Equilibrium Pricing in Incomplete Markets under Translation Invariant Preferences

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joint work with

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September 2012

Motivation

Price processes are martingales under $\mathbb{Q}$

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- Harrison–Kreps (1979). Martingales and arbitrage in multiperiod security markets.
- Harrison–Pliska. (1981). Martingales and stochastic integrals in the theory of continuous trading.
- Dalang–Morton–Willinger (1989). Equivalent martingale measures and no-arbitrage in stochastic securities market models.
- Delbaen (1992). Representing martingale measures when asset prices are continuous and bounded.
- Schachermayer (1993). Martingale measures for discrete time processes with infinite horizon.
- Delbaen–Schachermayer (1994). A general version of the fundamental theorem of asset pricing.
- Delbaen–Schachermayer (1998). The fundamental theorem of asset pricing for unbounded stochastic processes.

Usual approach to derivatives pricing:

1. Model the underlying securities as a J-dimensional stochastic process (R_t) on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$
2. Price derivatives by $\mathbb{E}_{\mathbb{Q}}\left[\,.\,\right]$ for some equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$

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In complete markets: $\mathbb{Q}$ is unique

binomial tree models, Black–Scholes model ...

In incomplete markets: $\mathbb{Q}$ is not unique

trinomial tree models, GARCH-type models,

stochastic volatility models, jump-diffusion models,

Levy-process models, more general semimartingale models

...

Motivation

Problem:

Choose a pricing measure $\hat{\mathbb{Q}}$ among all equivalent martingale measures

Some commonly used methods:

1. Parameterize $\hat{\mathbb{Q}}_\theta$, $\theta \in \Theta$ and calibrate to market data of traded derivatives ... via $\frac{d\hat{\mathbb{Q}}_\theta}{d\mathbb{P}}$ or without $\mathbb{P}$.
   E.g. build a stoch vol model or local vol model directly under $\hat{\mathbb{Q}}$
2. Choose $\hat{\mathbb{Q}}$ so that it minimizes some distance to $\mathbb{P}$,
   e.g. L^p-distance, relative entropy, f-divergence ...
3. Indifference pricing
4. ...

Our goal: derive $\hat{\mathbb{Q}}$ from equilibrium considerations

Some motivating Examples

Horst and Müller (2007).

On the spanning property of risk bonds priced by equilibrium

Bakshi, Kapadia and Madan (2003).

Stock return characteristics, skew laws, and the differential pricing of individual equity options

Garleanu, Pedersen and Poteshman (2009).

Demand-based option pricing

Carmona, Fehr, Hinz and Porchet (2010).

Market design for emission trading schemes.

Outline

1. Model
2. Existence of equilibrium
3. Uniqueness of equilibrium
4. Random walks and BSΔEs
5. Brownian motion and BSDEs
6. Option pricing under demand pressure

Ingredients

- filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^{T}, \mathbb{P})$

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- agent $a \in \mathbb{A}$ is endowed with an uncertain payoff $H^a = g^{a,R} R_T + g^{a,S} S_T + G^a$
- at time t agent a invests to optimize a preference functional

$$U_t^a : L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_t)$$

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$U_t^a(\lambda X + (1-\lambda)Y) \geq \lambda U_t^a(X) + (1-\lambda)U_t^a(Y)$ for all $X, Y \in L^\infty(\mathcal{F}_T)$ and $\lambda \in L^\infty(\mathcal{F}_t)$ such that $0 \leq \lambda \leq 1$

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(T) Translation property
$U_t^a(X+Y) = U_t^a(X) + Y$ for all $X \in L^\infty(\mathcal{F}_T)$ and $Y \in L^\infty(\mathcal{F}_t)$

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$U_t^a(X + Y) = U_t^a(X) + Y$ for all $X \in L^\infty(\mathcal{F}_T)$ and $Y \in L^\infty(\mathcal{F}_t)$

(TC) Time-consistency
$U_{t+1}^a(X) \geq U_{t+1}^a(Y) \quad \text{implies} \quad U_t^a(X) \geq U_t^a(Y)$

$$\Leftrightarrow \qquad U_t^a(X) = U_t^a(U_{t+1}^a(X))$$

Related to coherent and convex risk measures

- Artzner–Delbaen–Eber–Heath (1999).
  Coherent measures of risk.
- Föllmer–Schied (2002).
  Convex measures of risk and trading constraints
- Frittelli–Rosazza Gianin (2002).
  Putting order in risk measures.

Examples

1) $U_t^a(X) = -\frac{1}{\gamma}\log \mathbb{E}\left[e^{-\gamma X} \mid \mathcal{F}_t\right]$

2) $U_t^a(X) = \mathbb{E}\left[X \mid \mathcal{F}_t\right] - \lambda \mathbb{E}\left[(X - \mathbb{E}\left[X \mid \mathcal{F}_t\right])^2 \mid \mathcal{F}_t\right]$

3) $U_t^a(X) = (1-\lambda)\mathbb{E}\left[X \mid \mathcal{F}_t\right] - \lambda \rho_t(X)$

where ρ_t is a conditional convex risk measure

Model

An equilibrium of plans, prices and price expectations à la Radner (1972) consists of

- an adapted process $(S_t)_{t=0}^T$ with $S_T = S$
- trading strategies $(\hat{\vartheta}_t^a)_{t=1}^T$

such that the following hold:

(i) individual optimality

$$
\begin{aligned}
& U_t^a\left(H^a + \sum_{s=t+1}^{T} \hat{\vartheta}_s^{a,1}\Delta R_s + \hat{\vartheta}_s^{a,2}\Delta S_s\right) \\
\geq \quad & U_t^a\left(H^a + \sum_{s=t+1}^{T} \vartheta_s^{a,1}\Delta R_s + \vartheta_s^{a,2}\Delta S_s\right)
\end{aligned}
$$

for every t and all possible strategies (ϑ_s^a)

(ii) market clearing $\quad \sum_{a\in\mathbb{A}} \hat{\vartheta}_t^{a,2} = n$

Hart (1975) On the optimality of equilibrium when the market structure is incomplete:

In general, a Radner equilibrium does not exist, and if there is one, it is not unique.

One-step representative agents

Set $H_T^a = H^a$ and

$$H_{t+1}^a = U_{t+1}^a \left(H^a + \sum_{s=t+2}^{T} \hat{\vartheta}_s^{a,1} \Delta R_s + \hat{\vartheta}_s^{a,2} \Delta S_s \right)$$

the true representative agent would be

$$\hat{u}_t(x) = \operatorname*{ess\,sup}_{\substack{\vartheta^a \in L^\infty(\mathcal{F}_t)^2 \\ \sum_{a \in \mathbb{A}} \vartheta^{a,2} = x}} \sum_{a \in \mathbb{A}} U_t^a \left(H_{t+1}^a + \vartheta^{a,1} \Delta R_{t+1} + \vartheta^{a,2} \Delta S_{t+1} \right)$$

One-step representative agents

Set $H^a_T = H^a$ and

$H^a_{t+1} = U^a_{t+1}\left(H^a + \sum_{s=t+2}^T \hat{\vartheta}^{a,1}_s \Delta R_s + \hat{\vartheta}^{a,2}_s \Delta S_s\right)$

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But S_t is not known. So define

$$\hat{u}_t(x) = \operatorname*{ess\,sup}_{\substack{\vartheta^a \in L^\infty(\mathcal{F}_t)^2 \\ \sum_{a\in} \vartheta^{a,2} = x}} \sum_{a} U^a_t\left(H^a_{t+1} + \vartheta^{a,1}\Delta R_{t+1} + \vartheta^{a,2} S_{t+1}\right)$$

$\hat{u}_t$ is $\mathcal{F}_t$-concave

Convex dual characterization of equilibrium

Theorem A bounded, adapted process $(S_t)_{t=0}^T$ satisfying $S_T = S$ together with trading strategies $(\hat{\vartheta}_t^a)_{t=1}^T$, $a \in \mathbb{A}$, form an equilibrium $\iff$ for all t:

(i) $S_t \in \partial \hat{u}_t(n)$

(ii) $\sum_{a \in \mathbb{A}} U_t^a(H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,1} \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,2} S_{t+1}) = \hat{u}_t(n)$

(iii) $\sum_{a \in \mathbb{A}} \hat{\vartheta}_{t+1}^{a,2} = n$

Existence

Assumption (A)

For all $t = 0, \ldots, T-1$, $V^a \in L^\infty(\mathcal{F}_{t+1})$, $W \in L^\infty(\mathcal{F}_{t+1})$, there exist $\hat{\vartheta}^a_{t+1} \in L^\infty(\mathcal{F}_t)^2$, $a \in \mathbb{A}$, such that

$$\sum_{a \in \mathbb{A}} \hat{\vartheta}^{a,2}_{t+1} = 0$$

and

$$\sum_{a \in \mathbb{A}} U^a_t \left(V^a + \hat{\vartheta}^{a,1}_{t+1} \Delta R_{t+1} + \hat{\vartheta}^{a,2}_{t+1} W \right)$$
$$= \operatorname*{ess\,sup}_{\substack{\vartheta^a_{t+1} \in L^\infty(\mathcal{F}_t)^2 \\ \sum_{a \in \mathbb{A}} \vartheta^{a,2}_{t+1} = 0}} \sum_{a \in \mathbb{A}} U^a_t \left(V^a + \vartheta^{a,1}_{t+1} \Delta R_{t+1} + \vartheta^{a,2}_{t+1} W \right).$$

Lemma Under assumption (A) an equilibrium exists

Existence

Definition

U_0^a is sensitive to large losses if

$$\lim_{\lambda \to \infty} U_0^a(\lambda X) = -\infty$$

for all $X \in L^\infty(\mathcal{F}_T)$ such that $\mathbb{P}[X < 0] > 0$.

Theorem

If all U_0^a are sensitive to large losses,
then condition (A) is satisfied and an equilibrium exists.

Remark

The theorem also works with convex trading constraints.

Existence

Proposition

If the market is in equilibrium and at least one agent has strictly monotone preferences and open trading constraints, then there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that $R_t = \mathbb{E}_{\mathbb{Q}}\left[R_T \mid \mathcal{F}_t\right]$ and $S_t = \mathbb{E}_{\mathbb{Q}}\left[R_T \mid \mathcal{F}_t\right]$.

Differentiable preferences

We say U_t^a satisfies the differentiability condition (**D**) if for all $X, Y \in L^\infty(\mathcal{F}_{t+1})$, there exists $Z \in L^1(\mathcal{F}_{t+1})$ such that

$$\lim_{k\to\infty} k\left(U_t^a\left(X + \frac{Y}{k}\right) - U_t^a(X)\right) = \mathbb{E}\left[YZ \mid \mathcal{F}_t\right].$$

If such a Z exists, it has to be unique, and we denote it by $\nabla U_t^a(X)$.

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Theorem If at least one U_t^a satisfies (D), then there can exist at most one equilibrium price process $(S_t)_{t=0}^T$, and if the market is in equilibrium, then

$$\frac{d\mathbb{Q}_t^a}{d\mathbb{P}} := \nabla U_t^a\left(H^a+\sum_{s=1}^T \hat{\vartheta}_s^{a,1}\Delta R_s+\hat{\vartheta}_s^{a,2}\Delta S_s\right)$$

defines a pricing measure.

Random walks and BSΔEs

Fix $h > 0$ and $N \in \mathbb{N}$

Denote $\mathbb{T} = \{0, h, \dots, T = Nh\}$

$b_t^1, \dots, b_t^d$ d independent random walks with $P[\Delta b_{t+h}^i = \pm\sqrt{h}] = 1/2$

$b_t^{d+1}, \dots, b_t^D$ $2^d - (d+1)$ random walks orthogonal to $b_t^1, \dots, b_t^d$

Every $X \in L^\infty(\mathcal{F}_{t+h})$ can be represented as

$$X = \mathbb{E}\left[X \mid \mathcal{F}_t\right] + \pi_t(X) \cdot \Delta b_{t+h}$$

for

$$\pi_t(X) \cdot \Delta b_{t+h} = \sum_{i=1}^{D} \pi_t^i(X) \Delta b_{t+h}^i \quad \text{and} \quad \pi_t^i(X) = \frac{1}{h} \mathbb{E}\left[X \Delta b_{t+h}^i \mid \mathcal{F}_t\right].$$

Random walks and BSΔEs

$$U_t^a(X) = U_t^a\left(\mathbb{E}\left[X|\mathcal{F}_t\right] + \pi_t(X)\cdot\Delta b_{t+h}\right) = \mathbb{E}\left[X \mid \mathcal{F}_t\right] - f_t^a(\pi_t(X))h$$

for the $\mathcal{F}_t$-convex function $f_t^a : L^\infty(\mathcal{F}_t)^D \to L^\infty(\mathcal{F}_t)$ given by

$$f_t^a(z) := -\frac{1}{h}U_t^a(z\cdot\Delta b_{t+h})\,.$$

Assume condition (A) is satisfied and all U_t^a satisfy the differentiability condition (D).

Then there exists $\nabla f_t^a(z) \in L^\infty(\mathcal{F}_t)^D$ such that

$$\lim_{k\to\infty} k\left(f_t^a(z+z'/k) - f_t^a(z)\right) = z'\cdot\nabla f_t^a(z)$$

Random walks and BSΔEs

For given R_{t+h}, S_{t+h}, H^a_{t+h} denote

$$\begin{aligned} Z^R_{t+h} &:= \pi_t(R_{t+h}) \\ Z^S_{t+h} &:= \pi_t(S_{t+h}) \\ Z^a_{t+h} &:= \pi_t(H^a_{t+h}) \\ Z_{t+h} &= (Z^R_{t+h}, Z^S_{t+h}, Z^a_{t+h}, a \in \mathbb{A}) \,. \end{aligned}$$

and define the function $f_t : L^\infty(\mathcal{F}_t)^{(3+|\mathbb{A}|)D} \to L^\infty(\mathcal{F}_t)$ by

$$\begin{aligned} & f_t(v, Z_{t+h}) \\ = & \operatorname*{ess\,inf}_{\substack{\vartheta^a \in L(\mathcal{F}_t)^2 \\ \sum_{a\in\mathbb{A}} \vartheta^{a,2} = 0}} \sum_{a\in\mathbb{A}} f^a_t \left(\frac{v}{|\mathbb{A}|} + Z^a_{t+h} + \vartheta^{a,1}_{t+h} Z^R_{t+h} + \vartheta^{a,2}_{t+h} Z^S_{t+h} \right. \\ & \left. \qquad -\vartheta^{a,1}_{t+h} \frac{\mathbb{E}\left[\Delta R_{t+h} \mid \mathcal{F}_t\right]}{h} \right). \end{aligned}$$

Random walks and BSΔEs

Set

$$\begin{aligned} g_t^S(Z_{t+h}) &:= Z_{t+h}^S \cdot \nabla^v f_t(nZ_{t+h}^S, Z_{t+h}) \\ g_t^a(Z_{t+h}) &:= f_t^a\left(Z_{t+h}^a + \hat{\vartheta}_{t+h}^{a,1} Z_{t+h}^R + \hat{\vartheta}_{t+h}^{a,2} Z_{t+h}^S\right) \\ &\quad -\hat{\vartheta}_{t+h}^{a,1} \frac{1}{h} \mathbb{E}\left[\Delta R_{t+h} \mid \mathcal{F}_t\right] - \hat{\vartheta}_{t+h}^{a,2} g_t^S(Z_{t+h}). \end{aligned}$$

Random walks and BSΔEs

The processes (S_t) and (H^a_t) satisfy the following coupled system of BSΔEs

$$\begin{aligned}
\Delta S_{t+h} &= g^S_t(Z_{t+h})h + Z^S_{t+h} \cdot \Delta b_{t+h}\,, \quad S_T = S \\
\Delta H^a_{t+h} &= g^a_t\,(Z_{t+h})\,h + Z^a_{t+h} \cdot \Delta b_{t+h}\,, \quad H^a_T = H.
\end{aligned}$$

Example

Assume that the price of the exogenous asset is given by

$$\Delta R_{t+h} = R_t(\mu h + \sigma \Delta b^1_{t+h}), \quad R_0 > 0$$

and agent a's preference functional is

$$U^a_t(X) = -\frac{1}{\gamma^a} \log \mathbb{E}\left[\exp(-\gamma^a X) \mid \mathcal{F}_t\right] \quad \text{for some} \quad \gamma^a > 0\,.$$

Then

$$U^a_t(X) = \mathbb{E}\left[X \mid \mathcal{F}_t\right] - f^a_t(\pi_t(X))h$$

for

$$f^a_t(z) = \frac{1}{h\gamma^a} \log \mathbb{E}\left[\exp(-\gamma^a z \cdot \Delta b_{t+h})\right]\,.$$

Random walks and BSΔEs

Neglect the random walks $b^{d+1}, \ldots, b^D$

and use the approximation

$$\frac{1}{h\gamma^a} \sum_{i=1}^{d} \log\cosh\left(\sqrt{h}\gamma^a z^i\right) \approx \frac{\gamma^a}{2} \sum_{i=1}^{d} (z^i)^2$$

Then the BSΔE of the last theorem yields ...

Random walks and BSΔEs

... the recursive algorithm

$$\begin{aligned} S_t &= \mathbb{E}\left[S_{t+1} \mid \mathcal{F}_t\right] - g_t^S h\,, \qquad & S_T = S \\ H_t^a &= \mathbb{E}\left[H_{t+1}^a \mid \mathcal{F}_t\right] - g_t^a h\,, \qquad & H_T^a = H^a\,, \end{aligned}$$

where

$$\begin{aligned} g_t^S &= \frac{1}{c^{RR}}\left[c^{RS}\mu S_t + \gamma\left(n\left\{c^{RR}c^{SS} - c^{RS}c^{RS}\right\} + c^{RA}c^{RR} - c^{SR}c^{RA}\right)\right] \\ g_t^a &= \frac{\gamma^a}{2}\left\|Z_{t+h}^a + \hat{\vartheta}_{t+h}^{a,1}Z_{t+h}^R + \hat{\vartheta}_{t+h}^{a,2}Z_{t+h}^S\right\|_2^2 - \hat{\vartheta}_{t+h}^{a,1}\mu R_t - \hat{\vartheta}_{t+h}^{a,2}g_t^S \\ \hat{\vartheta}_{t+h}^{a,1} &= \frac{\mu S_t}{\gamma^a c^{RR}} + \frac{c^{SR}c^{Sa} - c^{Ra}c^{SS}}{c^{RR}c^{SS} - c^{RS}c^{RS}} - \frac{c^{RS}}{c^{RR}}\frac{\gamma}{\gamma^a}\left(n + \frac{c^{RR}c^{SA} - c^{RS}c^{AR}}{c^{RR}c^{SS} - c^{RS}c^{RS}}\right) \\ \hat{\vartheta}_{t+h}^{a,2} &= n\frac{\gamma}{\gamma^a} + \frac{c^{RS}c^{Ra} - c^{Sa}c^{RR} - \frac{\gamma}{\gamma^a}\left(c^{RS}c^{RA} - c^{RR}c^{SA}\right)}{c^{RR}c^{SS} - c^{RS}c^{RS}} \end{aligned}$$

for $\gamma := \left(\sum_a (\gamma^a)^{-1}\right)^{-1}$,
$c^{RR} := Z_{t+h}^S \cdot Z_{t+h}^S, \quad c^{SR} := Z_{t+h}^S \cdot Z_{t+h}^R, \quad c^{SA} := Z_{t+h}^S \cdot \sum_a Z_{t+h}^a, \quad \ldots$

Example

Let B_t^R, B_t^S, B_t^a, $a \in \mathbb{A}$, be independent Brownian motions

$$dR_t = \mu R_t dt + \sigma R_t dB_t^R\,, \quad R_0 > 0$$

and suppose agent a's preference functional is

$$U_t^a(X) = -\frac{1}{\gamma^a} \log \mathbb{E}\left[\exp(-\gamma^a X) \mid \mathcal{F}_t\right] \quad \text{for some} \quad \gamma^a > 0\,.$$

Brownian motion and BSDEs

The BSDE corresponding to the above BSDE is

$$\begin{aligned} dS_t &= g_t^S dt + Z_t^S \cdot dB_t\,, \qquad S_T = S \\ dH_t^a &= g_t^a dt + Z_t^a \cdot dB_t\,, \qquad H_T^a = H^a\,, \end{aligned}$$

where

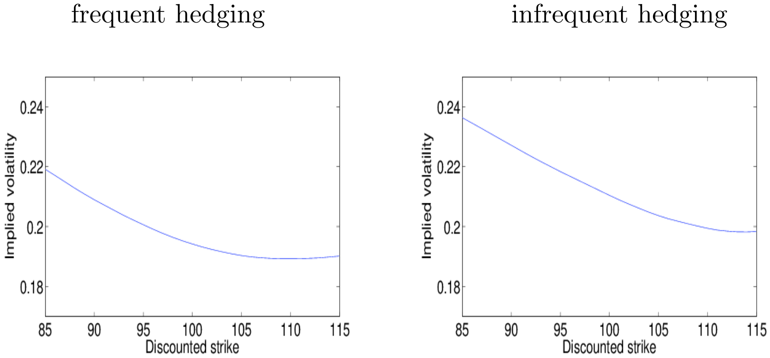
$$\begin{aligned} g_t^S &= \frac{1}{c^{RR}}\left[c^{RS}\mu S_t + \gamma\left(n\left\{c^{RR}c^{SS} - c^{RS}c^{RS}\right\} + c^{RA}c^{RR} - c^{SR}c^{RA}\right)\right] \\ g_t^a &= \frac{\gamma^a}{2}\left\|Z_t^a + \hat{\vartheta}_t^{a,1}Z_t^R + \hat{\vartheta}_t^{a,2}Z_t^S\right\|_2^2 - \hat{\vartheta}_t^{a,1}\mu R_t - \hat{\vartheta}_t^{a,2}g_t^S \\ \hat{\vartheta}_t^{a,1} &= \frac{\mu S_t}{\gamma^a c^{RR}} + \frac{c^{SR}c^{Sa} - c^{Ra}c^{SS}}{c^{RR}c^{SS} - c^{RS}c^{RS}} - \frac{c^{RS}}{c^{RR}}\frac{\gamma}{\gamma^a}\left(n + \frac{c^{RR}c^{SA} - c^{RS}c^{AR}}{c^{RR}c^{SS} - c^{RS}c^{RS}}\right) \\ \hat{\vartheta}_t^{a,2} &= n\frac{\gamma}{\gamma^a} + \frac{c^{RS}c^{Ra} - c^{Sa}c^{RR} - \frac{\gamma}{\gamma^a}\left(c^{RS}c^{RA} - c^{RR}c^{SA}\right)}{c^{RR}c^{SS} - c^{RS}c^{RS}} \end{aligned}$$

for

$$c^{RR} := Z_t^R \cdot Z_t^R, \quad c^{RS} := Z_t^R \cdot Z_t^S, \quad c^{RA} := Z_t^R \cdot \sum_a Z_t^a, \quad \ldots$$

Zero endowments and stochastic volatility

(I) No demand pressure

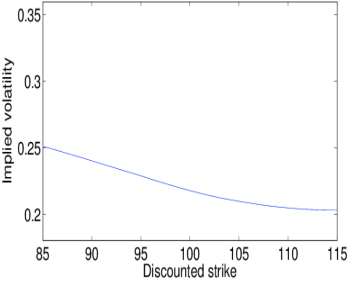


Zero endowments and stochastic volatility

(II) Positive demand pressure

frequent hedging

infrequent hedging



Zero endowments and stochastic volatility

(III) Positive demand pressure and short selling constraints