Mathematics of the Cube

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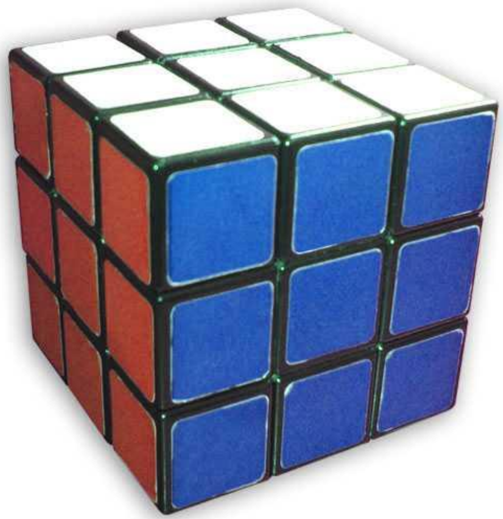
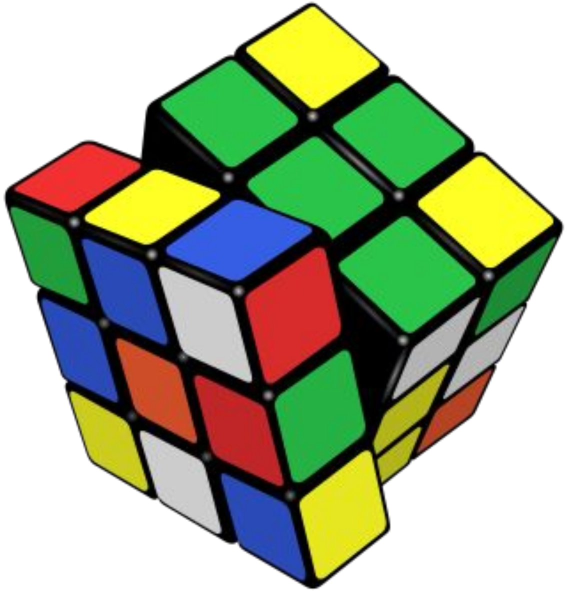
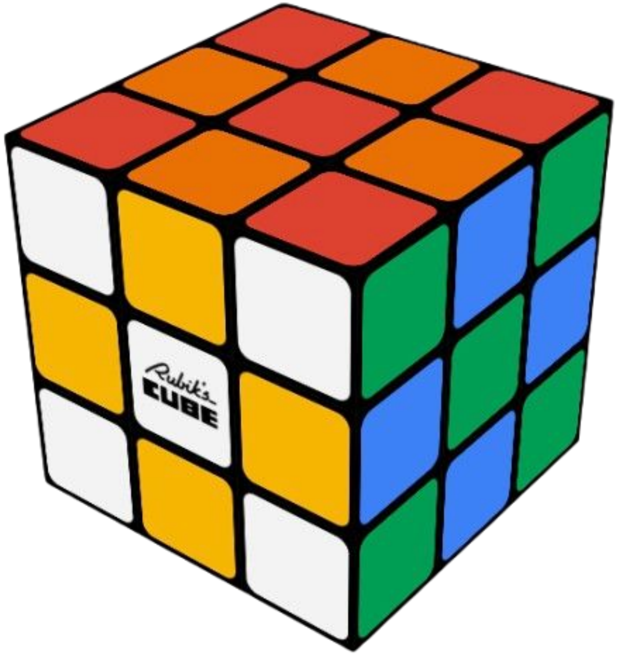
Rubik’s cube day

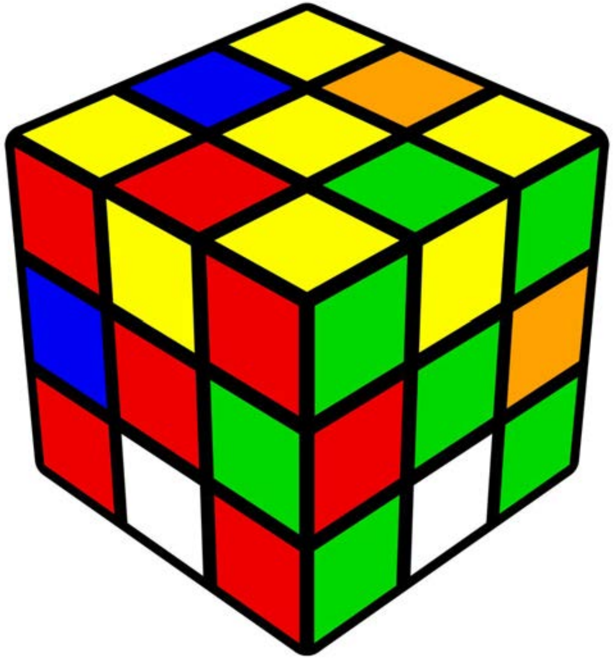
Rubik's cube



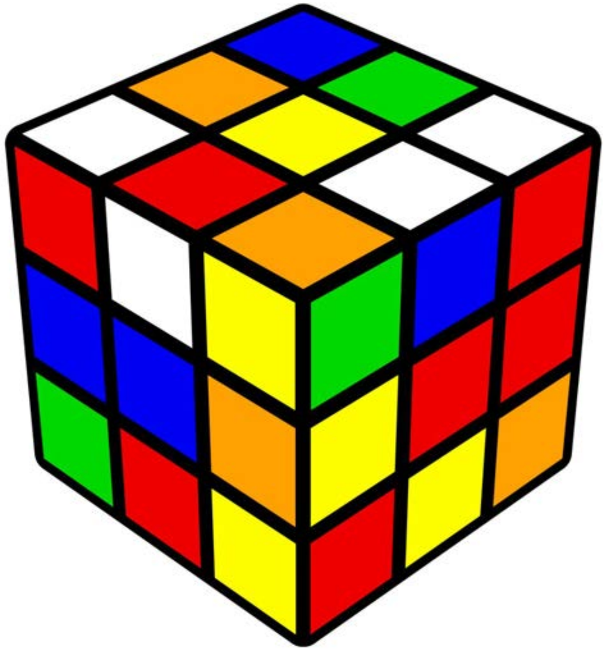


You can make nice patterns out of it



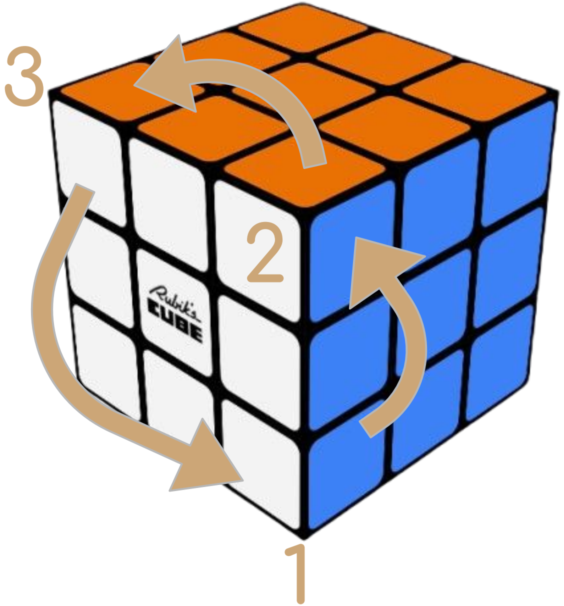


Or just scramble it



Some people can also solve it quickly

How to solve the Cube?



Step 1: Bring the cublet 1 to the place of cublet 2 without disrupting the top layer

Step 2: Rotate the top face counterclockwise

Step 3: Reverse Step 1

Step 4: Reverse Step 2

How to solve the Cube?

Step 1: Bring the cublet 1 to the place of cublet 2 without disrupting the top layer — X

Step 2: Rotate the top face counterclockwise — Y

Step 3: Reverse Step 1 — X^{-1}

Step 4: Reverse Step 2 — Y^{-1}

Mathematical perspective: Group theory

6 turns on a Rubik's cube - one for each face

Counterclockwise turns -

R, L, U, D, F, B (generators)

Sequences of turns make up a **group**

e.g. RU, FU are elements of the group

The Cube Group

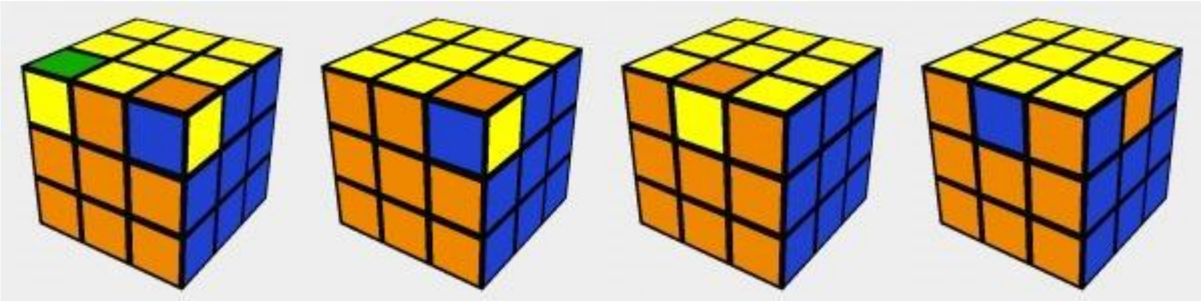
The Cube group is a subgroup of $S_8 \times S_{12} \times (Z/3Z)^8 \times (Z/2Z)^{12}$

Permutation of corners

Permutation of edges

Orientation of corners

Orientation of edges



Invariants of legal positions

Write 0, 1, 2 on corner cublets

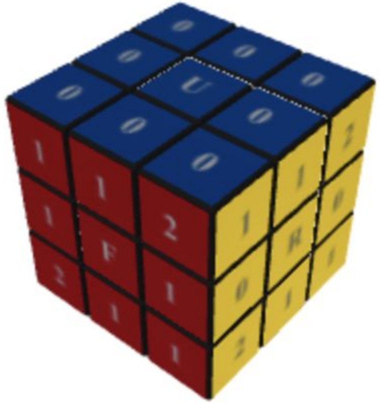
Write 0, 1 on edge cublets

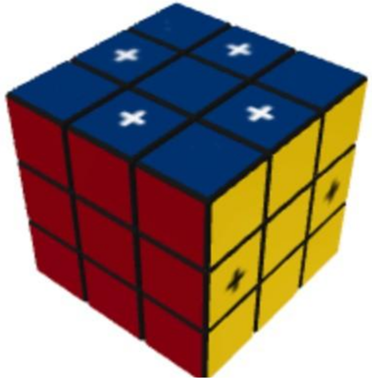
Place pluses on the cube (not moving)

Sum the numbers marked by +

Sum is divisible by 3 for legal configurations

Two conditions: corners and edges





Invariants of legal positions

Orientation of the corner i = c_i, Orientation of the edge j = e_j.

$c_1 + \dots + c_8 = 0$ modulo 3

$e_1 + \dots + e_{12} = 0$ modulo 2

sign(corner permutation) = sign(edge permutation)

How many configurations are there?

Recall Cube group is a subgroup of $S_8 \times S_{12} \times (Z/3Z)^8 \times (Z/2Z)^{12}$

Out of all configurations, only 1 in 12 is legal

Hence, there are $8! \times 12! \times 3^8 \times 2^{12} / 12 \approx 4{,}3 \times 10^{19}$

So, about 43 quntillion legal positions

How many turns do we need to solve the Cube, in the worst case?

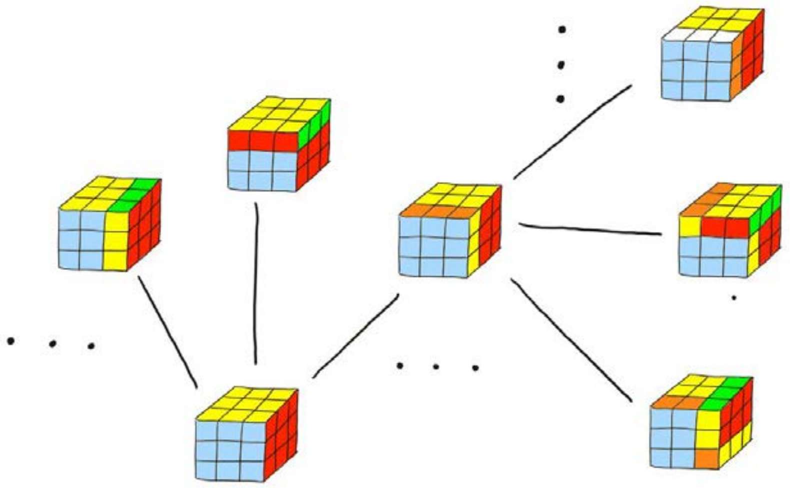
Can you always solve the Cube in 100 moves?

Cayley graphs

Vertices = legal cube configurations (= group elements)

Edges = turning a single face (generators of the group)

Metric: one face rotation = one move



How many moves do you need to solve a given cube?

Imagine our Cayley graph is a skyscraper

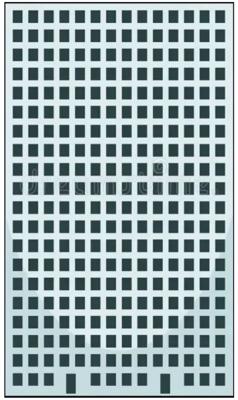
Turns U, D, R^2, L^2, F^2, B^2 move you around the same floor

Other turns allow you to change floors

Your goal is to find the exit

First get to the ground floor using any moves

Then, find the exit without leaving the ground floor



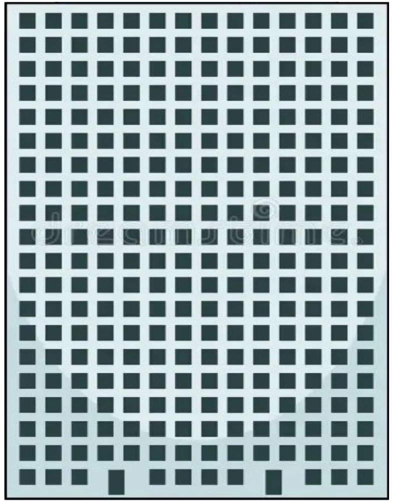
The subgroup H

Consider the subgroup $H = \langle U, D, R^2, L^2, F^2, B^2 \rangle$

Phase 1: From the starting configuration, find a path to H

Phase 2: Find an shortest path within H to the solved cube

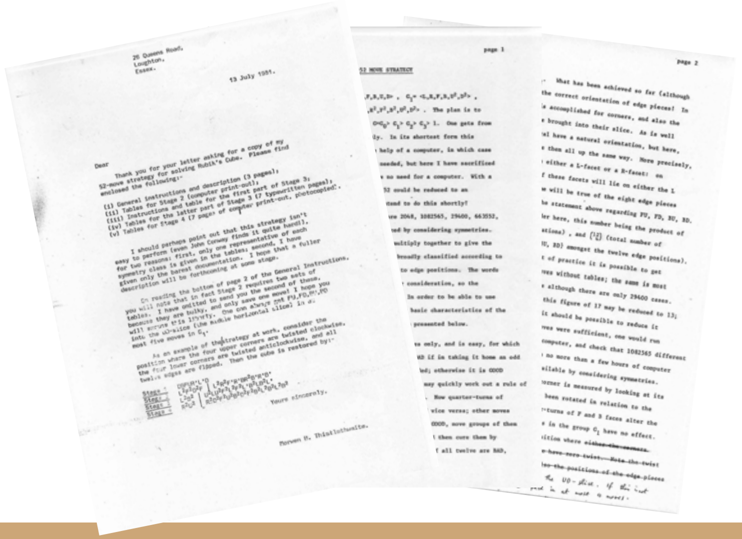
Trying many potential paths in Phase 1 and combining them with optimal solutions in Phase 2 usually gives good results

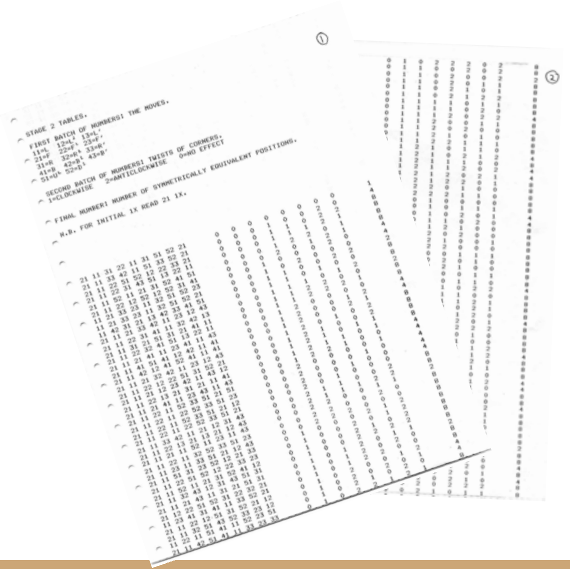


What bounds do we get?

In 1981, Thistlethwaite gave a 4-phase algorithm showing every cube can be solved in 52 moves

20 pages of hard-coded tables, 3 intermediate subgroups





Lower bounds: counting

Count the number of sequences S_n of n moves such that:

No two consecutive moves are on the same face

If R, L are adjacent, R comes before L (similarly for pairs U, D and F, B)

$$S_n = 12\, S_{n-1} + 18\, S_{n-2},$$

$$S_0 = 1,\ S_1 = 18,\ S_2 = 243$$

Solving the recurrence gives $S_{17} < 2 \times 10^{19}$

So, at least 18 moves are needed (most positions require 18 moves)

18 52

1981

18

42

1990

18 39

1992

18 37

1992

18 29

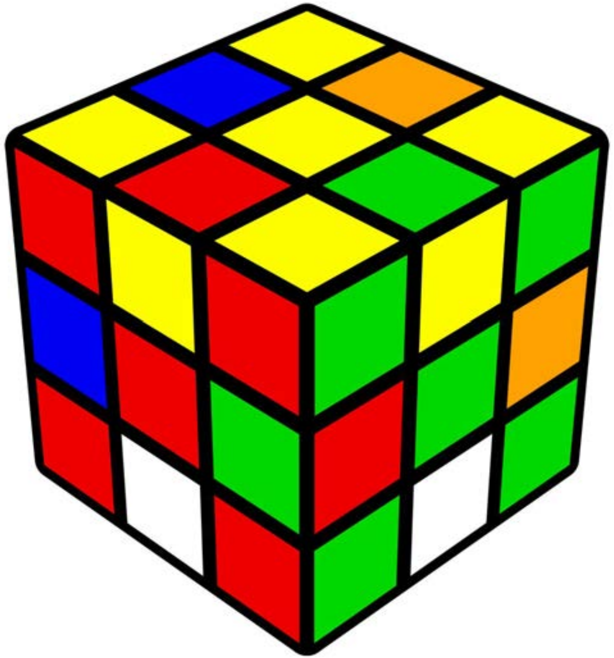
1995

20 29

1995

Computers come to the rescue

20 moves are needed to solve the superflip configuration



20 28

2005

20 27

2006

20 26

2007

20 25

2008

20 23

2008

20 22

2008

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THE DIAMETER OF THE RUBIK'S CUBE GROUP IS TWENTY*

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Abstract. We give an expository account of our computational proof that every position of Rubik's Cube can be solved in 20 moves or less, where a move is defined as any twist of any face. The roughly 4.3×10^{19} positions are partitioned into about two billion cosets of a specially chosen subgroup, and the count of cosets required to be treated is reduced by considering symmetry. The reduced space is searched with a program capable of solving one billion positions per second, using about one billion seconds of CPU time donated by Google. As a byproduct of determining that the diameter is 20, we also find the exact count of cube positions at distance 15.