2}} \underset{N}{\longrightarrow} 0,
\end{aligned} \tag{3.63}$$

and (3.51) is proved. In addition for $k \geq 1$, we see that:

$$
(3.64)\quad \begin{aligned}
|a_N(2k)| &= \mathrm{cap}_B(\Lambda)^{-1}\,|(V_N,(G_BV_N)(G_BV_N)^{2(k-1)}\,1)| \\
&\overset{\text{symmetry}}{=} \mathrm{cap}_B(\Lambda)^{-1}|(G_BV_N, V_N(G_BV_N)^{2(k-1)}\,1)| \\
&\overset{(2.6)}{\leq} c\,\log N\,\|G_BV_N\|_\infty\,\|V_N(G_BV_N)^{2(k-1)}\,1\|_\infty \\
&\overset{(3.56),(3.59)}{\leq} c\,c^k.
\end{aligned}
$$

This bound combined with the last line of (3.63) finishes the proof of (3.50). There remains to prove (3.52). We proceed by induction. We already know that $a_N(0)=1$, and

$$
(3.65)\quad \begin{aligned}
a_N(2) &= \mathrm{cap}_B(\Lambda)^{-1}\,(V_N, G_BV_N) \\
&\overset{\langle V\rangle=0}{=} \frac{\mathrm{cap}_B(\Lambda)^{-1}}{2\log N}\sum_{x,y\in\Lambda} V(x)(g_B(x,y)-g_B(x,x))V(y) \\
&\underset{N}{\overset{(2.6),(1.11)}{\longrightarrow}} -\frac{1}{\pi}\sum_{x,y\in\Lambda} V(x)\,a(x-y)\,V(y) \overset{(3.5)}{=} \frac{1}{\pi}\,\mathcal{E}(V).
\end{aligned}
$$

Moreover using a telescoping sum, we can write for $k \geq 1$,

$$
(3.66)\quad \begin{aligned}
&\|(G_BV_N)^{2k}\,1_\Lambda - \big(\tfrac{1}{\pi}\,\mathcal{E}(V)\big)^k 1_\Lambda\|_\infty \leq \\
&\sum_{m=0}^{k-1}\|(G_BV_N)^{2(m+1)}\big(\tfrac{1}{\pi}\,\mathcal{E}(V)\big)^{k-(m+1)}1_\Lambda - (G_BV_N)^{2m}\big(\tfrac{1}{\pi}\,\mathcal{E}(V)\big)^{k-m}1_\Lambda\|_\infty \overset{(3.59)}{\leq} \\
&\sum_{m=0}^{k-1}\big(\tfrac{1}{\pi}\,\mathcal{E}(V)\big)^{k-(m+1)}c_1^m\,\|(G_BV_N)^2\,1_\Lambda - \tfrac{1}{\pi}\,\mathcal{E}(V)\,1_\Lambda\|_\infty \leq \\
&\qquad\qquad \widetilde{c}^k\,\|(G_BV_N)^2\,1_\Lambda - \tfrac{1}{\pi}\,\mathcal{E}(V)\,1_\Lambda\|_\infty.
\end{aligned}
$$

Using once again the identity $\langle V\rangle = 0$, we see that for $z \in \Lambda$

$$
(3.67)\quad \begin{aligned}
G_BV_NG_BV_N(z) &= \sum_{x\in\Lambda}\frac{1}{2\log N}\,g_B(z,x)\,V(x)\sum_{y\in\Lambda}(g_B(x,y)-g_B(x,x))\,V(y) \\
&\underset{N}{\overset{(1.11),(1.12)}{\longrightarrow}} -\frac{1}{\pi}\sum_{x,y}V(x)\,a(x-y)\,V(y) = \frac{1}{\pi}\,\mathcal{E}(V).
\end{aligned}
$$

Hence the last term of (3.66) tends to zero. We have thus shown that

$$
(3.68)\quad \|(G_BV_N)^{2k}1_\Lambda - \big(\tfrac{1}{\pi}\,\mathcal{E}(V)\big)^k 1_\Lambda\|_\infty \underset{N}{\longrightarrow} 0,\ \text{for any } k \geq 1\,.
$$

We then write:

$$
\begin{aligned}
a(2(k+1)) &= \mathrm{cap}_B(\Lambda)^{-1}\,(V_N, G_BV_N(G_BV_N)^{2k}\,1) \\
&\overset{\text{symmetry}}{=} \mathrm{cap}_B(\Lambda)^{-1}(G_BV_N, V_N(G_BV_N)^{2k}\,1) \\
&= \mathrm{cap}_B(\Lambda)^{-1}\Big(G_BV_N, V_N\Big\{(G_BV_N)^{2k}\,1_\Lambda - \big(\tfrac{1}{\pi}\,\mathcal{E}(V)\big)^k\,1_\Lambda\Big\}\Big) + \\
&\quad \mathrm{cap}_B(\Lambda)^{-1}\big(\tfrac{1}{\pi}\,\mathcal{E}(V)\big)^k(G_BV_N, V_N).
\end{aligned}
$$

By (3.65) the last term converges to $(\frac{1}{\pi}\,\mathcal{E}(V))^{k+1}$, whereas the absolute value of the previous term is smaller than

$$
c\,\log N\,\|G_BV_N\|\,\|V_N\|_\infty\,\Big\|(G_BV_N)^{2k}\,1_\Lambda - \big(\tfrac{1}{\pi}\,\mathcal{E}(V)\big)^k\,1_\Lambda\Big\|_\infty \underset{N}{\overset{(3.56),(3.68)}{\longrightarrow}} 0\,.
$$

This shows that $\lim_N a(2(k+1)) = (\frac{1}{\pi}\mathcal{E}(V))^{k+1}$, and completes the proof of (3.52). We have thus proved Lemma 3.6. $\square$

This also completes the proof of Proposition 3.5. $\square$

As explained below (3.27), Propositions 3.4 and 3.5 yield (3.25) and we have thus proved Theorem 3.2. $\square$

The last main result in this section is

Theorem 3.7. *Under P, as N goes to infinity,*

$$\Big(\frac{L^y_{t_N}}{\log N}\Big)_{y\in\mathbb{Z}^2} \text{ converges in distribution to a flat random field with constant value distributed as } R^2, \text{ with } R \text{ as in (3.3).} \tag{3.69}$$

Proof. We recall the notation $\widetilde{V}_N$ from (2.9). By Theorem 3.1 we only need to concentrate on the case $y=0$ and show that with R as in (3.3), under P,

$$\int_0^{t_N} \widetilde{V}_N(X_s)\,ds \underset{N}{\overset{\text{law}}{\longrightarrow}} R^2. \tag{3.70}$$

The same argument used to infer Theorem 3.1 from Theorem 3.2 now shows that (3.70) will follow once we show that in the notation of (2.11)

$$\Big(\frac{R_2-R_1}{N^2\log N}, \widetilde{\xi}_1\Big) \underset{N}{\overset{\text{law}}{\longrightarrow}} (\tau,\widetilde{\zeta}), \tag{3.71}$$

where τ and $\widetilde{\zeta}$ are independent exponential variables with parameter $\frac{\pi}{2}$. In this last step we use the fact that when $\widetilde{\sigma}_k, k\ge 0$, are the partial sums of i.i.d. copies $\widetilde{\zeta}_\ell, \ell\ge 1$, of $\widetilde{\zeta}$, and J is an independent Poisson variable with parameter $\alpha\frac{\pi}{2}$ one has:

$$E[e^{-\lambda\widetilde{\sigma}_J}] = \exp\Big\{-\alpha\,\frac{\pi}{2}(1-E[e^{-\lambda\widetilde{\zeta}}])\Big\} = \exp\Big\{-\frac{\alpha\lambda}{1+\frac{2}{\pi}\lambda}\Big\}, \text{ for } \lambda\ge 0, \tag{3.72}$$

so that $\widetilde{\sigma}_J$ has same distribution as R^2, see (1.37).

Now the same argument used in the proof of Theorem 3.2 shows that (3.71) will follow once we show the corresponding statement to Proposition 3.5, namely

$$\int_0^{T_B} \widetilde{V}_N(X_s)\,ds \underset{N}{\overset{\text{law}}{\longrightarrow}} \widetilde{\zeta} \text{ under } P_0. \tag{3.73}$$

Proceeding as in (3.46), (3.47), we see that

$$E_0\big[e^{z\int_0^{T_B}\widetilde{V}_N(X_s),ds}\big] = \sum_{n\ge 0}\widetilde{a}_N(n)\,z^n, \text{ for } |z|\le \widetilde{c}_N, \text{ where} \tag{3.74}$$

$$\begin{aligned} &\widetilde{a}_N(0)=1, \text{ and for } n\ge 1,\\ &\widetilde{a}_N(n) = \mathrm{cap}_B(\{0\})^{-1}(\widetilde{V}_N,(G_B\widetilde{V}_N)^{n-1}\,1)\\ &\qquad = g_B(0,0)\,\frac{1}{(\log N)^n}\,g_B(0,0)^{n-1} \underset{N}{\overset{(1.12)}{\longrightarrow}} \Big(\frac{2}{\pi}\Big)^n. \end{aligned} \tag{3.75}$$

In particular we see that

$$(3.76) \qquad 0 \le \widetilde{a}_N(n) \le c^n, \text{ for } n \ge 0,$$

$$(3.77) \qquad \widetilde{a}_N(n) \underset{N}{\longrightarrow} \Big(\frac{2}{\pi}\Big)^n, \text{ for } n \ge 0.$$

The same argument used below the statement of Lemma 3.6 then shows that when N goes to infinity,

$$(3.78) \qquad \begin{array}{l} \displaystyle\int_0^{T_B} \widetilde{V}_N(X_s)\, ds \text{ converges in law to a distribution with} \\ \text{characteristic function } \Big(1 - i\,\dfrac{2b}{\pi}\Big)^{-1},\ b \in \mathbb{R}\,. \end{array}$$

This proves (3.73) and completes the proof of Theorem 3.7. □

Remark 3.8.

1) The results of this section answer positively a question raised in Remark 4.10 1) of [18]. They are, as we now explain, consistent at a heuristic level with the limit statements contained in Theorem 4.2 and 4.9 of [18]. It was shown there that the occupation field $\mathcal{L}_y$, $y \in \mathbb{Z}^2$, of continuous time random interlacements on $\mathbb{Z}^3$ at level $\beta\, \frac{\log N}{N}$, with $\beta > 0$, in long rods $J_y = \{y\} \times \{1, \dots, N\} \subseteq \mathbb{Z}^3$, $y \in \mathbb{Z}^2$, satisfy the limit theorems:

$$(3.79) \qquad \Big(\frac{\mathcal{L}_y - \mathcal{L}_0}{\sqrt{2 \log N}}\Big)_{y \in \mathbb{Z}^2} \underset{N}{\overset{\text{law}}{\longrightarrow}} (\widehat{R}\, \widehat{\psi}_y)_{y \in \mathbb{Z}^2}\,,$$

and

$$(3.80) \qquad \Big(\frac{\mathcal{L}_y}{\log N}\Big)_{y \in \mathbb{Z}^2} \underset{N}{\overset{\text{law}}{\longrightarrow}} \text{the flat field with constant value } \widehat{R}^2,$$

where $\widehat{R}$ and $(\widehat{\psi}_y)_{y \in \mathbb{Z}^2}$ are independent with

$$(3.81) \qquad \widehat{R} \text{ BES}^0\Big(\sqrt{\beta}, \frac{3}{2\pi}\Big)\text{-distributed},$$

$$(3.82) \qquad \begin{array}{l} (\widehat{\psi}_y)_{y \in \mathbb{Z}^2} \text{ a centered Gaussian field with} \\ \text{covariance } E[\widehat{\psi}_y \widehat{\psi}_{y'}] = \dfrac{3}{2}\, (a(y) + a(y') - a(y' - y)), \text{ for } y, y' \in \mathbb{Z}^2\,. \end{array}$$

These results can be heuristically reconciled with Theorems 3.1 and 3.7 as follows. Continuous time random interlacements on $\mathbb{Z}^3$ at level $u_N = \beta\, \frac{\log N}{N}$ (which tends to zero), ought to describe the local picture left by a random walk on $(\mathbb{Z}/N\mathbb{Z})^3$ with unit jump rate, uniform starting distribution, at time $u_N\, N^3 = \beta N^2 \log N$ (see for instance [20], [19], which however contain results pertaining to $u_N = \text{const.}$). The $(\mathbb{Z}/N\mathbb{Z})^2$-projection of this walk behaves as a walk on $(\mathbb{Z}/N\mathbb{Z})^2$ with jump rate $\frac{2}{3}$, uniform starting distribution, which runs up to time $\beta N^2 \log N$ (note however that the local trace left by this walk close to the origin involves more than the local trace close to the origin of the original walk on $(\mathbb{Z}/N\mathbb{Z})^3$). The occupation field on $(\mathbb{Z}/N\mathbb{Z})^2$ induced by this projection is thus distributed as $(\frac{3}{2}\, L^y_{t_N})_{y \in \mathbb{Z}/N\mathbb{Z}^2}$, with $t_N = \frac{2}{3}\, \beta N^2 \log N \overset{\text{def}}{=} \alpha\, N^2 \log N$. By Theorem 3.1 and 3.7 we know that

$$(3.83) \qquad \Big(\frac{3}{2}\, \frac{L^y_{t_N} - L^0_{t_N}}{\sqrt{2 \log N}}\Big)_{y \in \mathbb{Z}^2} \underset{N}{\overset{\text{law}}{\longrightarrow}} \Big(\sqrt{\frac{3}{2}}\, R\, \sqrt{\frac{3}{2}}\, \psi_y\Big)_{y \in \mathbb{Z}^2}, \text{ and}$$

$$(3.84) \qquad \Big(\frac{3}{2}\, \frac{L^y_{t_N}}{\log N}\Big)_{y \in \mathbb{Z}^2} \longrightarrow \frac{3}{2}\, R^2,$$

with R and $(\psi_y)_{y\in\mathbb{Z}^2}$ as in (3.3), (3.4). However $\sqrt{\frac{3}{2}}\,R$ is $\mathrm{BES}^0(\sqrt{\frac{3\alpha}{2}}, \frac{3}{2\pi})$-distributed, i.e. has the same distribution as $\widehat{R}$ in (3.81), and $(\sqrt{\frac{3}{2}}\,\psi_y)_{y\in\mathbb{Z}^2}$ has the same distribution as $(\widehat{\psi}_y)_{y\in\mathbb{Z}^2}$. So we recover the limiting fields in (3.79), (3.80). In other words the limit behavior of occupation times of long rods in $\mathbb{Z}^3$ by random interlacements at level $u_N = \beta\,\frac{\log N}{N}$ recovers the limit behavior of the occupation times close to the origin of the $(\mathbb{Z}/N\mathbb{Z})^2$-projection of the walk on $(\mathbb{Z}/N\mathbb{Z})^3$, with uniform starting distribution, and running up to time $u_N N^3 = \beta N^2 \log N$. This signals the existence of a link between random interlacements and the walk on a large two-dimensional torus.

2) It would be interesting to give a proof of Theorems 3.1 and 3.7 along the lines of the above stated heuristics, bringing into play some transfer between continuous time random interlacements and random walk on $(\mathbb{Z}/N\mathbb{Z})^3$, see also [19]. It would also be interesting to give a proof of Theorems 3.1 based on the generalized second Ray-Knight theorem, see [13], p. 372, which relates the field of local times of walk on $\mathbb{T}_N$, at the first time the local time at the origin goes beyond some given (deterministic) level, and the Gaussian free field on $\mathbb{T}_N$ with covariance the Green function on $\mathbb{T}_N$ killed at the origin. The Poissonian regime makes its application a bit impractical, and perhaps Eisenbaum's isomorphism theorem, see [13], p. 362, might be more adapted. □

4 The ergodic regime

In this section we analyze the asymptotic behavior of the field of occupation times of the walk on $\mathbb{T}_N$ close to the origin at times much larger than $N^2 \log N$ as N goes to infinity. We relate the limiting behavior of this occupation field to the Gaussian free field pinned at the origin, cf. (1.36). The situation is simpler than in the Poissonian regime. The main statements appear in Theorems 4.1 and 4.3. In Remark 4.4 we reconcile these theorems with the results from [18] concerning the occupation times of large rods by random interlacements in $\mathbb{Z}^3$. We use throughout this section the convention on constants stated at the beginning of Section 2. We recall the notation t'_N from (0.2) ii)

$$t'_N = \alpha_N\, N^2 \log N, \text{ where } \lim_N \alpha_N = \infty\,. \tag{4.1}$$

Theorem 4.1. *Under P, as N goes to infinity,*

$$\Big(\frac{L^y_{t'_N} - L^0_{t'_N}}{\sqrt{2t'_N N^{-2}}}\Big)_{y\in\mathbb{Z}^2} \text{ converges in distribution to } (\psi_y)_{y\in\mathbb{Z}^2} \text{ (see (1.36) for notation).} \tag{4.2}$$

This theorem will follow from the next proposition (see (2.11), (3.5) for notation).

Proposition 4.2.

$$\lim_N E[\xi_1^2] = \frac{2}{\pi}\,\mathcal{E}(V)\,. \tag{4.3}$$

$$\sup_N E[\xi_1^4] \le c\,. \tag{4.4}$$

Proof of Theorem 4.1 *(assuming Proposition* 4.2*):* Claim (4.2) will follow once we show that for V as in (2.3),

$$\int_0^{t'_N} V'_N(X_s)\,ds \xrightarrow[N]{\text{law}} N(0,\mathcal{E}(V)), \text{ (see (2.10) for notation).} \tag{4.5}$$

By (2.17) it suffices to show that

$$S'_{K'_N} \underset{N}{\overset{\text{law}}{\longrightarrow}} N(0,\mathcal{E}(V)). \tag{4.6}$$

By (1.26) we know that

$$\lim_N P[m_N^- \le K'_N \le m_N^+] = 1, \text{ if } m_N^\pm = \big[(1\pm\delta_N)\,\tfrac{\pi}{2}\,\alpha_N\big]\,. \tag{4.7}$$

It follows from Kolmogorov's inequality that for any $\varepsilon > 0$,

$$P\big[\sup_{m_N^- \le k \le m_N^+} |S'_k - S'_{m_N^-}| \ge \varepsilon\big] \le \varepsilon^{-2}(m_N^+ - m_N^-)\,\mathrm{var}(\xi'_1) \overset{(2.12)}{\le} c\,\varepsilon^{-2}(\delta_N\alpha_N + 1)\frac{\mathrm{var}(\xi_1)}{\alpha_N} \underset{N}{\overset{(4.3)}{\longrightarrow}} 0\,.$$

Claim (4.2) will thus follow once we show that

$$S'_{m_N^-} \underset{N}{\overset{\text{law}}{\longrightarrow}} N(0,\mathcal{E}(V))\,. \tag{4.8}$$

To this end we introduce the characteristic function of ξ_1 under P

$$\varphi(t) = E[e^{it\xi_1}],\; t \in \mathbb{R}\,. \tag{4.9}$$

By a similar reward renewal argument as above (1.21) we have

$$E[\xi_1] \overset{(2.11)}{=} \sum_{y\in\mathbb{T}_N} V_N(y)\, E_0\Big[\int_0^{H_0\circ\theta_{T_B}+T_B} 1\{X_s = y\}\, ds\Big] = h_0 \sum_{y\in\mathbb{T}_N} V_N(y)\,\pi(y) \overset{(2.3)}{=} 0\,. \tag{4.10}$$

By (3.7) p. 86 of [7] and (4.10) we find that for $u \in \mathbb{R}$,

$$\Big|\varphi(u) - 1 + \frac{1}{2}\, E[\xi_1^2]\, u^2\Big| \le \frac{|u|^3}{6}\, E[|\xi_1|^3] \overset{(4.4)}{\le} c\,|u|^3. \tag{4.11}$$

As a result, we see that for $t \in \mathbb{R}$, by (4.3), (4.11),

$$E[e^{itS'_{m_N^-}}] = \varphi\Big(\frac{t}{\sqrt{\alpha_N}}\Big)^{m_N^-} = \Big(1 - \frac{\mathcal{E}(V)}{\pi}\,\frac{t^2}{\alpha_N} + O\Big(\frac{|t|^3}{\alpha_N^{3/2}}\Big)\Big)^{m_N^-} \underset{N}{\overset{(4.7)}{\longrightarrow}} e^{-\frac{1}{2}\mathcal{E}(V)t^2}. \tag{4.12}$$

This proves (4.8) and Theorem 4.1 follows. □

Proof of Proposition 4.2*:* By (2.5) and the strong Markov property at time $T_B \circ \theta_{R_1} + R_1$,

$$\lim_N E\Big[\Big|\xi_1 - \int_{R_1}^{R_1+T_B\circ\theta_{R_1}} V_N(X_s)\, ds\Big|^4\Big] = 0\,. \tag{4.13}$$

We can thus replace ξ_1 by $\int_0^{T_B} V_N(X_s)\, ds$ and P by P_0 when proving (4.3), (4.4). Moreover observe that, see (3.42) for notation,

$$\lim_N E_\nu\Big[\Big|\int_0^{T_B} V_N(X_s)\, ds - 1\{H_0 < T_B\}\Big(\int_0^{T_B} V_N(X_s)\, ds\Big)\circ\theta_{H_0}\Big|^4\Big] = 0\,. \tag{4.14}$$

Indeed the expression under absolute value equals $\int_0^{H_0\wedge T_B} V_N(X_s)ds$, and we apply (2.5). Hence we find that for $p = 2, 4$

$$\lim_N \Big|E_\nu\Big[\Big(\int_0^{T_B} V_N(X_s)\, ds\Big)^p\Big]^{\frac{1}{p}} - P_\nu[H_0 < T_B]^{\frac{1}{p}}\, E_0\Big[\Big(\int_0^{T_B} V_N(X_s)\, ds\Big)^p\Big]^{\frac{1}{p}}\Big| = 0\,. \tag{4.15}$$

By (2.4) we know that $P_\nu[H_0 < T_B] \underset{N}{\longrightarrow} 1$, and by (3.46), (3.47) that $E_\nu[(\int_0^{T_B} V_N(X_s)ds)^p] = p!\, a_N(p)$. As a consequence of Lemma 3.6 we thus find that for $p = 2, 4$

$$\lim_N E_0\Big[\Big(\int_0^{T_B} V_N(X_s)\, ds\Big)^p\Big] = p!\,\Big(\frac{\mathcal{E}(V)}{\pi}\Big)^{\frac{p}{2}}. \tag{4.16}$$

In particular this completes the proof of Proposition 4.2. □

Our last result is

Theorem 4.3. *Under P, as N goes to infinity,*

$$\Big(\frac{L^y_{t'_N}}{t'_N N^{-2}}\Big)_{y\in\mathbb{Z}^2} \text{ converges in distribution to the flat random field with value } 1. \tag{4.17}$$

Proof. By Theorem 4.1 we only need to consider the single location $y = 0$, and prove that under P

$$\int_0^{t'_N} \widetilde{V}'_N(X_s)\, ds \xrightarrow[N]{\text{law}} 1, \text{ (see (2.10) for notation).} \tag{4.18}$$

We cannot simply invoque the ergodic theorem to prove (4.19) since we have to deal with a sequence of Markov chains on the various state spaces $\mathbb{T}_N$ as N varies. Instead we argue as follows. By Proposition 2.2 it suffices to show that under P,

$$\widetilde{S}_{K'_N} \xrightarrow[N]{\text{law}} 1\,. \tag{4.19}$$

By (4.7) and the fact that $k \ge 0 \to S_k$ is non-decreasing, we only need to show that

$$\widetilde{S}'_{m_N^-} \xrightarrow[N]{\text{law}} 1, \text{ and } \widetilde{S}'_{m_N^+} \xrightarrow[N]{\text{law}} 1\,. \tag{4.20}$$

By (2.12) and (1.3) we find that

$$E[\widetilde{\xi}'_1] = \frac{g_B(0,0)}{\alpha_N \log N} \overset{(1.12)}{\sim} \frac{2}{\pi}\, \alpha_N^{-1}, \tag{4.21}$$

and by a similar calculation as in (3.46),

$$E[\widetilde{\xi}'^2_1] = \frac{E_0[(\int_0^{T_B} 1\{X_s = 0\}ds)^2]}{(\alpha_N \log N)^2} = 2\frac{g_B(0,0)^2}{(\alpha_N \log N)^2} \overset{(1.12)}{\le} c\,\alpha_N^{-2}. \tag{4.22}$$

The variables $\widetilde{\xi}'_k$, $k \ge 1$, are i.i.d. and we thus find that:

$$E\Big[\Big(\widetilde{S}'_{m_N^+} - \widetilde{S}'_{m_N^-}\Big)^2\Big] \le c(m_N^+ - m_N^-)\,\mathrm{var}_P(\widetilde{\xi}'_1) + (m_N^+ - m_N^-)^2\, E[\widetilde{\xi}'_1]^2 \underset{N}{\longrightarrow} 0, \tag{4.23}$$

using (4.7), (4.21), (4.22). In addition we see that

$$\mathrm{var}_p(\widetilde{S}_{m_N^-}) \le c\, m_N^-\, \alpha_N^{-2} \underset{N}{\longrightarrow} 0\,. \tag{4.24}$$

If we now observe that

$$E[\widetilde{S}_{m_N^-}] = m_N^-\, E[\widetilde{\xi}'_1] \underset{(4.21)}{\overset{(4.7)}{\sim}} \frac{\pi}{2}\,\alpha_N \times \frac{2}{\pi}\,\alpha_N^{-1} = 1\,. \tag{4.25}$$

The claim (4.20) readily follows and this proves Theorem 4.3. □

Remark 4.4.

1) The results of this section can also, at a heuristic level, be reconciled with the limit statements of [18]. It was shown in Theorems 4.2 and 4.9 of [18] that the occupation field $\mathcal{L}'_y$, $y \in \mathbb{Z}^2$, of long rods $J_y = \{y\} \times \{1, \dots, N\}$, $y \in \mathbb{Z}^2$, by continuous time random interlacements in $\mathbb{Z}^3$ at level $u'_N = \beta_N \frac{\log N}{N}$, where $\lim_N \beta'_N = \infty$, satisfy the limit theorems:

$$\left(\frac{\mathcal{L}'_y - \mathcal{L}'_0}{\sqrt{2u'_N N}}\right)_{y\in\mathbb{Z}^2} \underset{N}{\overset{\text{law}}{\longrightarrow}} (\widehat{\psi}_y)_{y\in\mathbb{Z}^2}, \text{ (see (3.82) for notation),} \tag{4.26}$$

and

$$\left(\frac{\mathcal{L}'_y}{u'_N N}\right)_{y\in\mathbb{Z}^2} \underset{N}{\overset{\text{law}}{\longrightarrow}} \text{ the constant field with value 1.} \tag{4.27}$$

Once again the heuristic argument explained below (3.82) suggests that the above results reflect the presence of similar limit theorems for the occupation time of a walk on $\mathbb{T}_N$ with jump rate $\frac{2}{3}$, uniform starting distribution, and running up to time $u'_N N^3 = \beta_N N^2 \log N$, or equivalently for $(\frac{3}{2}\, L^y_{t'_N})_{y\in\mathbb{T}_N}$, when $t'_N = \frac{2}{3}\,\beta_N N^2 \log N \overset{\text{def}}{=} \alpha_N N^2 \log N$. But Theorems 4.1 and 4.3 indeed show that

$$\left(\frac{3}{2}\, \frac{L^y_{t'_N} - L^0_{t'_N}}{\sqrt{2\beta_N \log N}}\right)_{y\in\mathbb{Z}^2} \underset{N}{\overset{\text{law}}{\longrightarrow}} \left(\sqrt{\frac{3}{2}}\; \psi_y\right)_{y\in\mathbb{Z}^2} \overset{\text{distribution}}{=} (\widehat{\psi}_y)_{y\in\mathbb{Z}^2}, \tag{4.28}$$

and

$$\left(\frac{3}{2}\, \frac{L^y_{t'_N}}{\beta_N \log N}\right) \underset{N}{\overset{\text{law}}{\longrightarrow}} \text{ the constant field with value 1.} \tag{4.29}$$

So indeed the heuristic guess based on the results on random interlacements proven in [18] leads to the correct limit statement proven in this section.

2) The above discussion and Remark 3.8 1) point out the existence of a link, at least for the "local picture", between random walk with jump rate $\frac{2}{3}$ on $\mathbb{T}_N$, and random interlacements on $\mathbb{Z}^3$ via their presence in long rods $J_y = \{y\} \times \{1, \dots, N\}$, $y \in \mathbb{Z}^2$, when N goes to infinity. It would be interesting to explore whether this link extends to more global quantities. For instance one knows from [6] that the cover time of $\mathbb{T}_N$ by the above mentioned walk has order $\frac{6}{\pi}\, N^2 (\log N)^2$. By analogy one can introduce the random level $\mathcal{U}_N$, which is the smallest $u \geq 0$ such that all J_y, $y \in \{1, \dots, N\}^2$, meet $\mathcal{I}^u$, the random interlacement at level u, see [17]. How does $\mathcal{U}_N N^3$ compare to $\frac{6}{\pi}\, N^2 (\log N)^2$?

□

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