On the Probabilistic Method and Permutations

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Introduction

There is no doubt that probability theory has many direct applications. However, during the mid-twentieth century mathematicians developed a further potential of the theory: It can be used as a tool for proving theorems, which may have nothing to do with probabilities. This so-called „Probabilistic Method“ was pioneered by Paul Erdős, one of the most influential mathematicians of the era. The method has grown into one of the most powerful and widely used strategies in the field of combinatorics.

The probabilistic method is used for proving the existence of some mathematical object with certain properties. This is achieved by selecting some object at random and proving that the probability of it having the properties we want is strictly greater than 0. Although the method can get arbitrarily more complex, it always boils down to this basic principle.

Contents

In my paper I talk about the basic idea of the probabilistic method and introduce the reader to a range of useful concepts and strategies. I illustrate how they can be applied to more complex structures by using the established ideas to analyse three properties of what are called permutations, a very well-known mathematical object.

After having found many of their probabilistic properties, I then go on to describe how we can use the concept of higher moments. I prove a number of well-known inequalities and, using these higher moments, I deduce more concrete results. In the very last chapter I then give some famous examples of theorems from different areas of mathematics, which can be approached using the probabilistic method. In these proofs, I use the ideas discussed throughout the paper.

Expected Value

Let X be a variable whose value is dependent on the outcome of some random phenomenon. We define the expected value of X as

$$\mathbb{E}(X) = \sum_{x} \Big(x \cdot \mathbb{P}(X = x) \Big)$$

where the sum goes over all possible values of X. Here, the function P denotes the probability that some event occurs.

The expected value can be thought of as a kind of average of all possible values of the variable, where each outcome is weighted by its corresponding probability of occuring.

In my paper I try to give the reader some intuition about the concept of expectancy by showing many examples I came up with.

Linearity of Expectancy

Linearity is by far the most important property of the expected value. It states that for any two real variables X and Y we have:

$$\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)$$

This fact is, especially in the case where the two variables happen to be dependent on each other, not intuitive at all and turns out to be a strong tool.

The proof of this result which I give in my paper only involves a few clever manipulations of sums.

Mean-Inequality

Let X be a variable that takes real values. It is easy to show that the following always holds:

$$\mathbb{P}\Big(X \geq \mathbb{E}(X)\Big) > 0$$

This theorem basically states that not all possible values of the variable can be above its expected value. Intuitively this should make sense as there should always be at least one sampled value above and below any kind of average.

This is only one of the many inequalities I prove in my paper. Given enough higher moments of a probability distribution, it is then possible to bound specific probabilities.

Sum-Free Subsets

We call a set of real numbers sum-free if it does not contain two not necessarily distinct elements whose sum is also in that set.

Theorem. Given any set of integers A there is a sum-free subset of A with at least a third of its size.

Lemma 1. Given a sum-free set S of real numbers and a positive real value t, the set

$$\{t \cdot s \ \forall s \in S\}$$

is also sum-free. This follows from the fact that the real value t can just be pulled out of the condition for sum-freeness due to its linearity.

Lemma 2. For any set S of real numbers, the set

$$S \cap \Big([1,2) \cup [4,5) \cup [7,8) \cup \cdots\Big)$$

is sum-free. This fact can be checked easily.

Proof. Let A be a set of integers and let us pick a random real number t in the interval $(0, 3)$. Let us define the following sets:

$$B = \{a \cdot t \ \forall a \in A\}$$

$$C = (1,2) \cup (4,5) \cup (7,8) \cup \cdots$$

We further define so-called indicator variables:

$$\mathbb{I}_a = \begin{cases} 1, & \text{if } a \cdot t \in C \\ 0, & \text{otherwise} \end{cases}$$

Since t is chosen at random it follows that

$$\mathbb{P}\Big(a \cdot t \in C\Big) = \frac{1}{3}$$

and from linearity of expectation we deduce

$$\mathbb{E}(|B \cap C|) = \sum_{a \in A} \mathbb{E}(\mathbb{I}_a) = \frac{1}{3}|A|$$

since the expected value of any indicator variable is just the probability that its corresponding event occurs.

The mean-inequality now tells us that there has to be some choice of t such that the intersection of B and C is at least one third of the size of A. By Lemma 2 we know that this intersection is in fact sum-free.

Finally, it follows from Lemma 1 that A also has to have a sum-free subset of at least this size.

This concludes the proof. ■