B Banach space, T operator on B

$x \in B$, $(x, Tx, T^2x, \dots)$ orbit of x.

x cyclic if $\overline{\text{span}\{T^n x\}} = B$

Invariant Subspace Problem for B:
Is there a T s.t. every $x \neq 0$
is cyclic?

ISP does not ask how vectors are cyclic or ask about other properties of orbits. By asking such questions we are led to new problems in Operator Theory, Geometry of Banach spaces, Approximation Theory, Classical Analysis etc.

x is <u>hypercyclic</u> if $\overline{\{T^n x \mid n \geq 0\}} = B$

x is <u>supercyclic</u> if $\left\{ \frac{T^n x}{\|T^n x\|} \mid n \geq 0 \right\}$ is dense on the unit sphere

The orbit of x is <u>hyperful</u> if $\overline{\operatorname{span}\{T^{n_j} x \mid j \geq 0\}} = B$ for every subsequence (n_j).

How many elements of $\{T^n x \mid n \geq 0\}$ can be removed so that the remaining ones still span B?

If $\{T^{n_j} x \mid j \geq 0\}$ does not span B, what does it span?

EXAMPLES

$B = C(0,1)$ or $L_2(0,1)$

T : Multiplication by t.

Orbit of 1 is $(1, t, t^2, \dots)$

By Weierstrass, 1 is cyclic.

By Müntz-Szasz $\{t^{n_j} \mid j \geq 0\}$ spans $L_2(0,1)$ iff $\sum \frac{1}{n_j} = \infty$.

By Gurariy-Matsaev: Iff for some $\lambda > 1$, $n_{j+1} \geq \lambda n_j$ for all j, then (t^{n_j}) is a basic sequence in $C(0,1)$ and $\overline{\operatorname{span}\{t^{n_j}\}}$ is isomorphic to c_0.

The structure of $\overline{\operatorname{span}\{t^{n_j}\}}$ is unknown for many cases.

T integration operator on $L_2(0,1)$.

$$(Tf)(t) = \int_0^t f(u)\,du$$

Th: f is cyclic iff $f \not\equiv 0$ on every interval $[0, \varepsilon]$.

This generalizes Weierstrass.

Does Müntz-Szasz generalize?

JOEL BEIL

Th. If, on some interval $[0, \varepsilon]$ $f(t) \geq C \cdot t^j$ for some j, then $\forall N$

1) $\dfrac{a^N}{N!} \leq \| T^N f \| \leq \dfrac{b^N}{N!}$ for some a and b

2) $\left\| \dfrac{T^M f}{\|T^M f\|} - \dfrac{T^N f}{\|T^N f\|} \right\| \geq \delta$ if $M \geq (1+\gamma) N$

3) $\left\| \dfrac{T^{N+1} f}{\|T^{N+1} f\|} - \dfrac{T^N f}{\|T^N f\|} \right\| \sim \dfrac{1}{N}$.

With 2) we can generalize parts of Gurariy - Matsaev.

Orbits, where $f(t)>0$ on some interval $[0,\varepsilon]$, will have 1) and 2). Do they have 3) ?

There are orbits, where $T^N f$ changes signs infinitely many times in each interval $[0,\varepsilon]$, for every N.

Some $f(t)$ similar to $\sin\frac{1}{t}$ will have this property

Q: Do these orbits have similar properties as the others, or is there a new world to discover?

Diagonal Operators on ℓ_2 (and other sequence spaces)

$$T: \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \ddots \end{pmatrix}$$

MARCO KITEV

Th. If $|\lambda_i| \neq |\lambda_j|$ for $i \neq j$, then $x = (x_1, x_2, \dots)$ is cyclic, iff $x_i \neq 0$ for every i.

<u>Proof</u>: Assume M invariant for T. Orthogonal projection P from ℓ_2 onto M commutes with T. $TP = PT$

Now $Te_j = \lambda_j e_j$ if e_j is j:th natural basis vector in ℓ_2. So, $TPe_j = PTe_j =$
$= \lambda_j Pe_j$. So, either $Pe_j = e_j$ or $Pe_j = 0$. So, M is spanned by a subset of the e_j:s.

How dense subsequences of $(T^n x)$ do we need to span the whole space if x is cyclic? If $\lambda_i > 0$ for all i, then T^k will also satisfy the assumptions of the Theorem, so

$$\overline{\text{span}}\,\{T^{kn}x\} = \ell_2 .$$

If $\lambda_1 > \lambda_2 > \lambda_3 > \cdots > 0$ then

$$\frac{T^{n_j} x}{\| \quad \|} = \frac{(\lambda_1^{n_j} x_1 , \lambda_2^{n_j} x_2 , \cdots)}{\| \qquad \|} \to (\pm 1, 0, 0, \cdots)$$

From this we easily deduce that $\overline{\operatorname{span}\{T^{n_j} x \mid j \geq 0\}} = \ell_2$ for every subsequence (n_j), so the orbit of x is hyperful.

If $0 < \lambda_1 < \lambda_2 < \lambda_3 \cdots\cdots < a$ then

$$\frac{T^{n} x}{\| \quad \|} = \frac{(\lambda_1^{n} x_1 , \lambda_2^{n} x_2 , \cdots)}{\| \qquad \|} \to 0 \text{ weakly.}$$

Thus $\frac{T^n x}{\| \quad \|}$ has a subsequence which is a basic sequence and so the orbit cannot be hyperful.

With this we can prove

<u>Theorem</u>: If T is a diagonal operator on ℓ_2, then either

A) All cyclic vectors have hyperful orbits

or

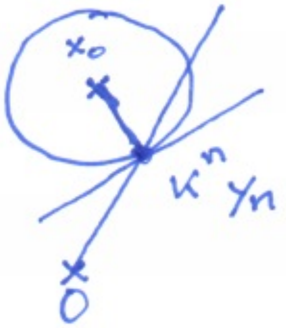
B) None of the cyclic vectors has a hyperful orbit.

If B) happens how dense subsequences do we need to span ℓ_2?

How much of these results carry over to other sequence spaces like ℓ_1 or c_0?

<u>Theorem</u>: Let K be compact operator on Hilbert space, $\overline{R(K)} = H$, $\sigma(K) = \{0\}$. Then $\exists\, v$ s.t. Kv is non-cyclic for every T commuting with K.

<u>Proof</u>. Let $\|x_0\| = 1$, $0 < \varepsilon < \frac{1}{10}$, and let y_n be the unique y of minimal norm s.t. $\|K^n y - x_0\| \le \varepsilon$



Then 1) $r_n \perp y_n \Rightarrow K^n r_n \perp x_0 - K^n y_n$

2) $\exists (n_\nu)$ s.t. $\dfrac{\|y_{n_\nu}\|}{\|y_{n_\nu+1}\|} \to 0$

and $K^{n_\nu} y_{n_\nu} \xrightarrow{w} v$ and $K^{n_\nu+1} y_{n_\nu+1} \xrightarrow{w} z$, $x_0 - z \neq 0$

Take integer m. We have

$$T^m y_{n_\nu} = \alpha_\nu y_{n_\nu+1} + r_{n_\nu+1}, \quad \alpha_\nu \to 0$$

$$T^m K K^{n_\nu} y_{n_\nu} = \alpha_\nu K^{n_\nu+1} y_{n_\nu+1} + K^{n_\nu+1} r_{n_\nu+1}$$

$$\langle T^m K K^{n_\nu} y_{n_\nu}, x_0 - K^{n_\nu+1} y_{n_\nu+1} \rangle = \alpha_\nu \langle K^{n_\nu+1} y_{n_\nu+1}, x_0 - K^{n_\nu+1} y_{n_\nu+1} \rangle + \langle K^{n_\nu+1} r_{n_\nu+1}, x_0 - K^{n_\nu+1} y_{n_\nu+1} \rangle$$

Let $\nu \to \infty$

$\langle T^m K v, x_0 - z \rangle = 0 + 0 = 0$, so Kv is non-cyclic for T.