An Algebraic Approach to the Ramer-Kusuoka Formula

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In this talk, I will present a new perspective to the Ramer-Kusuoka formula, an anticipative version of the Cameron- Martin- Maruyama- Girsanov formula, by giving a totally algebraic proof to it.

We understand and generalize the formula in terms of an action of a "generalized" Heisenberg algebra.

Introduction

Intuitions: 1. Heisenberg Action on Gaussian Probability Space

As is widely recognized among probabilists thanks to P. Malliavin's writings, the algebra generated by Hermite polynomials on a Gaussian probability space is a representation space of a Heisenberg algebra.
Let us briefly recall the fact. Let μ be the 1-dimensional Gaussian measure; $\mu(dx) = (2\pi)^{-1/2}e^{-x^2/2}dx =: p(x)dx$ and D be the differential operator. For polynomials f and g, we apply integration by parts

to get

$$\begin{aligned}\langle Df, g\rangle_{L^2(\mu)} &= \int (Df)(x)g(x)p(x)\,dx = -\int f(x)D(g(x)p(x))\,dx \\ &= \int f(x)(-Dg + p'/p)(x)p(x)\,dx \\ &= \int f(x)(-Dg(x)+x)p(x)\,dx = \langle f, D^*g\rangle_{L^2(\mu)},\end{aligned} \tag{2.1}$$

where we set D^*, which sends a polynomial to another one, by

$$D^*f(x) = -Df(x) + xf(x), \qquad x \in \mathbf{R}.$$

Here the operator D^* behaves as an adjoint operator, and more importantly, it satisfies the *canonical commutation relation* (CCR) against D; $(DD^* - D^*D)f(x) = f(x)$ for any polynomial f, or equivalently

$$[D, D^*] := DD^* - D^*D = 1.$$

This can be easily generalized to multi-, or even infinite dimensional cases.

Let $\mathcal{W}$ be a classical Wiener space, H be its Cameron-Martin space, and $\{h_i\}$ be an orthonormal basis of H. Let D_i be the derivative in the direction of h_i, acting on the space of (Wiener) Hermite polynomials, and define

$$D_i^* f(w) = -D_i f(w) + \left(\int \dot{h}_i dw \right) f(w), \quad w \in \mathcal{W} \qquad (2.2)$$

$i = 1, 2, \cdots$. Then, it can be easily checked that they satisfy the CCR;

$$[D_i, D_j] = [D_i^*, D_j^*] = 0, \text{ and } \quad [D_i, D_j^*] = \delta_{ij}. \qquad (2.3)$$

Further, we still have

$$E[(Df)g] = \langle Df, g \rangle = \langle f, D^* g \rangle = E[f(D^* g)]. \qquad (2.4)$$

Intuitions: 1. Heisenberg Action on Gaussian Probability Space and Cameron- Martin Formula

Let T_h be the translation by $h \in H$; $T_h f(w) = f(w+h)$. The action of the Heisenberg algebra in turn enables one to understand the translation as

$$T_h f = e^{D_h} f = e^{\sum \langle h, h_i \rangle D_{h_i}} f. \tag{2.5}$$

This should be understood as Taylor expansion for "analytic function", and can be extended to more general classes of random variables.

Intuitions: 1. Heisenberg Action on Gaussian Probability Space and Cameron- Martin Formula

The trivial expression

$$f = e^{-D_h} e^{D_h} f = \sum_{n=0}^{\infty} \frac{(-D_h)^n}{n!} T_h f$$

together with the adjoint relation
$E[(Df)g] = \langle Df, g \rangle = \langle f, D^* g \rangle = E[f(D^* g)]$ leads to

$$E[\sum_{n=0}^{\infty} \frac{(D^*_{-h})^n(1)}{n!} T_h f] = E[f],$$

which is nothing but the Cameron-Martin formula since one has

$$\sum_{n=0}^{\infty} \frac{(D^*_{-h})^n(1)}{n!} = \exp\left(-\int \dot{h}\, dw - \frac{1}{2}\|h\|^2\right). \qquad (2.6)$$

In the case with translation by an adapted map $w \mapsto h(w) \in H$, which corresponds to Girsanov-Maruyama formula, the exponential map does not define a translation since $\langle h, h_i \rangle$'s are not constant anymore. We need to modify it as

$$T_h = \sum_{n=0}^{\infty} \sum_{i_1,\cdots,i_n} \langle h, h_{i_1} \rangle \cdots \langle h, h_{i_n} \rangle \frac{D_{h_{i_1}} \cdots D_{h_{i_n}}}{n!} = \text{``}: e^{\sum \langle h, h_i \rangle D_{h_i}} :\text{''}. \tag{2.7}$$

The last expression is a so-called *normal order product*; inside the colons, the multiplication operators should always be left and the differential operators be right.

Intuitions: 1. Heisenberg Action on Gaussian Probability Space and Girsanov- Maruyama Formula

Since T_{-h} is not the inverse of T_h anymore, we cannot use the same argument as above. In this talk, I show how the Girsanov-Maruyama can be retrieved in the algebraic way. In the course, things get rather easier without the requirement of adaptedness. This is the starting points of the present study.

If the requirement of the adaptedness is dropped to include non-anticipative maps, then the Girsanov-Maruyama density turns to a Ramer-Kusuoka one;

$$\text{“}\det\text{”}(I+\nabla h)\exp\left(-\int \dot{h}\,\text{“}dw\text{”}-\frac{1}{2}\|h\|^2\right), \tag{3.1}$$

where “det” can be understood either as the Fredholm determinant or the Carleman Fredholm one, and $\int\cdot\,\delta w$ as a Skorohod integral or an Ogawa integral, respectively. If ∇h is *quasi-nilpotent*, meaning that $\mathrm{Tr}(\nabla h)^n = 0$ for all n, the two densities coincides.

Intuitions: 2. Where the Density of Ramer-Kusuoka Formula comes from?

The look of the Ramer-Kusuoka density is more familiar since it involves a "Jacobi-determinant". The trick is made clear if we look at a finite dimensional case. The standard change-of-variables formula goes like

$$\int |\det(1 + \nabla h)| F(x + h(x))\, dx = \int F(x)\, dx$$

for an integrable function F. Rewriting $F(x) = f(x)p(x)$ with a probability density function p, we have

$$\int |\det(1 + \Delta h)| f(x + h(x)) \frac{p(x + h(x))}{p(x)} p(x)\, dx = \int f(x)p(x)\, dx.$$

Intuitions: 2. Where the Density of Ramer-Kusuoka Formula comes from?

If p is the Gaussian function; $p(x) = (2\pi)^{-d/2} \exp(-|x|^2/2)$, then we have

$$\frac{p(x+h(x))}{p(x)} = \exp\left(-\langle h, x\rangle - \frac{|x|^2}{2}\right), \tag{3.2}$$

which actually is a prototype of the Ramer-Kusuoka density

$$\text{``det''}(I + \nabla h) \exp\left(-\int \dot{h} \text{ ``}dw\text{''} - \frac{1}{2}\|h\|^2\right),$$

Intuitions: 2. Where the Density of Ramer-Kusuoka Formula comes from?

Here we notice that the density formula might be obtained for other cases than Gaussian since the expression $p(x+h(x))/p(x)$ of the density is fairly general, though at this stage it is valid only for the above finite dimensional cases. To extend it in infinite dimensional settings, we will rely on an algebraic approach, as we have seen that the action of Heisenberg algebra extends the absolute continuity to infinite dimensional one.

Intuitions: 3. How Should the Heisenberg Action be Generalized?

To generalize the CCR

$$[D_i, D_j] = [D_i^*, D_j^*] = 0, \text{ and } \quad [D_i, D_j^*] = \delta_{ij}.$$

still to get a density formula, we need to look into the following integration by parts more deeply.

$$\begin{aligned}\langle Df, g\rangle_{L^2(\mu)} &= \int (Df)(x)g(x)p(x)\,dx = -\int f(x)D(g(x)p(x))\,dx \\ &= \int f(x)(-Dg + p'/p)(x)p(x)\,dx \\ &= \int f(x)(-Dg(x) + x)p(x)\,dx = \langle f, D^*g\rangle_{L^2(\mu)},\end{aligned}$$

One may notice that (i) $p'/p = (\log p)'$ is crucial to the expression of the density and (ii) is understood to be $D + D^*$ as a multiplication operator, where D is the differential operator and D^* is its "adjoint" with respect to $p(x)dx$.

Intuitions: 3. How Should the Heisenberg Action be Generalized?

By considering a multi-dimensional case we notice that

$$[D_i, D_j] = [D_i^*, D_j^*] = 0, \text{ and } \quad [D_i, D_j^*] = \partial_{ij}(\log p). \tag{4.1}$$

Since we are thinking of more general situations with "non-linear transformation" $f(x) \mapsto f(x + h(x))$, we read the last relation so as that they are multiplication operators.

Intuitions: 3. How Should the Heisenberg Action be Generalized?

With these observations at hand, we will work on the following algebra(s). Let $\mathcal{B}$ be a commutative topological algebra over $\mathbf{R}$, $D_i, D_i^*, i \in \mathbf{N}$ are linear operators acting on $\mathcal{B}$ in the manner that

$$[D_i, D_j] = [D_i^*, D_j^*] = 0, \tag{4.2}$$

$$D_i + D_i^* \in \mathscr{M}(\mathcal{B}), \tag{4.3}$$

and

$$[D_i, \mathscr{M}(\mathcal{B})],\ [D_i^*, \mathscr{M}(\mathcal{B})] \subset \mathscr{M}(\mathcal{B}). \tag{4.4}$$

Here $\mathscr{M}(\mathcal{B})$ denotes the algebra of multiplication operators.

Intuitions: 3. How the Heisenberg Action Should be Generalized

The algebra generated by $\{D_i, D_i^*\}, \mathscr{M}(\mathcal{B})$ will be denoted by $\mathscr{D}(\mathcal{B})$, which is dependent on the choice of $\mathcal{B}$ and linear operators $D_i, D_i^*, i \in \mathbf{N}$ on it.
The requirements are an abstraction of D and D^* being differential operators, and being adjoint to each other.

When $\mathcal{B}$ is continuously and densely embedded in a Hilbert space and D^* is actually an adjoint operator of D, we call the embedding a *representation* of $\mathscr{D}(\mathcal{B})$.

To avoid the tedious argument on convergence, we sometimes work in $\mathscr{D}(\mathcal{B})[\![t]\!]$, the ring of formal series in t with coefficients in $\mathscr{D}(\mathcal{B})$.

Let $h := \{h_i : i \in \mathbf{N}\}$ be a sequence in $\mathcal{B}$. With (2.7) in mind, we define "translation operator" in $\mathscr{D}(\mathcal{B})[\![t]\!]$ associated with h by

$$T_h(t) := \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{i_1,\cdots,i_n} h_{i_1} \cdots h_{i_n} D_{i_1} \cdots D_{i_n} =: \sum_{n=0}^{\infty} \frac{t^n}{n!} T_{h,n}$$

if $T_{h,n} f$ converges in $\mathcal{B}$ for every f and n.

In the sequels we use as a conventional notation analogue to the normal order product for (possibly infinite) sum of the monomials in $D_i, D_i^*, h_i, i \in \mathbf{N}$, $:\sum a_n: \in \mathscr{D}(\mathcal{B})$ for $a_n \in \mathscr{D}(\mathcal{B})$ in the following way: (i)$:\sum a_n := \sum :a_n:$ (ii) for monomials a and b, $:aD_ib := :ab: D_i$, and $:ah_ib := h_i :ab:$. Here $D_i, D_i^*, h_i, i \in \mathbf{N}$ are treated as symbols with $[D_i, D_j] = [D_i^*, D_j^*] = [h_i, h_j] = 0$. Thus, for example, we have

$$:(\sum_i h_i D_i)^2 (\sum_i D_i^* h_i)^2 := \sum_{i,j,k,l} h_i h_j h_k h_l D_k^* D_l^* D_i D_j.$$

With this notation, we have $T_h(t) =: e^{t\sum h_i D_i}:$ by understanding e^X to be the infinite series $\sum X^n/n!$, which is again a conventional notation.

We also introduce a right ideal in $\mathscr{D}(\mathcal{B})$ as

$$\mathcal{I} := \{\sum_i c_i D_i^* a_i : a_i \in \mathscr{D}(\mathcal{B}), c_i \in \mathbf{R},\ \exists i \text{ s.t.} c_i \neq 0.\ \},$$

and one in $\mathscr{D}(\mathcal{B})[\![t]\!]$ as

$$\mathcal{I}[\![t]\!] := \{\sum_{n=1}^{\infty} t^n I_n : I_n \in \mathcal{I}\}.$$

The Scenario

What we are looking for is a formula like

$$1 + \Phi(t) = M(t) T_h(t),$$

where Φ is in $\mathcal{I}[\![t]\!]$ and M is in $\mathscr{M}(\mathcal{B})[\![t]\!]$, or preferably,

$$1 + \Phi = M T_h, \quad \Phi \in \mathcal{I}, M \in \mathscr{M}(\mathcal{B}). \tag{5.1}$$

The reason why it works is as follows. If we have a representation of $\mathscr{D}(\mathcal{B})$ in L^2 space of a probability measure, and if we have sufficient regularity to take expectation of the operators in both sides of (5.1) applied to a random variable f, we have

$$E[f] + E[\Phi f] = E[Mf(\omega + h(\omega))]. \tag{5.2}$$

The adjoint operator of Φ is, with a proper regularity, an annihilation one; differential operators in the right. Thus we would have

$$E[\Phi f] = E[\Phi^*(1) f] = 0.$$

Hence the M in the right-hand-side of (5.2) is the density if it exists and is positive.

The Results

Let $h = \{h_i\}$ be such that $h_i = 0$ for all but finite i. For such h set $\Phi_h := \{D_i^* h_j : i, j \in \mathbf{N}\}$ and $\Psi_h := \{[D_i^*, h_j] : i, j \in \mathbf{N}\}$. We can define their monomials, and hence polynomials, by for example

$$\Phi_h \Psi_h = \{\sum_k D_i^* h_k [D_k^*, h_j] : i, j \in \mathbf{N}\},$$

and so on. For such a polynomial P we can define $\mathrm{Tr} P$ as usual. With these notations, we have the following results.

Proposition

The operator

$$\mathcal{E}_h := \Big(: \exp\Big(t \sum h_i (D_i + D_i^*)\Big) :\Big)$$

is in $\mathscr{M}(\mathcal{B})$.

The Results

Theorem

In $\mathscr{D}(\mathcal{B})[\![t]\!]$, we have

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} t^n s_n(\mathrm{Tr}\Phi, \mathrm{Tr}\Phi(-\Psi), \cdots, \mathrm{Tr}\Phi(-\Psi)^{n-1}) \\ = \exp\left(\sum_{n=1}^{\infty} \frac{t^n}{n}(-1)^{n-1}\mathrm{Tr}\Psi_h^n\right) \mathcal{E}_h\, T_{-h}, \end{aligned} \tag{5.3}$$

where s_n, $n \in \mathbf{N}$ are a non-commutative version of Schur functions (character polynomials) defined inductively by $s_0 \equiv 1$ and

$$s_{n+1}(x_0, x_1, \cdots, x_n) = \frac{1}{n+1}\sum_{k=0}^{n} x_k s_{n-k}.$$

Here x_i's are non-commutative indefinite variables.

The Results

Since apparently the operator in the left-hand-side of

$$
\begin{aligned}
1 + \sum_{n=1}^{\infty} t^n s_n(\mathrm{Tr}\Phi, \mathrm{Tr}\Phi(-\Psi), \cdots, \mathrm{Tr}\Phi(-\Psi)^{n-1}) \\
= \exp\left(\sum_{n=1}^{\infty} \frac{t^n}{n}(-1)^{n-1}\mathrm{Tr}\Psi_h^n\right) \mathcal{E}_h\, T_{-h},
\end{aligned}
$$

is in $\mathcal{I}[\![t]\!]$ and the first factor in the right-hand-side is in $\mathscr{M}(\mathcal{B})$, Proposition 5.1 asserts that (5.3) is an abstract version of Ramer-Kusuoka type density formula.

Historical Remark

Cameron-Martin (1949)
Gross (1960)
Shepp (1966)
Kuo (1971)
Ramer (1974)
Kusuoka (1982)
Buckdhan (1991)
Enchev (1993)
Buckdhan-Föllmer (1993)
Üstünel-Zakai (1994)
Kallianpur-Karandikar (1994)
and still going on...

Proof of the Proposition

We will show that

$$:e^{t\sum h_i(D_i+D_i^*)}: = \sum_{n=0}^{\infty} \frac{t^n}{n!} :\{\sum_i h_i(D_i + D_i^*)\}^n:$$

is a multiplication operator. Since

$$:\{\sum_i h_i(D_i + D_i^*)\}^n := \sum h_{i_1} \cdots h_{i_n} :(D_{i_1} + D_{i_1}^*) \cdots (D_{i_n} + D_{i_n}^*):$$

it suffices to show that

$$:(D_{i_1} + D_{i_1}^*) \cdots (D_{i_n} + D_{i_n}^*):$$

is a multiplication operator.

Proof of the Proposition

We show it by induction. We have

$$\begin{aligned}
&: (D_{i_1} + D_{i_1}^*) \cdots (D_{i_n} + D_{i_n}^*) : \\
&\qquad = D_{i_1}^* : (D_{i_2} + D_{i_2}^*) \cdots (D_{i_n} + D_{i_n}^*) : \\
&\qquad\quad + : (D_{i_2} + D_{i_2}^*) \cdots (D_{i_n} + D_{i_n}^*) : \ D_{i_1}.
\end{aligned}$$

Since we have

$$\begin{aligned}
&D_i^* \mathscr{M}(\mathcal{B}) + \mathscr{M}(\mathcal{B}) D_i \\
&\qquad = [D_i^*, \mathscr{M}(\mathcal{B})] - [D_i, \mathscr{M}(\mathcal{B})] + \mathscr{M}(\mathcal{B})(D_i^* + D_i) \subset \mathscr{M}(\mathcal{B}),
\end{aligned}$$

the proof is complete.

Outline of the Proof of Main Theorem

In the final step we solve a *differential equation* in $\mathscr{D}(\mathcal{B})[\![t]\!]$. Here differentiation in $\mathscr{D}(\mathcal{B})[\![t]\!]$ is the formal one; it sends $\sum_{n=0}^{\infty} t^n a_n$ to $\sum_{n=0}^{\infty} t^n (n+1) a_{n+1}$, and the *indefinite integral* sends $\sum_{n=0}^{\infty} t^n a_n$ to $\sum_{n=1}^{\infty} t^n (n-1) a_{n-1} + c$, where c is the indefinite constant. The differentiation of $a(t)$ is denoted by $a'(t)$ and the indefinite integral is denoted by $\int_0^t a(s)\, ds$ if the indefinite constant is taken to be zero.

Outline of the Proof of Main Theorem

Linear equations in $\mathscr{D}(\mathcal{B})[\![t]\!]$ is solved in the following way.

Lemma

The differential equation

$$a'(t) = b(t)a(t), \textit{with } a(0) = 1 \tag{7.1}$$

in $\mathscr{D}(\mathcal{B})[\![t]\!]$, where $b(t) = \sum_{n=0}^{\infty} t^n b_n$ have the unique solution

$$a(t) = \sum_{n=0}^{\infty} t^n s_n(b_0, \cdots, b_{n-1}),$$

where $s_n, n = 1, 2, \cdots$ are the non-commutative Schur functions defined in the statement of Theorem 1.

Outline of the Proof of Main Theorem

Thus, once we have the differential equation

$$a'(t) = b(t)a(t), \text{with } a(0) = 1 \tag{7.2}$$

for

$$a(t) := \exp\left(\sum_{n=1}^{\infty} \frac{t^n}{n}(-1)^{n-1}\mathrm{Tr}\Psi_h^n\right) \mathcal{E}_h(t)\, T_{-h}(t)$$

and

$$b(t) := \sum_{n=0}^{\infty} t^n \mathrm{Tr}\Phi(-\Psi)^n,$$

the proof will be complete.

Outline of the Proof of Main Theorem

The first step to the project is the following

Lemma

$$\mathcal{E}_h(t)\, T_{-h}(t)(=: e^{t\sum h_i(D_i^*+D_i)} :: e^{-t\sum h_i D_i} :) =: e^{t\sum h_i D_i^*} : .$$

Proof.

The coefficient of t^n in the right-hand-side is

$$\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\sum_{i_1,\cdots,i_n} h_{i_1}\cdots h_{i_k} : (D_{i_1}+D_{i_1}^*)\cdots(D_{i_k}+D_{i_k}^*) : \\ h_{i_{k+1}}\cdots h_{i_n}(-D)_{i_{k+1}}\cdots(-D)_{i_n}$$

□

Outline of the Proof of Main Theorem

The first step to the project is the following

Lemma

$$\mathcal{E}_h(t)\,T_{-h}(t)(=: e^{t\sum h_i(D_i^*+D_i)} :: e^{-t\sum h_i D_i} :) =: e^{t\sum h_i D_i^*} : .$$

Proof.

The coefficient of t^n in the right-hand-side is

$$= \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\sum_{i_1,\cdots,i_n} h_{i_1}\cdots h_{i_k}h_{i_{k+1}}\cdots h_{i_n}$$
$$: (D_{i_1}+D_{i_1}^*)\cdots(D_{i_k}+D_{i_k}^*)(-D)_{i_{k+1}}\cdots(-D)_{i_n} :$$

□

Outline of the Proof of Main Theorem

The first step to the project is the following

Lemma

$$\mathcal{E}_h(t)\,T_{-h}(t)(=:e^{t\sum h_i(D_i^*+D_i)}::e^{-t\sum h_i D_i}:)=:e^{t\sum h_i D_i^*}:.$$

Proof.

The coefficient of t^n in the right-hand-side is

$$=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}:\sum_{i_1,\cdots,i_n} h_{i_1}\cdots h_{i_k}h_{i_{k+1}}\cdots h_{i_n}$$
$$(D_{i_1}+D_{i_1}^*)\cdots(D_{i_k}+D_{i_k}^*)(-D)_{i_{k+1}}\cdots(-D)_{i_n}:$$

□

Outline of the Proof of Main Theorem

The first step to the project is the following

Lemma

$$\mathcal{E}_h(t)\, T_{-h}(t)(=:e^{t\sum h_i(D_i^*+D_i)}::e^{-t\sum h_i D_i}:)=:e^{t\sum h_i D_i^*}:\,.$$

Proof.

The coefficient of t^n in the right-hand-side is

$$=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}:\sum_{i_1,\cdots,i_n} h_{i_1}\cdots h_{i_k}(D_{i_1}+D_{i_1}^*)\cdots(D_{i_k}+D_{i_k}^*)$$
$$h_{i_{k+1}}\cdots h_{i_n}(-D)_{i_{k+1}}\cdots(-D)_{i_n}:$$

□

Outline of the Proof of Main Theorem

The first step to the project is the following

Lemma

$$\mathcal{E}_h(t)\, T_{-h}(t) (=: e^{t \sum h_i (D_i^* + D_i)} :: e^{-t \sum h_i D_i} :) =: e^{t \sum h_i D_i^*} : .$$

Proof.

The coefficient of t^n in the right-hand-side is

$$= \frac{1}{n!} : \sum_{k=0}^{n} \binom{n}{k} \{\sum h_i (D_i + D_i^*)\}^k (-\sum h_i D_i)^{n-k} :$$

$\square$

Outline of the Proof of Main Theorem

The first step to the project is the following

Lemma

$$\mathcal{E}_h(t)\,T_{-h}(t)(=:e^{t\sum h_i(D_i^*+D_i)}::e^{-t\sum h_i D_i}:)=:e^{t\sum h_i D_i^*}:\ .$$

Proof.

The coefficient of t^n in the right-hand-side is

$$=\frac{1}{n!}:\{\sum h_i(D_i+D_i^*-D_i)\}^n:=\frac{1}{n!}:\{\sum h_i D_i^*\}^n:\ .$$

□

Outline of the Proof of Main Theorem

With an approximation where the "random" operator Ψ is compact, we can use the expressions

$$\exp\left(\sum_{n=1}^{\infty}\frac{t^n}{n}(-1)^{n-1}\mathrm{Tr}\Psi_h^n\right)(=\det(1+t\Psi))=\sum_{n=0}^{\infty}t^n\mathrm{Tr}\wedge^n\Psi$$

$$=\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{i_1,\cdots,i_n}\langle\Psi_{i_i}\wedge\cdots\Psi_{i_n},e_{i_i}\wedge\cdots e_{i_n}\rangle=:\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{i_1,\cdots,i_n}\det[\Psi_{i_i},\cdot,\Psi_{i_n}].$$

The last two expressions would make sense with $\Psi_i=([D_i^*,h_1],\cdots,[D_i^*,h_n],\cdots)$.

Outline of the Proof of Main Theorem

The key is the following

Lemma

$$\det(1+t\Psi) : e^{t\sum h_i D_i^*} := 1 + \int_0^t g'(s) \ : e^{t\sum h_i D_i^*} : \ ds,$$

where

$$g(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{i_1,\cdots,i_n} D_{i_1}^* \det[h, \Psi_{i_1}, \cdots, \Psi_{i_n}].$$

Our proof is very involved (needs several pages...so I omit it here) but this lemma already solves the problem.

Outline of the Proof of Main Theorem

The following lemma completes the proof of the main theorem.

Lemma

$$g'(t) = \sum_{n=0}^{\infty} t^n \mathrm{Tr}\{\Phi(-\Psi)^{n-1}\} \det(1 + t\Psi).$$

Proof.

(Outline) We show by induction

$$\sum_{i_1,\cdots,i_n} D^*_{i_1} \det[h, \Psi_{i_1}, \cdots, \Psi_{i_n}] = (n-1)! \sum_{k=1}^{n} \mathrm{Tr}\{\Phi(-\Psi)^{k-1}\} \mathrm{Tr} \wedge^{n-k} \Psi.$$

□

Comments on Absolute Continuity

If we have a representation of the algebra in a space of random variables, the multiplication operator in the formula applied to constant function, that is,

$$\exp\left(\sum_{n=1}^{\infty}\frac{t^n}{n}(-1)^{n-1}\mathrm{Tr}\Psi_h^n\right)\ :e^{t\sum h_iD_i^*}:(1),$$

describes the density if it exists (for sufficiently small $t \in \mathbf{R}$).

Comments on Absolute Continuity

To see "if it exists" or not, it is better to rewrite $:e^{t\sum h_i D_i^*}:(1)$ as an exponential function formally as

$$:e^{t\sum h_i D_i^*}:(1) = \exp \int_0^t b(s)\, ds = \exp \sum_{n=1}^{\infty} \frac{t^n}{n} b_{n-1},$$

by the relation

$$(n+1)a_{n+1} = \sum_{k=0}^{n} b_k a_{n-k},$$

where

$$a_n = \frac{1}{n!} \sum_{i_1,\cdots,i_n} h_{i_1} \cdots h_{i_n} D_{i_1}^* \cdots D_{i_n}^*(1).$$

Comments on Absolute Continuity

We have, by the definition of the algebra,

$$D_i^* f = [D_i^*, f] + q_i f,$$

where we put $q_i = D_i^*(1)$. Using this relation, we can inductively obtain an expression of $D_{i_1}^* \cdots D_{i_n}^*(1)$.
We have, for example,

$$D_i^* D_j^*(1) = [D_i^*, q_j] + q_i q_j,$$

etc and

$$\sum_{i,j} h_i h_j D_i^* D_j^*(1) = \langle h, q \rangle^2 + \langle \Psi_{q,2}, h \otimes h \rangle,$$

where $\Psi_{q,2}$ is the matrix $[D_i^*, q_j]$.

Comments on Absolute Continuity

The "density" is now expressed as

$$\exp\{t(\mathrm{Tr}\Psi_h - \langle h, q\rangle) + \frac{t^2}{2}(\mathrm{Tr}\Psi_h^2 - \langle h \otimes h, \Psi_{q,2}\rangle)$$
$$+ \frac{t^3}{3}(\mathrm{Tr}\Psi_h^3 - \langle h \otimes h \otimes h, \Psi_{q,3}\rangle + \cdots) + \cdots\}.$$

Thank you!