# The Last Arrival Problem and Stochastic Processes with Proportional Increments

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**in honour of Freddy Delbaen**

Joint work with Marc Yor

(Stoch.Proc.Th.Appl., 2012)

# Objectives

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(c) No information except $N < \infty$ *a.s.* (**The l.a.p.**)

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**Central question**: Can we prove that the **l.a.p.** is an ill-posed problem?

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**Conclusion**: As we understand *no-information*, it is *not possible* to prove that the **l.a.p.** is ill-posed.

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(v) Sequential observation $\implies$ relevant process is $(N_t)_{0<t\leq 1}$

$$N_t = \sum_{k=1}^{N} \mathbf{1}\{X_k \leq t\}$$

$$\mathcal{F}_t = \sigma\{N_u : 0 \leq u \leq t\}.$$

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Indeed:

Using "no-formation" , i.e. at time $t$ no other information than that contained in $\mathcal{F}_t$, and

$$\mathrm{E}(N_t) = tN$$

$$\mathrm{E}\Big(\mathrm{E}(N_{t+s}|\mathcal{F}_t)\Big) = (t+s)N = (t+s)\mathrm{E}\Big(\frac{N_t}{t}\Big)$$

we can show:

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$$\forall 0 < t \leq t + s \leq 1 \text{ with } N_t \neq 0, \qquad (1)$$

$$\mathrm{E}(N_{t+s} - N_t | \mathcal{F}_t) = \frac{s}{t} N_t \ a.s.$$

More generally:

### Definition

Let $(N_t)_{t>0}$ be a stochastic process on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ with natural filtration $\mathcal{F}_t = \sigma\{N_u : u \leq t\}$.

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$$\begin{aligned} \mathrm{E}(N_{t+s} - N_t | \mathcal{F}_t) &= \mathrm{E}\Big((t+s)\mathcal{B}_{t+s} - t\mathcal{B}_t | \mathcal{F}_t\Big) \\ &= \mathrm{E}\left((t+s)(\mathcal{B}_t + (\mathcal{B}_{t+s} - \mathcal{B}_t)) - t\mathcal{B}_t | \mathcal{F}_t\right) \\ &= s\mathrm{E}(\mathcal{B}_t | \mathcal{F}_t) = \frac{s}{t} t\mathcal{B}_t = \frac{s}{t} N_t. \end{aligned} \tag{2}$$

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But .... is there anything deeper to all this?

# A related martingale

## Theorem

*Let $(N_t)_{t>0}$ be a p.i.- counting process and $R_t = N_t/t$.*
*If $N_{t_0} > 0$ and $\mathrm{E}(N_{t_0}) < \infty$ for some $t_0 > 0$ then $(R_t)$ is a $\mathcal{F}_t$-martingale on $[t_0, \infty[$.*

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*Proof.* Use $|N_t| = N_t$, $\mathcal{F}_t \supseteq \mathcal{F}_{t_0}$, and p.i.-property:

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\begin{aligned}
\text{(i)}\quad \mathrm{E}(|R_t|) &= \frac{1}{t}\mathrm{E}(N_{t_0} + (N_t - N_{t_0})) \\
&\leq \frac{1}{t_0}\mathrm{E}(N_{t_0}) + \frac{1}{t}\mathrm{E}(N_t - N_{t_0}) \\
&= \mathrm{E}(R_{t_0}) + \frac{1}{t}\mathrm{E}\left[\mathrm{E}\left(N_t - N_{t_0} \middle| \mathcal{F}_{t_0}\right)\right] \\
&= \mathrm{E}(R_{t_0}) + \frac{1}{t}\mathrm{E}\left((t - t_0)\frac{N_{t_0}}{t_0}\right) \leq 2\mathrm{E}(R_{t_0}) < \infty.
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&= \frac{1}{t+s}\left(N_t + \mathrm{E}(N_{t+s} - N_t|\mathcal{F}_t)\right) \\
&= \frac{1}{t+s}\left(N_t + \frac{s}{t}N_t\right) \\
&= \frac{N_t}{t} = R_t.
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□

# Reverse martingale

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Carr, Geman, Madan and Yor (2011):

$$\forall 0 \leq t \leq T : \mathrm{E}(N_T/T|\mathcal{F}_t^+) = N_t/t \ a.s.$$

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A little challenge:

Distributional prescription

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**Definition**

$(\Pi_t)_{t\geq 0}$ counting process such that for all $T > 0$ and $0 \leq t \leq T$

$$P(\Pi_T = n|\mathcal{F}_t) = \binom{n}{\Pi_t} p(t,T)^{\Pi_t+1}(1-p(t,T))^{n-\Pi_t}$$

where $\Pi_0 = 0$ and $(\mathcal{F}_t) = \sigma(\{\Pi_u : u \leq t\})$. Then $(\Pi_t)$ is called a Pascal process with parameter function $p(t,T)$.

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A: Think in terms of odds "future/past"!!!

## Theorem

*Every Pascal process* $(\Pi_t)$ *augmented by* $1$ *has odds-proportional increments with odds* $r(t, T) := (1 - p(t, T))/p(t, T)$, *where* $p(t, T)$ *is the corresponding parameter function, that is*

$$E(\Pi_T - \Pi_t | \mathcal{F}_t) = r(t, T)(\Pi_t + 1) \text{ a.s.}$$

### Theorem

*If* $(\Pi_t)_{t\geq 0}$ *is a Pascal process with parameter function* $p(t, T)$ *and filtration* $\mathcal{F}_t = \sigma(\{\Pi_u : 0 \leq u \leq t\})$*, then the process* $(R_t)_{t\geq 0}$ *defined by*

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Further generalizations: **"f-increment processes"**

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Conclusion: Quite some room for discovering p.i.-processes!

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Odds-Theorem of optimal stopping

# Solution of the l.a.p.

**Theorem**

*Let* $T_k = X_{<k,N>}, k = 1, 2, \cdots, N$ *be the (a.s) strictly increasing jump times of* $(N_t)$*. Further let*

$$\tau = \inf\left\{ T_k \in ]0,1] : k \le \frac{T_k}{1 - T_k} \right\},$$

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# Proof (sketch)

(i) Odds theorem of opt. stop. ( B., Ann. of Probab. (2000))

**Theorem**

*Let $I_1, I_2, \cdots, I_n$ be independent indicators on some $(\Omega, \mathcal{A}, P)$ with known $p_k = \mathrm{E}(I_k)$. We want to stop (online) with maximum probability on the last "success". An optimal strategy $\tau$ exists and is as follows:*

$r_k := p_k/(1-p_k)$

$s :=$ *largest* $k$ *with* $r_n + r_{n-1} + \cdots + r_k \geq 1$

*(*$s := 1$ *if no such* $1 \leq k \leq n$ *exists)*

$$\tau = \min\{s \leq k \leq n : I_k = 1\} \text{ is optimal}$$

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If all odds sum up to at least one, then $\tau$ always succeeds with probability $\geq 1/e$.

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This is easy for any counting process with *independent* increments;

(ii) Addendum to the Odds theorem ( B., Ann. of Probab. (2003))

If all odds sum up to at least one, then $\tau$ always succeeds with probability $\geq 1/e$.

(iii) How to pass from discrete time and fixed $n$ to continuous time and unknown $N$ ?

This is easy for any counting process with *independent* increments; Specifically in Poisson process case:

Take Riemann sum limit for limiting odds

$$\lim_{dt \to 0} \frac{1}{dt} (\lambda_t \, dt + o(dt)) / (1 - \lambda_t dt - o(dt)) = \lambda_t$$

( $\Longrightarrow$ integral version of odds-algorithm (B. (2000)))

(iv) Confine interest to stopping times $\tau < 1$.

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(iv) Slightly more general integral version of the odds algorithm (adapted to the l.a.p.):

Let $(Y_t)$ be a counting process on $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$. Suppose there exists $s > 0$ such that $(Y_t)_{t \geq s}$ is a PP with rate $\Lambda_t$ possibly depending on $\mathcal{G}_s$, then ......

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(a) Let $\tau$ be any $\mathcal{F}_t-$stopping time and define

$$M_t^\tau := \mathbf{1}_{\{t\le\tau\}}N_t + \mathbf{1}_{\{t>\tau\}}\Big(N_\tau + \mu_{t-\tau}(\Lambda_\tau)\Big),$$

where $\mu$ denotes a homogeneous Poisson Process of rate (.).

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where $\mu$ denotes a homogeneous Poisson Process of rate (.).

(b) We want $(M_t^\tau)/t$ to satisfy the martingale property of $(N_t/t)$.

(c) A *necessary* condition for $M_t^\tau$ to be a martingale is to impose $\Lambda_\tau = N_\tau/\tau$ ("Poisson shadow" of $(N_t)$ in $\tau$)

**Theorem**

*If $N$ turns out to be $n$ then the win probability equals*

$$w_n = \frac{n!}{n+1} \int_0^{1/2} \int_{x_1}^{2/3} \int_{x_2}^{3/4} \cdots \int_{x_{n-2}}^{(n-1)/n} dx_{n-1} \cdots dx_2 dx_1 .$$

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$(w_n)_{n\geq 3} \uparrow 1/e.$ (Conjecture solved on MathOverview!)

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but they are tractable and possibly broader than one might think and interesting as a modelling tool giving easily acces to martingales.

# References

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