Teaching Algebra and Computer Algebra Using Rubik's Cube

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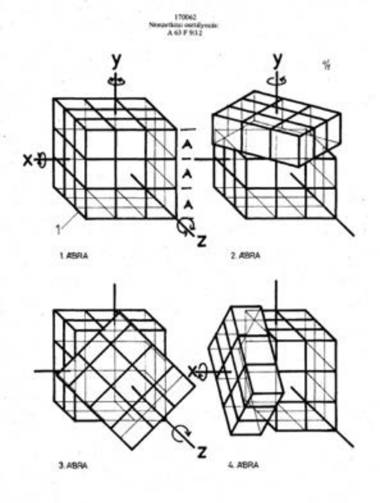
1. A Brief History of the Cube

There is an old saying about those who forget history.
I don't remember it, but it's good.
(Forgotten Author)

1974 **Ernő Rubik**, a Hungarian professor of architecture, invents the **Magic Cube**.

1975 The inventor registers his Hungarian patent **HU170062** which is granted in 1977.


170062
Nemzetközi osztályozás:
A 63 F 9/12
y
y
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1
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1. ÁBRA
2. ÁBRA
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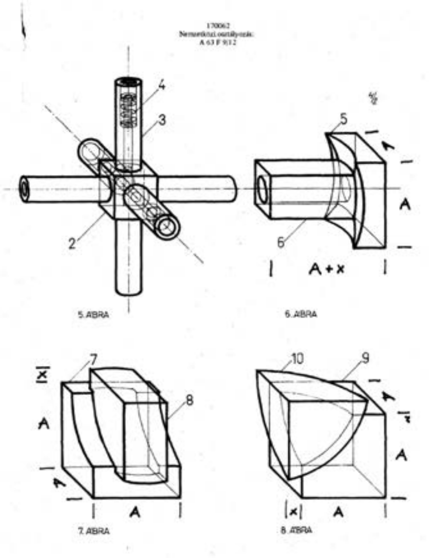
4. ÁBRA


170062
Nemzetközi osztályozás:
A 63 F 9/12
4
3
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5. ÁBRA
6. ÁBRA
7
8
x
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7. ÁBRA
10
9
A
x
A
8. ÁBRA

1977 The Hungarian company **Politechnika** (later renamed to **Poli-toys**) produces about 5000 magic cubes.

1978-1980 The companies **Pentangle (GB)** and **Ideal Toys (USA)** take care of international sales and order 500 000 magic cubes. For marketing reasons it is renamed **Rubik's Cube**.

1980 Rubik's cube is presented at **International Games Fairs** and chosen **Game of the Year** in Germany. A worldwide boom starts and within one year more than **100 million** cubes are sold.

The Rubik's Cube Today

11.6.2023 **Max Park** sets a new world record in speedcubing. He solves Rubik's Cube in **3.13 seconds**.

2024 Rubik's Cube is the best-selling toy of all times. More than 500 million cubes have been sold.

2024 In the online museum of `twistypuzzles.com` more than 12000 twisty puzzles are described. Almost daily new versions of Rubik's Cube are invented.

2. A Mathematical Model

What is Higher Mathematics?

If you awake in the morning with an Unknown.

First we need a lot of **definitions**. We assign names to the **moves**:

R rotate the **Right** side clockwise by 90^o

L rotate the **Left** side clockwise by 90^o

F rotate the **Front** side clockwise by 90^o

B rotate the **Back** side clockwise by 90^o

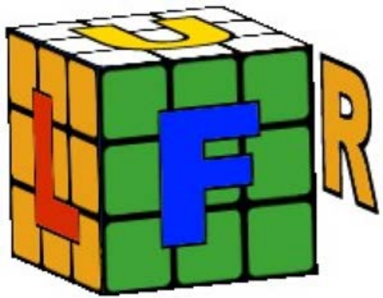
U rotate the **Upper** side clockwise by 90^o

D rotate the **Down** side clockwise by 90^o

More Moves

R', L', F', B', U', D' corresponding anticlockwise 90^o rotations

$\mathbf{R^2}$, $\mathbf{L^2}$, $\mathbf{F^2}$, $\mathbf{B^2}$, $\mathbf{U^2}$, $\mathbf{D^2}$ corresponding 180^o rotations



Even More Terminology

The outside of the Rubik's cube consists of 26 little **cubies**.

On the outward faces of the cubies we have coloured **stickers**.

By turning the sides repeatedly and randomly, the cube is **scrambled**.

Goal: Find suitable moves to return a scrambled cube to its original state where the stickers on each face all have the same colour.

Idea: Apply certain **sequences of moves** (sometimes called **algorithms**) which achieve intermediate goals such as positioning some corners or edges.

The Rubik's Cube Group

The sequences of moves of a Rubik's cube form a set which has the mathematical structure of a **group**. This means:

(1) We can **compose** two sequences s_1, s_2 and get a sequence $s_1 \cdot s_2$.

(2) For three sequences s_1, s_2, s_3, the **associative law** $(s_1 \cdot s_2) \cdot s_3 = s_1 \cdot (s_2 \cdot s_3)$ holds.

(3) There is a sequence, namely the **trivial sequence** e ("do nothing") such that $e \cdot s = s \cdot e = s$ for every sequence s.

(4) Every sequence can be undone by the **inverse sequence**. E.g., the inverse of **R** is **R'**, and the inverse of **F' U** is **U' F**.

Note: To undo a sequence of moves, we have to perform the inverse moves in the reverse order.

More About the Rubik's Cube Group

Example

(a) Another group is the set of integers $\mathbb{Z}$ with respect to **addition**.

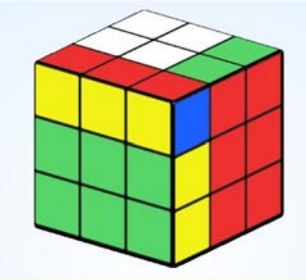
(b) The set of non-zero rational numbers $\mathbb{Q} \setminus \{0\}$ is a group with respect to **multiplication**.

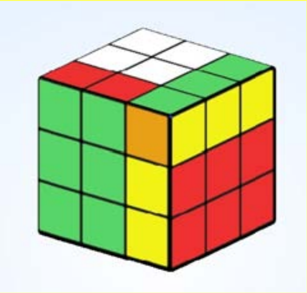
The Rubik's cube group is a **subgroup** of the group of **permutations** of the 54 stickers. The Rubik's cube group has

$$\mathbf{43,252,003,274,489,856,000} \quad \text{elements!}$$

The Commutator

Let us compare the results of the sequences **R F'** and **F' R**:

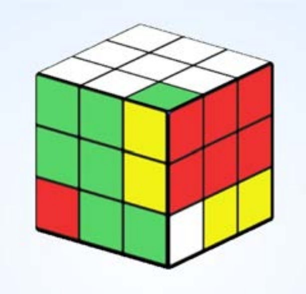


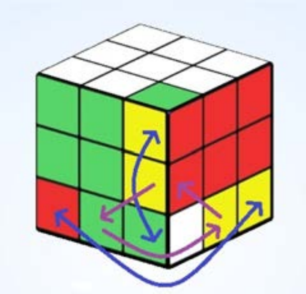


They are different! This means that the order of the moves matters, i.e., the Rubik's cube group is **not commutative**.

In contrast, the groups $(\mathbb{Z}, +)$ and $(\mathbb{Q} \setminus \{0\}, \cdot)$ are commutative.

Now let us look at the sequence **C = R F' R' F** (**out, out, in, in**).
It is called the **commutator**. What did it do to the cubies?





We get a **double transposition** of corners and a **3-cycle** of edges.

The commutator C has **order** 6, i.e., we have $\mathbf{C}^6 = e$.

4. A Slow-Cubing Tutorial

Look! I solved this puzzle in 4 hours!

So what?

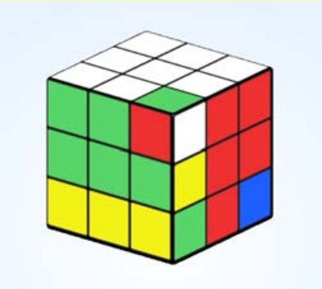
On the box it says 3-6 years!

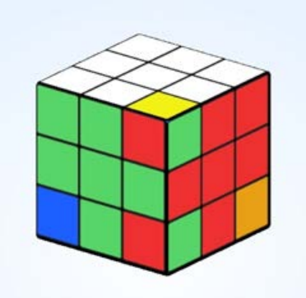
It is estimated that less than 5.8% of the world's population can solve the Rubik's Cube.

Goal: Solve the cube using only **one** type of sequence, namely commutators like **C = R F' R' F**.

Note: The inverse sequence of **C** is **C' = F' R F R'**. Thus it is **out, out, in, in** again, but this time starting with the **F** face.

Looking at the action of the commutator **C**, it is natural to check out what $\mathbf{C}^2$ and $\mathbf{C}^3$ do.



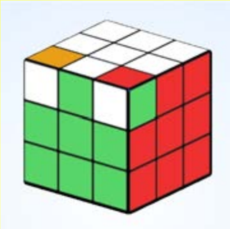


Here $\mathbf{C}^2$ is a **3-cycle** of edges together with rotations of some corners, and $\mathbf{C}^3$ is a **double transposition** of corners.

Idea: These operations are enough to solve the cube!

First Applications of Commutators

Suppose we want to solve the following position:



(1) Apply $\mathbf{C}^2$ to rotate the **FRU** corner by 120^o.

(2) Use the **setup move** **U'**.

(3) Apply $(\mathbf{C}')^2$ to rotate the **FRU** corner by -120^o.

(4) Undo the setup move using **U**.

Further Applications of Commutators

(1) Using $\mathbf{C^3}$ we can position the corners. We can create a 3-cycle of the corners via $\mathbf{C^3}$ **U'** $\mathbf{C^3}$ **U**.

(2) We can position the edges using $\mathbf{C^2}$.

(3) Another easy 3-cycle of edges is given by the commutator **M** $\mathbf{D}^2$ **M'** $\mathbf{D}^2$ where M is a 90^o turn of the **middle band**.

(4) Finally, to orient the edges, we can use $\mathbf{C}^2$ and check what it does to the orientation of the edges. Then turn the cube by 120^o around the space diagonal and undo using $(\mathbf{C}')^2$.

Remark

(a) *All products of commutators form a subgroup of the Rubik's cube group called the* **commutator subgroup** *which accounts for exactly half of all sequences.*

(b) *Let* **A** *be a sequence of moves, and let* **S** *be a setup move. Then the product* **S A S'** *is called a* **conjugation** *in group theory.*

Conjugations have many good properties: the conjugation of a product is a product of conjugations, the conjugation of a commutator is a commutator, etc. They are important tools in group theory.

4. God's Number

All those who believe in telekinesis, raise my hand.

For every position of the cube, we can look for the **shortest** sequence of moves which solves it. The maximal length of such a shortest sequence is called **God's number**. Clearly, this number depends on how we count moves.

Definition

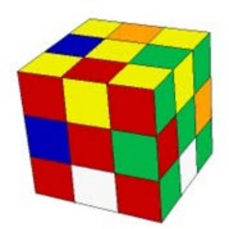
Counting **R**, **R'**, and $\mathbf{R^2}$ as **one** move each, and doing the same for the other face turns, we obtain the **face-turn metric (FTM)**.

God's Number in the Face-Turn Metric

Remark

In **2010** *it was shown that God's number in the FTM is* **20**.

One position which provably needs 20 moves is the **superflip**.



One shortest solution of the superflip is

U R^2 F B R B^2 R U^2 L B^2 R U' D' R^2 F R' L B^2 U^2 F^2.

God's Number in the Quarter-Turn Metric

Definition

Counting **R** and **R'** as **one** move, and $\mathbf{R^2}$ as **two** moves, and doing the same for the other face turns, we get the **quarter-turn metric**.

Remark

In **2014***, it was shown by* **Tomas Rokicki** *and* **Morley Davidson** *that God's number in the quarter-turn metric is* **26***. They needed about 29 CPU years for the calculation which involved solving 55,882,296 positions.*

One of the positions which provably needs 26 moves to be solved is the **superflip plus fourspot***.*

The Superflip Plus Fourspot

One shortest solution of the superflip plus fourspot is
U U F U U   R' L F F U   F' B' R L U   U R U D' R L' D R' L' D D.

The Hilbert-Dehn Function

Definition

For every $i \geq 0$, the number $\mathrm{HD}_G(i)$ of all positions requiring at least i moves for their solution in the quarter turn metric is called the **Hilbert-Dehn function** of the Rubik's cube group G.

Some values of the Hilbert-Dehn function of the Rubiks' cube group have been calculated:

$\mathrm{HD}_G(1) = 12$	$\mathrm{HD}_G(7) = 8,221,632$	$\mathrm{HD}_G(13) = 5,442,351,625,028$
$\mathrm{HD}_G(2) = 114$	$\mathrm{HD}_G(8) = 76,843,595$	$\mathrm{HD}_G(14) = 50,729,620,202,582$
$\mathrm{HD}_G(3) = 1,068$	$\mathrm{HD}_G(9) = 717,789,576$	$\mathrm{HD}_G(15) = 472,495,678,811,004$
$\mathrm{HD}_G(4) = 10,011$	$\mathrm{HD}_G(10) = 6,701,836,858$	$\mathrm{HD}_G(16) = 4,393,570,406,220,123$
$\mathrm{HD}_G(5) = 93,840$	$\mathrm{HD}_G(11) = 62,549,615,248$	$\mathrm{HD}_G(17) = 40,648,181,519,827,392$
$\mathrm{HD}_G(6) = 878,880$	$\mathrm{HD}_G(12) = 583,570,100,997$	$\mathrm{HD}_G(18) = 368,071,526,203,620,348$

The values $\mathrm{HD}_G(19), \ldots, \mathrm{HD}_G(26)$ are not known.

5. God's Algorithm

An ideal math talk should have theorems and jokes, and they should not be the same.

Goal: Find an algorithm which computes for every position of Rubik's cube the shortest sequence of moves which solves it.

Since it is assumed that such an algorithm is humanly inconceivable, it is called **God's algorithm**.

However, using the theory of non-commutative **Gröbner bases** in **Computer Algebra**, we can find such an algorithm!

A Mathematical Excursion

Definition

(a) Given a field K and **unknowns** $x_1, \ldots, x_n$, we can form the **non-commutative polynomial ring** $K\langle x_1, \ldots, x_n\rangle$. Its elements are formal sums $f = c_1\, t_1 + \cdots + c_s\, t_s$ where $c_i \in K$ and t_i is a **word** in $x_1, \ldots, x_n$.

(b) For the Rubik's cube group G, we can form the **group ring** $K\langle G\rangle = \{\sum_{g_i \in G} c_i\, g_i \mid c_i \in K\}$. Since G is generated by the face turns R, L, F, B, U, D and their inverses R', L', F', B', U', D', we have a **presentation**

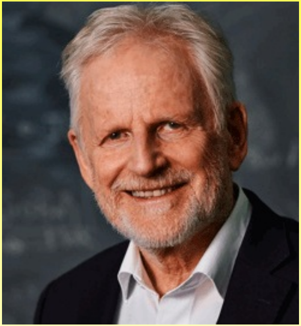
$$K\langle G\rangle \;\cong\; K\langle R, R', L, L', F, F', B, B', U, U', D, D'\rangle / I$$

where $I = \langle R^4 - 1,\ (R')^4 - 1,\ RR' - 1,\ R'R - 1, \ldots\rangle$.

Three Basic Algorithms of Algebra

Euclid

C.F. Gauß



B. Buchberger

Buchberger's Algorithm

Sequences of moves of Rubik's cube fulfil certain **relations**. These are sequences which result in no change of the position, e.g., $\mathbf{R}^4$ or $\mathbf{C}^6$ for a commutator **C**.

Theorem

Starting from a set of **defining relations** *of the Rubik's cube,* **Buchberger's Algorithm** *computes a* **complete set of relations** S *with the following property:*
Given a sequence of moves (representing a position of the cube), use the relations in S *to simplify the sequence as much as possible. Then the unique result represents the shortest solution of the position.*

A set of relations S as in this theorem is called a **Gröbner basis** of the 2-sided ideal I of all relations of the Rubik's cube.

THE END

Life is short.

So, smile while you still have teeth.

Thank you for your attention!