RAMSEY THEORY: COLOURING OF COMPLETE GRAPHS

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A random edge colouring of a complete graph with two colours does not result in a completely unpredictable chaos, does it? But is it possible to calculate the numbers of colourings that all show the same specific number of monochromatic triangles? In fact it is..

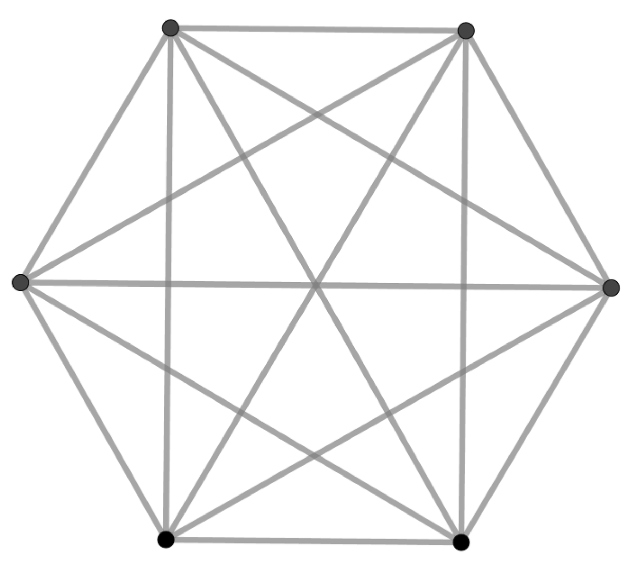
Ramsey's Theory

The Theory says, that complete disorder is impossible. In other words, it looks for the needed amount of objects in order that a certain subobject occurs. One of the simplest methods for solving such problems is the **pigeonhole principle**. This principle bases on the fact, that if there is one pigeon more than there are holes, then in at least one hole, there is more than one pigeon.

Definitions

Complete Graph

A complete graph, denoted K_n, is a graph with n nods that are all connected to each others by d edges.



A K_6 with $n = 6$ nodes and $d = \frac{n(n-1)}{2} = 15$ edges

Edge Colouring

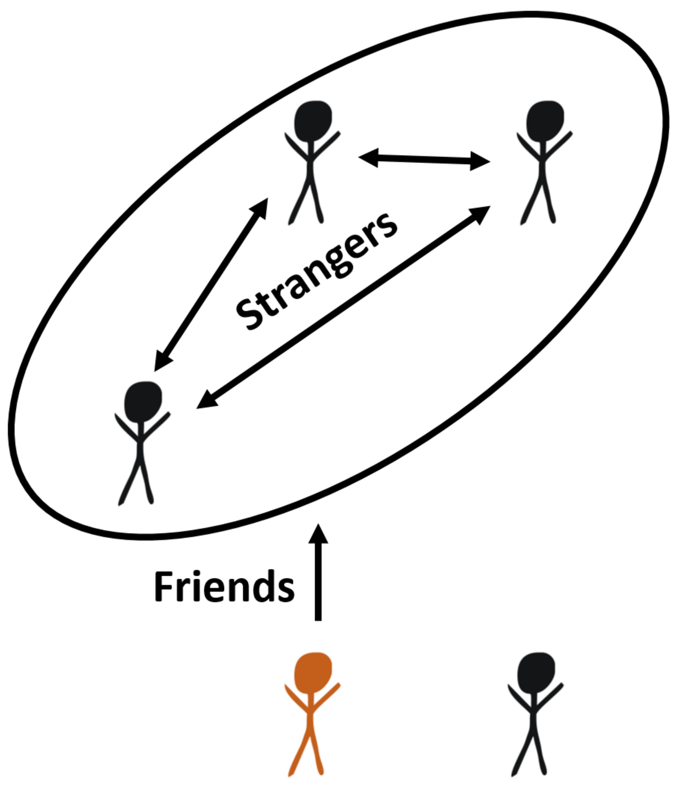
In a r-coloured graph, every edge is assigned to one of the r colours. The number of different colourings with given r and d can be calculated by $N = r^d$.

Monochromatic Triangle

A monochromatic triangle in a coloured complete graph is a triangle that consists of three edges that are all assigned to the same colour. The maximal number of monochromatic triangles is equal to $\binom{n}{3}$.

Example

To visualize the problem of proving that it is impossible to obtain zero monochromatic triangles in a K_6, the nods are exchanged by people and the two colours by the conditions that two people either know each other or not. With the help of the pigeonhole principle it is clear, that there are at least three friends or three strangers. If among this group of three two are friends/strangers as well, there is a triangle. If not, they are the triangle themselves (see figure).

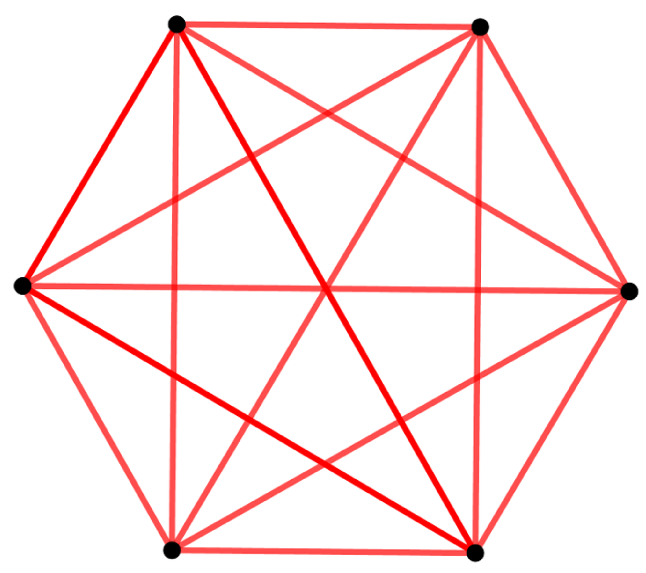


number of K_3	number of colourings
0	0
1	0
11	160
12	90
13	120
14	0
15	0
16	30
17	0
18	0
19	0
20	2

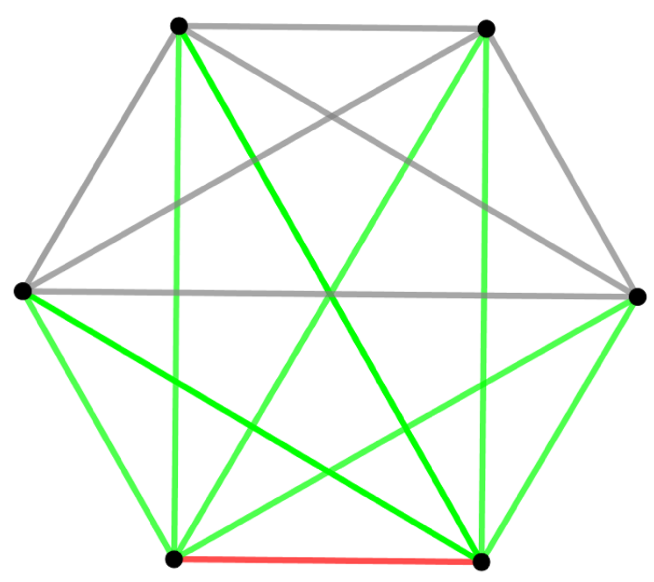
All proven numbers for n=6

First Steps

The maximal amount of monochromatic triangles occurs, if all the edges are either red or blue. Then, all the $\binom{n}{3}$ triangles are monochromatic and of course there are two different colourings: a red and a blue complete graph.



But what happens if just one edge is coloured differently from the others? As every edge belongs to $n-2$ different triangles, as many are no longer monochromatic.



One edge can be chosen in $\frac{n(n-1)}{2}$ ways and in two different colours. Therefore there are $n(n-1)$ different colourings that show $\binom{n}{3} - n + 2$ monochromatic triangles.

If this procedure is repeated, the following formulas can be deduced. And with the help of a computerised algorithm, the calculated numbers can be verified for $n \leq 9$.

number of K_3	number of colourings
$\binom{n}{3}$	$2 \; ; n \geq 3$
$(n-3)$ zeros inbetween $; n \geq 3$	
$\binom{n}{3} - n + 2$	$n(n-1) \; ; n \geq 3$
$(n-4)$ zeros inbetween $; n \geq 4$	
$\binom{n}{3} - 2n + 5$	$n(n-1)(n-2) \; ; n \geq 5$
$\binom{n}{3} - 2n + 4$	$\frac{1}{4}n(n-1)(n-2)(n-3) \; ; n \geq 6$
$(n-6)$ zeros inbetween $; n \geq 6$	
$\binom{n}{3} - 3n + 9$	$\frac{1}{3}n(n-1)(n-2)^2 \; ; n \geq 6$
$\binom{n}{3} - 3n + 8$	$n(n-1)(n-2)(n-3) \; ; n \geq 7$