Equilibrium Pricing in Incomplete Markets under Translation Invariant Preferences

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Joint work with
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Price processes are martingales under $\mathbb{Q}$
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- **Harrison–Kreps (1979).** Martingales and arbitrage in multiperiod security markets.
- **Dalang–Morton–Willinger (1989).** Equivalent martingale measures and no-arbitrage in stochastic securities market models.
- **Delbaen (1992).** Representing martingale measures when asset prices are continuous and bounded.
- **Schachermayer (1993).** Martingale measures for discrete time processes with infinite horizon.
- **Delbaen–Schachermayer (1994).** A general version of the fundamental theorem of asset pricing.
Usual approach to derivatives pricing:

1. Model the underlying securities as a $J$-dimensional stochastic process $(R_t)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$

2. Price derivatives by $\mathbb{E}_Q[.]$ for some equivalent martingale measure $Q \sim \mathbb{P}$
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2. Price derivatives by $\mathbb{E}_Q[. ]$ for some equivalent martingale measure $Q \sim \mathbb{P}$

In complete markets: $Q$ is unique
- binomial tree models, Black–Scholes model ...

In incomplete markets: $Q$ is not unique
- trinomial tree models, GARCH-type models,
- stochastic volatility models, jump-diffusion models,
- Levy-process models, more general semimartingale models ...

Problem:

Choose a pricing measure $\hat{Q}$ among all equivalent martingale measures.

Some commonly used methods:

1. Parameterize $\hat{Q}_\theta, \theta \in \Theta$ and calibrate to market data of traded derivatives ... via $\frac{d\hat{Q}_\theta}{d\mathbb{P}}$ or without $\mathbb{P}$. E.g. build a stochastic vol model or local vol model directly under $\hat{Q}$.

2. Choose $\hat{Q}$ so that it minimizes some distance to $\mathbb{P}$, e.g. $L^p$-distance, relative entropy, $f$-divergence ...

3. Indifference pricing

4. ...
Our goal: derive $\hat{Q}$ from equilibrium considerations

Some motivating Examples

Horst and Müller (2007).
On the spanning property of risk bonds priced by equilibrium

Bakshi, Kapadia and Madan (2003).
Stock return characteristics, skew laws, and the differential pricing of individual equity options

Demand-based option pricing

Carmona, Fehr, Hinz and Porchet (2010).
Market design for emission trading schemes.
1 Model

2 Existence of equilibrium

3 Uniqueness of equilibrium

4 Random walks and BSΔEs

5 Brownian motion and BSDEs

6 Option pricing under demand pressure
Ingredients

- filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})\)
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- structured product in external supply \(n\) with final payoff \(S \in L_\infty(\mathcal{F}_T)\)
- a group of finitely many agents \(\mathbb{A}\)
- agent \(a \in \mathbb{A}\) is endowed with an uncertain payoff \(H^a = g^{a,R} R_T + g^{a,S} S_T + G^a\)
Model

Ingredients

- filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$
- money market account with $r \equiv 0$
- exogenous asset $(R_t)_{t=0}^T$ satisfying (NA)
- structured product in external supply $n$ with final payoff $S \in L^\infty(\mathcal{F}_T)$
- a group of finitely many agents $A$
- agent $a \in A$ is endowed with an uncertain payoff $H^a = g^{a,R} R_T + g^{a,S} S_T + G^a$
- at time $t$ agent $a$ invests to optimize a preference functional $U^a_t : L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_t)$
We assume $U_t^a$ has the following properties:

(N) Normalization $U_t^a(0) = 0$
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(C) **$\mathcal{F}_t$-Concavity** \[ U_t^a(\lambda X + (1 - \lambda)Y) \geq \lambda U_t^a(X) + (1 - \lambda)U_t^a(Y) \text{ for all } X, Y \in L^\infty(\mathcal{F}_T) \text{ and } \lambda \in L^\infty(\mathcal{F}_t) \text{ such that } 0 \leq \lambda \leq 1 \]
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(T) Translation property
\[ U^a_t(X + Y) = U^a_t(X) + Y \] for all $X \in L^\infty(\mathcal{F}_T)$ and $Y \in L^\infty(\mathcal{F}_t)$
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(TC) Time-consistency
$$U_{t+1}^a(X) \geq U_{t+1}^a(Y)$$ implies $$U_t^a(X) \geq U_t^a(Y)$$
$$\Leftrightarrow \quad U_t^a(X) = U_t^a(U_{t+1}^a(X))$$
Related to coherent and convex risk measures

  Coherent measures of risk.

- **Föllmer–Schied (2002).**
  Convex measures of risk and trading constraints

- **Frittelli–Rosazza Gianin (2002).**
  Putting order in risk measures.
Examples

1) $U^a_t(X) = -\frac{1}{\gamma} \log \mathbb{E} [e^{-\gamma X} \mid \mathcal{F}_t]$

2) $U^a_t(X) = \mathbb{E} [X \mid \mathcal{F}_t] - \lambda \mathbb{E} [(X - \mathbb{E} [X \mid \mathcal{F}_t])^2 \mid \mathcal{F}_t]$

3) $U^a_t(X) = (1 - \lambda) \mathbb{E} [X \mid \mathcal{F}_t] - \lambda \rho_t(X)$

where $\rho_t$ is a conditional convex risk measure
An equilibrium of plans, prices and price expectations à la Radner (1972) consists of

- an adapted process \((S_t)_{t=0}^T\) with \(S_T = S\)
- trading strategies \((\hat{\vartheta}_t^a)_{t=1}^T\)

such that the following hold:

(i) individual optimality

\[
U^a_t \left( H^a + \sum_{s=t+1}^T \hat{\vartheta}_{s,1}^a \Delta R_s + \hat{\vartheta}_{s,2}^a \Delta S_s \right) \\
\geq U^a_t \left( H^a + \sum_{s=t+1}^T \vartheta_{s,1}^a \Delta R_s + \vartheta_{s,2}^a \Delta S_s \right)
\]

for every \(t\) and all possible strategies \((\vartheta_s^a)\)

(ii) market clearing

\[
\sum_{a \in A} \hat{\vartheta}_t^{a,2} = n
\]
Hart (1975) On the optimality of equilibrium when the market structure is incomplete:

In general, a Radner equilibrium does not exist, and if there is one, it is not unique.
One-step representative agents

Set $H^a_T = H^a$ and

$$H^a_{t+1} = U^a_{t+1} \left( H^a + \sum_{s=t+2}^T \hat{\varphi}^{a,1}_s \Delta R_s + \hat{\varphi}^{a,2}_s \Delta S_s \right)$$

the true representative agent would be

$$\hat{u}_t(x) = \text{ess sup} \sum_{a \in \mathbb{A}} U^a_t \left( H^a_{t+1} + \varphi^{a,1}_t \Delta R_{t+1} + \varphi^{a,2}_t \Delta S_{t+1} \right)$$

$$\sum_{a \in \mathbb{A}} \varphi^{a,2} = x$$
Existence

One-step representative agents

Set $H_T = H$ and

$$H^a_t = U^a_{t+1} \left( H^a + \sum_{s=t+2}^{T} \hat{\vartheta}_{s,1}^a \Delta R_s + \hat{\vartheta}_{s,2}^a \Delta S_s \right)$$

the true representative agent would be

$$\hat{u}_t(x) = \text{ess sup} \sum_{a \in A} U^a_t \left( H^a_{t+1} + \vartheta_{a,1}^a \Delta R_{t+1} + \vartheta_{a,2}^a \Delta S_{t+1} \right)$$

$$\sum_{a \in A} \vartheta_{a,2}^a = x$$

But $S_t$ is not known. So define

$$\hat{u}_t(x) = \text{ess sup} \sum_{a} U^a_t \left( H^a_{t+1} + \vartheta_{a,1}^a \Delta R_{t+1} + \vartheta_{a,2}^a S_{t+1} \right)$$

$$\sum_{a \in A} \vartheta_{a,2}^a = x$$

$\hat{u}_t$ is $\mathcal{F}_t$-concave
Convex dual characterization of equilibrium

Theorem A bounded, adapted process \((S_t)^T_{t=0}\) satisfying \(S_T = S\) together with trading strategies \((\hat{\vartheta}_t^a)^T_{t=1}, a \in A\), form an equilibrium \(\iff\) for all \(t\):

(i) \(S_t \in \partial \hat{u}_t(n)\)

(ii) \(\sum_{a \in A} U_t^a (H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,1} \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,2} S_{t+1}) = \hat{u}_t(n)\)

(iii) \(\sum_{a \in A} \hat{\vartheta}_{t+1}^{a,2} = n\)
Existence

Assumption (A)

For all $t = 0, \ldots, T - 1$, $V^a \in L^\infty(\mathcal{F}_{t+1})$, $W \in L^\infty(\mathcal{F}_{t+1})$, there exist $\hat{\vartheta}^a_{t+1} \in L^\infty(\mathcal{F}_t)^2$, $a \in \mathbb{A}$, such that

$$\sum_{a \in \mathbb{A}} \hat{\vartheta}^a_{t+1} = 0$$

and

$$\sum_{a \in \mathbb{A}} U^a_t \left( V^a + \hat{\vartheta}^a_{t+1} \Delta R_{t+1} + \hat{\vartheta}^a_{t+1} W \right) = \text{ess sup} \sum_{a \in \mathbb{A}} U^a_t \left( V^a + \vartheta^a_{t+1} \Delta R_{t+1} + \vartheta^a_{t+1} W \right).$$

Lemma Under assumption (A) an equilibrium exists
**Definition**

$U_0^a$ is sensitive to large losses if

$$\lim_{\lambda \to \infty} U_0^a(\lambda X) = -\infty$$

for all $X \in L^\infty(\mathcal{F}_T)$ such that $\mathbb{P}[X < 0] > 0$.

**Theorem**

If all $U_0^a$ are sensitive to large losses, then condition (A) is satisfied and an equilibrium exists.

**Remark**

The theorem also works with convex trading constraints.
Existence

**Proposition**

If the market is in equilibrium and at least one agent has strictly monotone preferences and open trading constraints, then there exists a probability measure $Q \sim P$ such that

$$R_t = \mathbb{E}_Q [R_T | \mathcal{F}_t] \quad \text{and} \quad S_t = \mathbb{E}_Q [R_T | \mathcal{F}_t].$$
Differentiable preferences

We say $U_t^a$ satisfies the differentiability condition \((D)\) if for all $X, Y \in L^\infty(\mathcal{F}_{t+1})$, there exists $Z \in L^1(\mathcal{F}_{t+1})$ such that

$$\lim_{k \to \infty} k \left( U_t^a \left( \frac{X + Y}{k} \right) - U_t^a(X) \right) = \mathbb{E}[YZ | \mathcal{F}_t].$$

If such a $Z$ exists, it has to be unique, and we denote it by $\nabla U_t^a(X)$. 
**Differentiable preferences**

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If such a $Z$ exists, it has to be unique, and we denote it by $\nabla U_t^a(X)$.

**Theorem** If at least one $U_t^a$ satisfies (D), then there can exist at most one equilibrium price process $(S_t)_{t=0}^T$, and if the market is in equilibrium, then

$$\frac{dQ_t^a}{d\mathbb{P}} := \nabla U_t^a \left( H^a + \sum_{s=1}^T \hat{\vartheta}_s^{a,1} \Delta R_s + \hat{\vartheta}_s^{a,2} \Delta S_s \right)$$

defines a pricing measure.
Fix $h > 0$ and $N \in \mathbb{N}$

Denote $T = \{0, h, \ldots, T = Nh\}$

$b^1_t, \ldots, b^d_t$ $d$ independent random walks with $P[\Delta b^i_t \pm h = \pm \sqrt{h}] = 1/2$

$b^{d+1}_t, \ldots, b^D_t$ $2^d - (d + 1)$ random walks orthogonal to $b^1_t, \ldots, b^d_t$

Every $X \in L^\infty(\mathcal{F}_{t+h})$ can be represented as

$$X = \mathbb{E}[X | \mathcal{F}_t] + \pi_t(X) \cdot \Delta b_{t+h}$$

for

$$\pi_t(X) \cdot \Delta b_{t+h} = \sum_{i=1}^D \pi^i_t(X) \Delta b^i_{t+h} \quad \text{and} \quad \pi^i_t(X) = \frac{1}{h} \mathbb{E}[X \Delta b^i_{t+h} | \mathcal{F}_t].$$
\[
U^a_t(X) = U^a_t(\mathbb{E}[X|\mathcal{F}_t] + \pi_t(X) \cdot \Delta b_{t+h}) = \mathbb{E}[X|\mathcal{F}_t] - f^a_t(\pi_t(X))h
\]

for the $\mathcal{F}_t$-convex function $f^a_t : L^\infty(\mathcal{F}_t)^D \to L^\infty(\mathcal{F}_t)$ given by
\[
f^a_t(z) := -\frac{1}{h}U^a_t(z \cdot \Delta b_{t+h}).
\]

Assume condition (A) is satisfied and all $U^a_t$ satisfy the differentiability condition (D).

Then there exists $\nabla f^a_t(z) \in L^\infty(\mathcal{F}_t)^D$ such that
\[
\lim_{k \to \infty} k \left( f^a_t(z + z'/k) - f^a_t(z) \right) = z' \cdot \nabla f^a_t(z)
\]
For given $R_{t+h}, S_{t+h}, H^a_{t+h}$ denote

$$
Z^R_{t+h} := \pi_t(R_{t+h})
$$

$$
Z^S_{t+h} := \pi_t(S_{t+h})
$$

$$
Z^a_{t+h} := \pi_t(H^a_{t+h})
$$

$$
Z_{t+h} = (Z^R_{t+h}, Z^S_{t+h}, Z^a_{t+h}, a \in A).
$$

and define the function $f_t : L^\infty(\mathcal{F}_t)^{3+|A|}D \to L^\infty(\mathcal{F}_t)$ by

$$
f_t(v, Z_{t+h}) = \text{ess inf}_{\vartheta^a \in L(\mathcal{F}_t)^2} \sum_{a \in A} f^a_t \left( \frac{v}{|A|} + Z^a_{t+h} + \vartheta^a,1_{t+h} Z^R_{t+h} + \vartheta^a,2_{t+h} Z^S_{t+h} \right)
$$

$$
- \vartheta^a,1_{t+h} \frac{\mathbb{E}[\Delta R_{t+h} \mid \mathcal{F}_t]}{h}.
$$
Set

\[
\begin{align*}
    g_t^S(Z_{t+h}) & := Z_{t+h}^S \cdot \nabla^v f_t(nZ_{t+h}^S, Z_{t+h}) \\
    g_t^a(Z_{t+h}) & := f_t^a \left( Z_{t+h}^a + \hat{\varphi}_{t+h}^{a,1} Z_{t+h}^R + \hat{\varphi}_{t+h}^{a,2} Z_{t+h}^S \right) \\
    & \quad - \hat{\varphi}_{t+h}^{a,1} \frac{1}{h} \mathbb{E} [\Delta R_{t+h} \mid \mathcal{F}_t] - \hat{\varphi}_{t+h}^{a,2} g_t^S(Z_{t+h}).
\end{align*}
\]
The processes \((S_t)\) and \((H^a_t)\) satisfy the following coupled system of BS\(\Delta\)Es

\[
\begin{align*}
\Delta S_{t+h} &= g_t^S (Z_{t+h}) h + Z_{t+h}^S \cdot \Delta b_{t+h}, \\
\Delta H_{t+h}^a &= g_t^a (Z_{t+h}) h + Z_{t+h}^a \cdot \Delta b_{t+h},
\end{align*}
\]

\(S_T = S\) \quad \text{and} \quad \(H_T^a = H\).
Example

Assume that the price of the exogenous asset is given by

$$\Delta R_{t+h} = R_t(\mu h + \sigma \Delta b^1_{t+h}), \quad R_0 > 0$$

and agent a’s preference functional is

$$U^a_t(X) = -\frac{1}{\gamma^a} \log \mathbb{E} [\exp(-\gamma^a X) \mid \mathcal{F}_t] \quad \text{for some} \quad \gamma^a > 0.$$

Then

$$U^a_t(X) = \mathbb{E} [X \mid \mathcal{F}_t] - f^a_t(\pi_t(X)) h$$

for

$$f^a_t(z) = \frac{1}{h \gamma^a} \log \mathbb{E} [\exp(-\gamma^a z \cdot \Delta b_{t+h})].$$
Neglect the random walks $b^{d+1}, \ldots, b^D$

and use the approximation

$$\frac{1}{\hbar \gamma^a} \sum_{i=1}^{d} \log \cosh \left( \sqrt{\hbar \gamma^a} z^i \right) \approx \frac{\gamma^a}{2} \sum_{i=1}^{d} (z^i)^2$$

Then the BSΔEs of the last theorem yields ...
Random walks and BSΔEs

... the recursive algorithm

\[ S_t = \mathbb{E}[S_{t+1} \mid \mathcal{F}_t] - g_t^S h, \quad S_T = S \]
\[ H_t^a = \mathbb{E}[H_{t+1}^a \mid \mathcal{F}_t] - g_t^a h, \quad H_T^a = H^a, \]

where

\[
g_t^S = \frac{1}{c^{RR}} \left[ c^{RS} \mu S_t + \gamma \left( n \left\{ c^{RR} c^{SS} - c^{RS} c^{RS} \right\} + c^{RA} c^{RR} - c^{SR} c^{RA} \right) \right]
\]
\[
g_t^a = \frac{\gamma^a}{2} \left\| Z^a_{t+h} + \hat{\theta}^a_{t+h} Z^R_{t+h} + \hat{\theta}^a_{t+h} Z^S_{t+h} \right\|^2_2 - \hat{\theta}^a_{t+h} \mu R_t - \hat{\theta}^a_{t+h} g_t^S
\]
\[
\hat{\theta}^a_{t+h} = \frac{\mu S_t}{\gamma^a c^{RR}} + \frac{c^{SR} c^{Sa} - c^{Ra} c^{SS}}{c^{RR} c^{SS} - c^{RS} c^{RS}} - \frac{c^{RS}}{c^{RR}} \frac{\gamma^a}{\gamma^a} \left( n + \frac{c^{RR} c^{SA} - c^{RS} c^{AR}}{c^{RR} c^{SS} - c^{RS} c^{RS}} \right)
\]
\[
\hat{\theta}^a_{t+h} = n \frac{\gamma}{\gamma^a} + \frac{c^{RS} c^{Ra} - c^{Sa} c^{RR} - \frac{\gamma}{\gamma^a} \left( c^{RS} c^{RA} - c^{RR} c^{SA} \right)}{c^{RR} c^{SS} - c^{RS} c^{RS}}
\]

for \( \gamma := (\sum a (\gamma^a)^{-1})^{-1} \),

\( c^{RR} := Z_t^S \cdot Z_t^S, \quad c^{SR} := Z_t^S \cdot Z_t^R, \quad c^{SA} := Z_t^S \cdot \sum a Z_t^a \).
Example

Let $B^R_t$, $B^S_t$, $B^a_t$, $a \in A$, be independent Brownian motions

$$dR_t = \mu R_t dt + \sigma R_t dB^R_t, \quad R_0 > 0$$

and suppose agent a’s preference functional is

$$U^a_t(X) = -\frac{1}{\gamma^a} \log \mathbb{E}[\exp(-\gamma^a X) \mid \mathcal{F}_t] \quad \text{for some} \quad \gamma^a > 0.$$
The BSDE corresponding to the above BSDE is

\[
\begin{align*}
    dS_t &= g_t^S \, dt + Z_t^S \cdot dB_t, \quad S_T = S \\
    dH_t^a &= g_t^a \, dt + Z_t^a \cdot dB_t, \quad H_T^a = H^a,
\end{align*}
\]

where

\[
\begin{align*}
    g_t^S &= \frac{1}{c_{RR}} \left[ c_{RS} \mu_S t + \gamma \left( n \left\{ c_{RR} c_{SS} - c_{RS} c_{RS} \right\} + c_{RA} c_{RR} - c_{SR} c_{RA} \right) \right] \\
    g_t^a &= \frac{\gamma^a}{2} \left\| Z_t^a + \hat{\gamma}_t^a,1 Z_t^R + \hat{\gamma}_t^a,2 Z_t^S \right\|^2_2 - \hat{\gamma}_t^a,1 \mu R_t - \hat{\gamma}_t^a,2 g_t^S \\
    \hat{\gamma}_t^a,1 &= \frac{\mu_S t}{\gamma^a c_{RR}} + \frac{c_{SR} c_{Sa} - c_{Ra} c_{SS}}{c_{RR} c_{SS} - c_{RS} c_{RS}} - \frac{c_{RS}}{c_{RR}} \frac{\gamma}{\gamma^a} \left( n + \frac{c_{RR} c_{SA} - c_{RS} c_{AR}}{c_{RR} c_{SS} - c_{RS} c_{RS}} \right) \\
    \hat{\gamma}_t^a,2 &= n \frac{\gamma}{\gamma^a} + \frac{c_{RS} c_{Ra} - c_{Sa} c_{RR} - \frac{\gamma}{\gamma^a} \left( c_{RS} c_{RA} - c_{RR} c_{SA} \right)}{c_{RR} c_{SS} - c_{RS} c_{RS}} \\
\end{align*}
\]

for

\[
\begin{align*}
    c_{RR} &= Z_t^R \cdot Z_t^R, \quad c_{RS} = Z_t^R \cdot Z_t^S, \quad c_{RA} = Z_t^R \cdot \sum_a Z_t^a, \quad \ldots
\end{align*}
\]
Zero endowments and stochastic volatility

(I) No demand pressure

frequent hedging     infrequent hedging
Zero endowments and stochastic volatility

(II) Positive demand pressure

frequent hedging  infrequent hedging
Zero endowments and stochastic volatility

(III) Positive demand pressure and short selling constraints