LATTICE POINTS ON SPHERES,

TORAL EIGENFUNCTIONS AND THEIR NODAL SETS

- **Moment Inequalities**
- **Rectifiability of Nodal Sets**
- **Restriction Theorems**
- **Nodal Domain**
- **Distribution of Lattice Points on Spheres**

M compact smooth manifold.

Eigenfunctions of the Laplacian

$$-\Delta\varphi = E\varphi$$

Nodal sets

$$N_\varphi = \varphi^{-1}(0) = \{x : \varphi(x) = 0\}$$

Nodal domains=
connected components of $M \backslash N_\varphi$

GENERAL PROBLEM

How do eigenfunctions and nodal sets behave when $E \to \infty$?

THE FLAT TORUS $\mathbb{T}^d$

Eigenfunctions

$$-\Delta\varphi = 4\pi^2 E\varphi$$

are explicit

$$\varphi(x) = \sum_{|n|^2=E} \hat{\varphi}(n) e^{2\pi i n.x}$$

$$|n|^2 = n_1^2 + \cdots + n_d^2$$

Dimension of eigenspace

$$||$$

lattice points on sphere $\{x \in \mathbb{R}^d; |x| = \lambda\}$ with $\lambda^2 = E$

MOMENT INEQUALITIES

(M, g) compact, smooth Riemannian manifold of dimension d without boundary

Theorem (Hormander, Sogge)

$$\|\varphi_E\|_p \leq c\lambda^{\delta(p)}\|\varphi_E\|_2$$

where

$$\delta(p) = \begin{cases} \frac{d-1}{2}\Big(\frac{1}{2} - \frac{1}{p}\Big) & \textit{if} \quad 2 \leq p \leq \frac{2(d+1)}{d-1} \\ d\Big(\frac{1}{2} - \frac{1}{p}\Big) - \frac{1}{2} & \textit{if} \quad \frac{2(d+1)}{d-1} \leq p \leq \infty \end{cases}$$

Estimates are sharp for $M = S^d$

THE CASE OF $\mathbb{T}^d$

Conjecture

$$\|\varphi_E\|_p \le c_p \|\varphi_E\|_2 \quad \textit{if } p < \frac{2d}{d-2}$$

and

$$\|\varphi_E\|_p \le c_p \lambda^{\left(\frac{d-2}{2} - \frac{d}{p}\right)} \|\varphi_E\|_2 \quad \textit{if } p > \frac{2d}{d-2}$$

Theorem (B, 93)

Conjecture holds for $d \ge 4$ *and* $p > \frac{2(d+1)}{d-3}$

Based on Hardy-Littlewood circle method and Kloosterman's refinement

ESTIMATES ON $\mathbb{T}^2$

E = sum of 2 squares

multiplicity of E = number N of representations of N as sum of two squares

Theorem (**Zygmund–Cook**) $\|\varphi_E\|_4 \leq C\|\varphi_E\|_2$

Estimating the L^6-norm $\Leftrightarrow$

number of solutions of $P_1 + P_2 + P_3 = A + iB$ in

$$P_1, P_2, P_3 \in \mathbb{Z} + i\mathbb{Z},\ |P_1|^2 = |P_2|^2 = |P_3|^2 = E$$

$\longrightarrow$ arithmetical problems about integral points on elliptic curves

Theorem (**Bombieri–B, 012**)

- $\|\varphi_E\|_6 \ll N^{\frac{1}{12}+\varepsilon}\|\varphi_E\|_2$

- *For all* $p < \infty, \|\varphi_E\|_p \le C\|\varphi_E\|_2$ *for 'most'* E

- *Conditional to GRH and* **Birch, Swinnerton-Dyer** *conjecture*

  $\|\varphi_E\|_6 \ll N^{\varepsilon}\|\varphi_E\|_2$ *for 'most' smooth numbers* E

$\mathbb{T}^3$

$$\|\varphi_E\|_4 \ll E^{\varepsilon} \|\varphi_E\|_2$$

Based on uniform bounds for integral points on ellipses.

Theorem (B, 011)

$d \geq 4, p = \frac{2d}{d-1}$

$$\|\varphi_E\|_p \ll E^{\varepsilon} \|\varphi_E\|_2$$

Based on the theory of multi-linear oscillatory integral operators (joint work with L. Guth, GAFA (011))

Theorem (B, 011)

Let $0 < \delta \ll 1, R > C(\varepsilon)\delta^{-2}$ *and* $\{U_\alpha\}$ *a partition of* S^{d-1} *in* δ*-caps.*

Then, if μ *is a measure on* S^{d-1} *and* $p = \frac{2d}{d-1}$

$$\|\widehat{\mu}\|_{L^p_{B(0,R)}} \ll \delta^{-\varepsilon} \Big[\sum_{\alpha} \|(\mu|_{U_\alpha})^\wedge\|^2_{L^p_{B(0,R)}} \Big]^{\frac{1}{2}}$$

Problem *Establish uniform L^p-bound for some $p > 2$ on higher dimensional toral eigenfunctions*

Important to PDE's

Control for Schrodinger operators in tori

Theorem (**B–Burq–Zworski, 012**)

Let $d \leq 3, V \in L^\infty(\mathbb{T}^d), \Omega \subset \mathbb{T}^d, \Omega \neq \phi$, an open set. Let $T > 0$. There is a constant $C = C(\Omega, T, V)$ such that for any $u_0 \in L^2(\mathbb{T}^d)$

$$\|u_0\|_{L^2(\mathbb{T}^d)} \leq C\|e^{it(\Delta+V)}u_0\|_{L^2(\Omega\times[0,T])}$$

(uses Zygmund's inequality)

GEOMETRY OF NODAL SETS

Yau's Conjecture (1982)

$$c_2\sqrt{E} < H^{d-1}(N) < C_1\sqrt{E}$$

with constants C_1, c_2 depending on (M, g)

Donnelly–Fefferman (1988) M real analytic

(smooth case is largely open)

Nodal set from local point of view

'Can a given real analytic hypersurface be part of the nodal set of infinitely many eigenfunctions?'

Theorem (**B–Rudnick, 011**) *A given real analytic hypersurface* Σ *in* $\mathbb{T}^d$ *with non-vanishing curvature cannot be part of the nodal set of eigenfunctions of arbitrary large* E

NODAL LINES IN $\mathbb{T}^2$

Theorem (**B–Rudnick, 011**)

Let $\Sigma \subset \mathbb{T}^2$ *be a smooth curve which is not a segment of a closed geodesic.*

Assume $E < E'$ *and* $\Sigma \subset N_{\varphi_E} \cap N_{\varphi_{E'}}$. *Then*

$$E' \ll E^{2+\varepsilon}$$

Based on abc-theorem in function fields

(**Voloch, Brownawell-Masser**)

Role of curvature

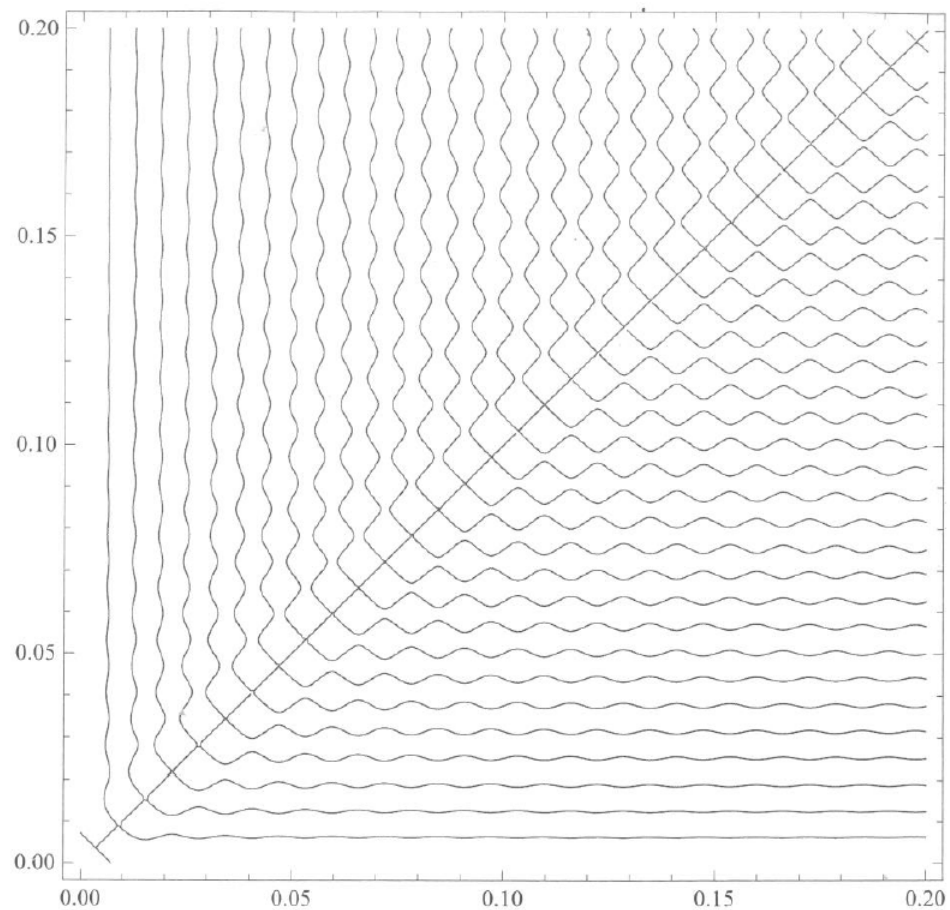
$$[y = 0] \subset N_\varphi \qquad \varphi(x, y) = (\sin mx)(\sin ny)$$

Theorem (M. Berry)

Nodal lines of ‘random’ plane waves have curvature

$$\sim \sqrt{E}$$

How to reformulate for individual eigenfunctions on $\mathbb{T}^2$?



$$\phi(x, y) = (\sin nx) \sin y - (\sin x) \sin ny$$

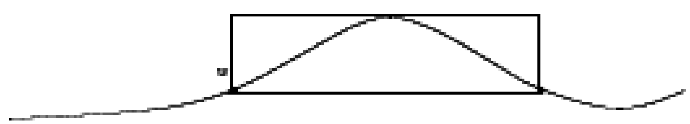
WIDTH OF REGULAR NODAL ARC C

$$\gamma : [0, \ell] \to C$$

$$|\dot{\gamma}| = 1, |\ddot{\gamma}| \sim \kappa \left(0 < \kappa < \frac{1}{\ell}\right)$$

WIDTH

$$w(C) \sim \ell^2 \kappa$$



Conjecture (B–Rudnick)

If $C \subset N_{\varphi_E}$ *is a regular arc, then* $w(C) \ll E^{-\frac{1}{2}+\varepsilon} = \lambda^{-1+\varepsilon}$

Theorem (B–R)

- $w(C) < \lambda^{-\frac{1}{3}+\varepsilon}$

- Let $\{C_\alpha\}$ be a collection of disjoint regular arcs from N such that $w(C_\alpha) > \lambda^{-\frac{1}{2}+\varepsilon}$

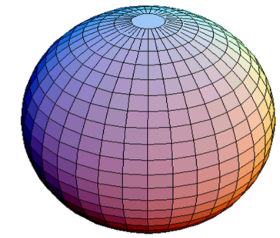
Then

$$\sum_\alpha |C_\alpha| < \lambda^{1-\varepsilon'}$$

- Let $\{C_\alpha\}$ be a collection of disjoint regular arcs from N such that $w(C_\alpha) > \lambda^{-1+\varepsilon}$. Then

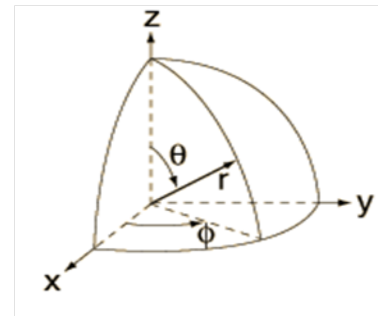
$$|N \backslash \bigcup C_\alpha| < c\lambda$$

EXAMPLE OF THE SPHERE



<u>zonal</u> spherical harmonics $Y_n(\theta, \varphi) = P_n(\cos\theta)$

$P_n(x)$=Legendre polynomial



$$P_n(x) = \frac{1}{2^n} \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{n}{j} \binom{2n-2j}{n-2j} x^{n-2j}$$

nodal lines = parallels e.g. the tropic of Capricorn

correspond to zeros of $P_n(x)$

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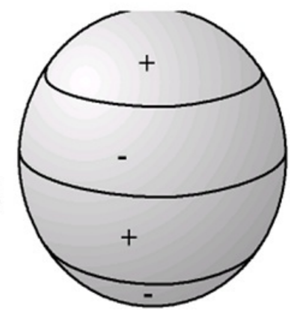
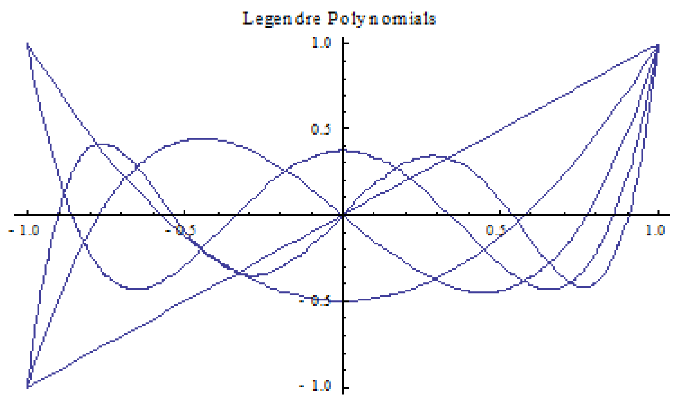


Special case of our problem:

Show that at most **<u>finitely many</u>** zonal spherical harmonics $Y_n(\theta, \varphi = P + n(\cos\theta)$ can vanish on a **<u>fixed</u>** parallel (except the equator $\cos\theta = 0$)

Stieltjes' conjecture for Legendre polynomials

Conj: $P_m(x), P_n(x)$ have **no** common zero if $m \neq n$ (except at $x = 0$)



Stieltjes' conjecture (1890)

The Legendre polynomials $P_{2n}(x), P_{2n+1}(x)/x$ are irreducible over the rationals ! (still open)

- Studied by Schur & his students
- Stieltjes' conjecture implies that the nodal lines of distinct zonal harmonics are **disjoint** (except for the equator), in particular a given parallel can lie on the nodal set of **at most ONE** horizontal harmonic.

RESTRICTION THEOREMS

Theorem (**Burq–Gerard–Tzvetkov, 07**)

M *compact, smooth Riemannian manifold*

$\Sigma =$ *codimension one submanifold*

$$\int_{\Sigma} |\varphi_E|^2 \ll E^{1/4} \|\varphi_E\|_2^2$$

If moreover Σ *is curved*

$$\int_{\Sigma} |\varphi_E|^2 \ll E^{1/6} \|\varphi_E\|_2^2$$

Estimates are sharp for the sphere

Motivated by work of A. Reznikov in hyperbolic case

RESTRICTION BOUNDS AND CONVEXITY BREAKING $(d = 2)$

Theorem (**B, 09**)

If $p \geq 2$ *and* $\gamma : [0, 1] \to M$ *a geodesic segment, then*

$$\|\varphi_E\|_{L^2(\gamma)} \leq C\lambda^{\frac{1}{2p}} \|\varphi_E\|_{L^p(M)}$$

Remark There is an integrable perturbation M of the flat torus $\mathbb{T}^2$ and a geodesic segment γ such that for a sequence of eigenfunctions

$$\|\varphi_E\|_6 \sim \lambda^{\frac{1}{6}} \text{ and } \|\varphi_E\|_{L^2(\gamma)} \sim \lambda^{\frac{1}{4}}$$

Theorem (B, 09)

$$\|\varphi_E\|_{L^4(M)} \ll \lambda^{\frac{1}{16}+\varepsilon} \max\{\|\varphi_E\|_{L^2(\gamma)} : \gamma \text{ geodesic}\}^{\frac{1}{4}}$$

Variants obtained by C. Sogge

SOME APPLICATIONS

Theorem (**S. Marshall, 012**)

$M = \mathbb{H}/\Gamma$ = *arithmetic hyperbolic surface*

ψ = *Hecke-Maass eigenfunction with spectral parameter* t

- $\|\psi\|_{L^2(\gamma)} \leq t^{3/14+\varepsilon}$ *for* γ *geodesic unit segment*
- $\|\psi\|_{L^4(M)} \ll t^{\frac{1}{8}-\frac{1}{112}+\varepsilon}$

(**Sarnak–Watson**: $\|\varphi\|_4 \ll t^{\varepsilon}$ conditional to Ramanujan conjecture)

Theorem (**Sogge–Zelditch, 011**)

$(M, g) =$ *compact, smooth* $2D$ *manifold with non-positive curvature. Then*

$$\|\varphi_E\|_{L^4(M)} = o(\lambda^{\frac{1}{8}})\|\varphi_E\|_{L^2(M)} \text{ for } E \to \infty$$

Theorem (**B–Rudnick, 011**)

Let $d = 2$ *or* $d = 3$

$\Sigma \subset \mathbb{T}^d$ *real analytic with non-vanishing curvature. There are constants* $C > c > 0$, $E_0 > 0$ *(depending on* Σ*) s.t. for all eigenfunctions with* $E > E_0$

$$c\|\varphi\|_2 \leq \left(\int_\Sigma |\varphi(x)|^2 \right)^{\frac{1}{2}} \leq C\|\varphi\|^2$$

Upper Bound conjectured by **Burq, Gerard, Tzvetkov**

Problem $(\mathbb{T}^2)$

Is there a constant such that

$$\|\varphi_E\|_{L^2(\gamma)} \leq C\|\varphi_E\|_2$$

if $\gamma \subset \mathbb{T}^2$ is a unit segment?

$$\Updownarrow$$

Problem

Is the number of lattice points

$$x^2 + y^2 = E$$

on an arc of size $E^{\frac{1}{4}}$ bounded by an absolute constant?

An arc of size $E^{\frac{1}{4}-\varepsilon}$ contains at most C_ε lattice points and is conjectured to be true for arcs of size $E^{\frac{1}{2}-\varepsilon}$

Remark

For most E, minimal spacing of lattice points on

$$x^2 + y^2 = E$$

is at least $E^{\frac{1}{2}-\varepsilon}$

Remark

For most E, there is inequality

$$\|\varphi_E\|_{L^2(\gamma)} \leq C\|\varphi_E\|_2$$

whenever $\gamma \subset \mathbb{T}^2$ is a smooth curve

COURANT NODAL DOMAIN THEOREM

Courant Theorem

n*-th eigenstate has at most* n *nodal domains*

Some improvements (**A. Pleijel, 1956**) for $2D$

$$\limsup_{n\to\infty} \frac{N(n)}{n} \leq 0,691...$$

No nontrivial lower bounds in general:

$N = 2$ for a sequence of eigenfunctions of $\mathbb{T}^2$

BOGOMOLNY–SCHMIT CONJECTURE AND 'WORD EXCHANGEABILITY' PRINCIPLE

Bogomolny–Schmit (02)

Asymptotic distribution of nodal domains of chaotic manifold (2D) is universal and described by percolation theory.

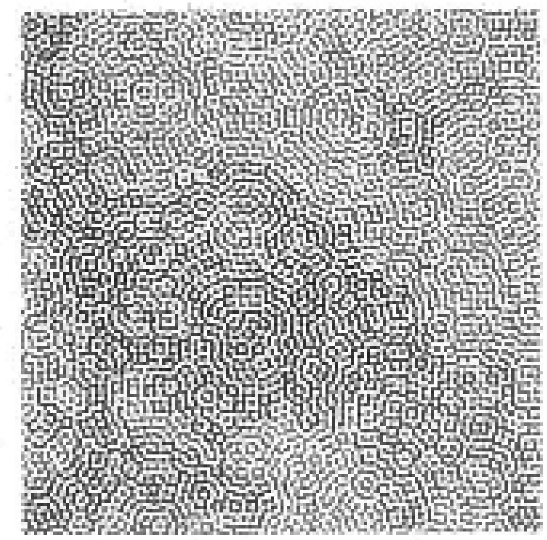
In particular

$$N(n) \sim n$$

EXCHANGEABILITY heuristic

Properties satisfied by generic eigenfunctions on (all) completely integrable manifolds are also satisfied by **ALL** *eigenfunctions on a generic chaotic manifold*

BOGOMOLNY–SCHMIDT RANDOM WAVE MODEL



Theorem (**Nazarov–Sodin, 09**)

Behavior of $\frac{N(n)}{n}$ *for random eigenfunctions on the sphere*

Theorem (**B–Rudnick, 011**)

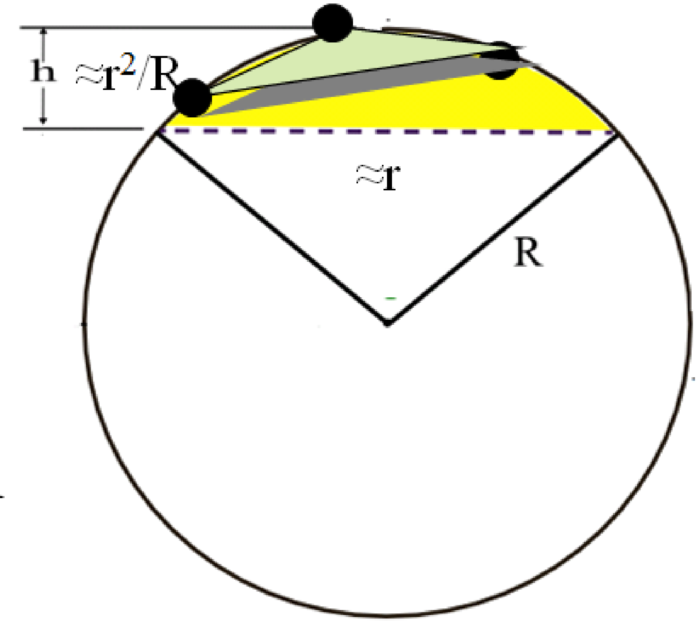
Similar results for tori, i.e. $N(n) \sim n$

Distribution of lattice points on spheres

- Jarnik type results
- Bounds on number of lattice points in caps
- Equidistribution theorems

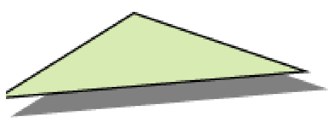
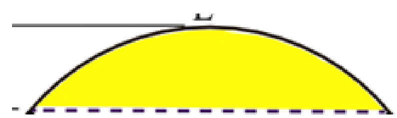
Jarnik's theorem

<u>**Jarnik (1926):**</u> Any arc of length $r << R^{1/3}$ on the circle $x_1^2+x_2^2=R^2$ contains at most TWO lattice points !

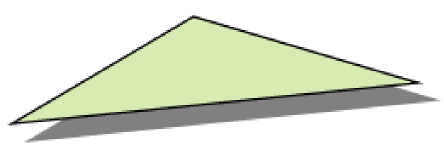


Given three points on an arc of size r, compute the area of the triangle they span.

On one hand, it is bounded above by the area of the cap spanned by the arc

 $<$  $\approx r^3/R$

On the other hand, being an INTEGER triangle implies its area is at least ½:

$1/2 \leq$ 

We get: $½ << r^3/R \;\longrightarrow\; r >> R^{1/3}$. QED

<u>**Higher dimensional version**</u>: On the sphere $\{|x| = r\}$ in $\mathbb{R}^d$, the lattice points in any spherical cap of diameter $<< r^{1/d+1}$ are co-planar

CLUSTER STRUCTURE

$$\mathcal{E} = \mathbb{Z}^d \cap \lambda S_{d-1}$$

There are constants $0 < \delta, C$ depending on d s.t. if $1 < \rho < R^\delta$, then there is a decomposition

$$\mathcal{E} = \bigcup \mathcal{E}_\alpha$$

dist $(\mathcal{E}_\alpha, \mathcal{E}_\beta) > \rho$ for $\alpha \neq \beta$
diam $(\mathcal{E}_\alpha) < \rho^C$

Corollary *Given $\tau > 0$, there is $\delta > 0$ s.t.*

$$\|\varphi_E|_I\|_2 \geq \delta \|\varphi_E\|_2$$

whenever $I \subset \mathbb{T}^d$ is a ball of size τ and φ an eigenfunction

LATTICE POINTS IN CAPS

$$\mathcal{E} = \mathbb{Z}^d \bigcap \lambda S_{d-1} \qquad \#\mathcal{E} \ll \lambda^{d-2+\varepsilon}$$

$F_d(\lambda, r)$ = maximal number of points in $\mathcal{E} \cap C_r$, where $1 < r < R$
and C_r a spherical cap of size r

Theorem (**Cilleruelo–Cordoba, 92**)

$$F_2(\lambda, r) < C_\varepsilon \text{ for } r < \lambda^{\frac{1}{2}-\varepsilon}$$

$$\prod_{1\le i<j\le m} |P_i - P_j| \ge \begin{cases} \lambda^{\frac{m}{2}(\frac{m}{2}-1)} & \text{if } m \text{ is even} \\ \lambda^{\frac{1}{4}(m-1)^2} & \text{if } m \text{ is odd} \end{cases}$$

Conjecture (**Cilleruelo–Granville, 07**)

$$F_2(\lambda, r) < C_\varepsilon \text{ for } r < \lambda^{1-\varepsilon}$$

Theorem (**B–Rudnick, 011**)

- $F_3(\lambda, r) \ll \lambda^\varepsilon\Big(r\big(\frac{r}{\lambda}\big)^\eta + 1\Big)$ for $\eta < \frac{1}{15}$
- $F_4(\lambda, r) \ll \lambda^\varepsilon\big(\frac{r^3}{\lambda} + r^{3/2}\big)$
- $F_d(\lambda, r) \ll \lambda^\varepsilon\big(\frac{r^{d-1}}{\lambda} + r^{d-3}\big)$ for $d \ge 5$

(λ^ε factor redundant for large d)

EQUIDISTRIBUTION IN 3D

$$\mathcal{E} = \{x \in \mathbb{Z}^3, |x|^2 = x_1^2 + x_2^2 + x_3^2 = E\}$$

$$|\mathcal{E}| \ll E^{\frac{1}{2}+\varepsilon}$$

$$\mathcal{E} \neq \phi \Leftrightarrow E \neq 4^a(8b+7) \quad \text{(Legendre-Gauss)}$$

$\mathcal{E}$ has primitive points $\Leftrightarrow E \neq 0, 4, 7 \pmod 8$

Then there is lower bound $|\mathcal{E}| > E^{\frac{1}{2}-\varepsilon}$ and projection $\frac{1}{\sqrt{E}}\mathcal{E}$ on S_2 becomes equidistributed for $E \to \infty$

(conjectured by **Linnik** and proven by **Duke–Iwaniec**)

DISTRIBUTION IN SMALLER REGIONS?

Divide $\sqrt{E}S_2$ in boxes A_j of size $\sim E^{\frac{1}{4}}$

Theorem $\sum_j |A_j \bigcap \mathcal{E}|^2 \ll E^{\frac{1}{2}+\varepsilon}$

'average' separation between points is $\sim E^{\frac{1}{4}\pm\varepsilon}$

Follows from Siegel mass formula applied to

$$\{(x,y) \in \mathbb{Z}^3 \times \mathbb{Z}^3; |x|^2 = E = |y|^2 \text{ and } \langle x,y \rangle = b\}$$

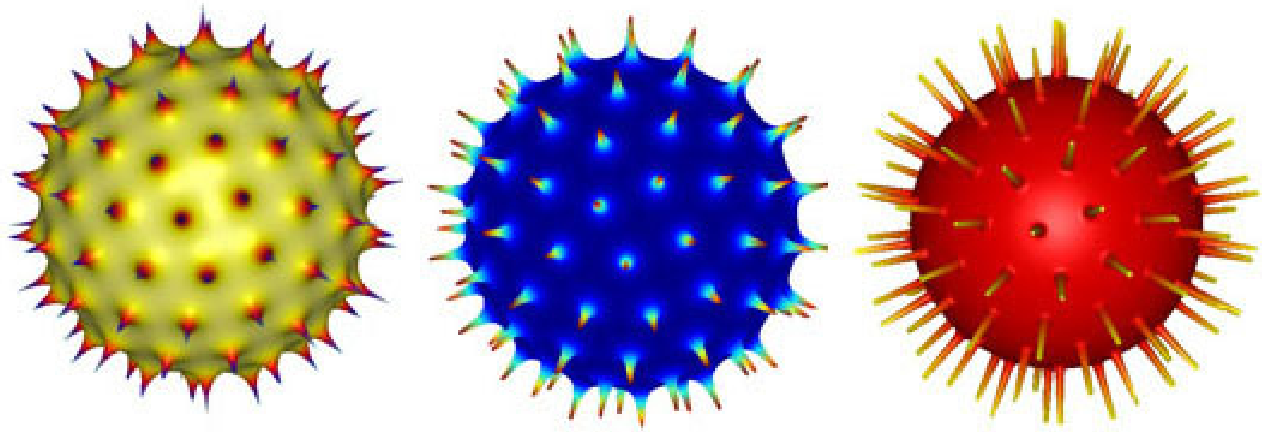
APPLICATION TO THE ELECTROSTATIC ENERGY

Problem Distribute point charges $x_1, \ldots, x_N \in S_2$ as to minimize the electrostatic energy

$$E(x_1, \ldots, x_N) = \sum_{i \neq j} \frac{1}{|x_i - x_j|}$$

and, more generally, the s-energy $(0 < s < 2)$

$$E_s(x_1, \ldots, x_N) = \sum_{i \neq j} \frac{1}{|x_i - x_j|^s}$$



The minimum energy problem is to find a set of m points on the unit sphere which minimize their Riesz energy. The images above correspond to the energy when one extra point is added to the set of 100 points illustrated above for (from the left to the right) $s = 0.1, s = 1$ and $s = 4$

Theorem (**G. Wagner, 92**)

$$E_s(x_1, \ldots, x_N) \geq I(s)N^2 - cN^{3/2}$$

where

$$I(s) = \int \|x-y\|^{-s} d\sigma(x) d\sigma(y) = \frac{2^{1-s}}{2-s}$$

Theorem (**B-R-S, 09**) *Let* $\widehat{\mathcal{E}}(n) = \frac{1}{\sqrt{n}}\mathcal{E}(n) \subset S_2$ *and* $N = \#\mathcal{E}(n)$*. Suppose* $n \to \infty$ *and* $n \neq 0, 4, 7(\mathit{mod}\, 8)$*. Then*

$$E\big(\widehat{\mathcal{E}}(n)\big) = I(s)N^2 + 0\big(N^{2-\delta}\big)$$

RESTRICTION ESTIMATES

$$\int_{\Sigma} |\varphi|^2 d\sigma \sim \sum_{m,n \in \mathcal{E}} \widehat{\varphi}(m)\overline{\widehat{\varphi}(n)} \; \frac{e^{i\|m-n\|_*}}{|m-n|^{\frac{d-1}{2}}}$$

$\| \ \|_*$ = support functional of Σ

Σ smooth, compact hyper-surface of (positive) curvature

Estimates on bilinear exponential sums using various information on lattice point distribution on spheres