On Quasi-convex Risk Measures

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Outline

- Prologue on Risk Measures and acceptance sets
- Motivation for Quasi-convex Risk Measures
- The dual representation of quasi-convex dynamic risk measures - a Vector Space approach
- Conditional evenly convex sets
- The complete duality for the class of evenly quasi-convex dynamic risk measures -a Module approach
- Risk measures on distributions, generated by acceptance families

Coherent Risk Measures

Artzner, Delbaen, Eber and Heath (1999)

This theory was developed to assess the riskiness of financial positions and specify a method to compute the capital requirement to be reported to the regulatory agency.

The key idea was to provide a set of axioms that any reasonable risk measure should have.

Definition

A map $\rho : L^\infty \to \mathbb{R}$ is a coherent risk measure if the following properties hold: for any $X, Y \in L^\infty$

- decreasing monotonicity: $X \leq Y$ a.s. $\Rightarrow \rho(X) \geq \rho(Y)$
- cash additivity: $\rho(X + m) = \rho(X) - m, \quad \forall\ m \in \mathbb{R}$
- positive homogeneity : $\rho(\lambda X) = \lambda \rho(X), \quad \forall\ \lambda \geq 0, \quad \lambda \in \mathbb{R}$
- subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

Convex Risk Measures

Föllmer and Schied (2002) and F. and Rosazza (2002)

Later, the concept of coherent risk measure was extended by relaxing the subadditivity and positive homogeneity conditions in favour of the weaker **convexity** requirement, which allows to *control the risk* of a convex combination by the combination of each single risk:

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y),\ \forall\ \lambda \in [0,1], \quad \lambda \in \mathbb{R}.$$

Definition

A map $\rho : L^\infty \to \mathbb{R}$ is a convex risk measure if satisfies the following conditions:

- decreasing monotonicity
- cash additivity
- convexity
- normalization: $\rho(0) = 0$

Relation with Acceptance Sets

- Given a set $\mathcal{A} \subset L^\infty$ of "acceptable positions", it is possible to associate a *cash additive* map $\rho_\mathcal{A}$

$$\rho_\mathcal{A}(X) := \inf \{ m \in \mathbb{R} \mid m + X \in \mathcal{A} \} .$$

- Viceversa, any *cash additive* map $\rho : L^\infty \to \mathbb{R}$ can be written as the minimal capital requirement:

$$\rho(X) := \inf \{ m \in \mathbb{R} \mid m + X \in \mathcal{A}_\rho \}$$

  where the set
$$\mathcal{A}_\rho = \{ X \in L^\infty : \rho(X) \leq 0 \} .$$
  is called the *acceptance set* of ρ.

- The properties of a risk measure can be deduced by those of the acceptance set and viceversa.

Cash-subadditive Risk Measures

El Karoui and Ravanelli (2009)

In a dynamic framework (with stochastic interest rate), El Karoui and Ravanelli suggested that cash additivity should be replaced by:

Definition

Cash-subadditive property:

$$\rho(X - m) \leq \rho(X) + m, \quad \forall\ m \in \mathbb{R}_+.$$

If m \$ are subtracted form a future position X then the present capital requirement $\rho(X - m)$ should not increase more than m \$.

Time consistency and convexity ... imply entropic risk measure

Recall the following result by Kupper and Schachermayer (2009):

- The only law invariant relevant convex and time consistent dynamic risk measure is the entropic risk measure.

This also suggest - specially in the dynamic case - the enlargement of the class of convex risk measures, provided we maintain the principle that diversification should not increase the risk .

Quasiconvex Risk Measures

Cerreia-Vioglio, Maccheroni, Marinacci, Montrucchio (2011)

These authors underlined that the concept:

"diversification should not increase the risk"

is **exactly** expressed by the **quasi-convexity** requirement:

$$\rho(\lambda X + (1-\lambda)Y) \leq \max\{\rho(X); \rho(Y)\}, \ \forall \ \lambda \in [0,1] \ \text{ and } \lambda \in \mathbb{R},$$

or equivalently:

the lower level sets $\{X \in L^\infty : \rho(X) \leq c\}, \ \forall c \in \mathbb{R},$ *are convex.*

Fact

Quasiconvexity and cash additivity $\Rightarrow$ *convexity.*
This is not true for quasiconvexity and cash-subadditivity
This is not true for ρ *defined on distributions*

Convexity, lsc and "cash additivity" are incompatible properties for maps on P(R)

Many risk measures adopted in practice are law invariant, i.e.:

$$X \sim_{\mathcal{D}} Y \Rightarrow \rho(X) = \rho(Y).$$

In this case one may reformulate the theory considering maps $\rho : \mathcal{P}(\mathbb{R}) \to \overline{\mathbb{R}}$ defined on the convex set of distributions $\mathcal{P}(\mathbb{R})$. However, we shall see that

- there are no convex lsc "cash-additive" maps $\rho : \mathcal{P}(\mathbb{R}) \to \overline{\mathbb{R}}$ (except $\rho = \infty$).
- There are plenty of quasi-convex lsc "cash-additive" maps $\rho : \mathcal{P}(\mathbb{R}) \to \overline{\mathbb{R}}$.

Acceptability Indices (Cherny - Madan (2009))

Definition

A map $\Phi : L^{\infty} \to [0, \infty]$ satisfying Quasiconcavity, MON ($\uparrow$), Scale Invariance and the Fatou property is called Acceptability Index.

Quoting (Cherny-Madan (2009)):

- *For a* **risk measure**, *all the positions are split in two classes: acceptable and not acceptable.*
- *In contrast, for an* **acceptability index** *we have a whole continuum of degrees of acceptability defined by the system* $(\mathcal{A}^m)$, $m \in \mathbb{R}$, *and the index* Φ *measures the degree of acceptability of a trade.*

Representation in terms of acceptance sets

- If $\mathcal{A}$ is an acceptance set of random variables, then the map $\rho_{\mathcal{A}}$

$$\begin{aligned}\rho_{\mathcal{A}}(X) &\triangleq \inf\{m \in \mathbb{R} \mid X + m \in \mathcal{A}\}\\ &= \inf\{m \in \mathbb{R} \mid X \in \{\mathcal{A} - m\}\}\\ &= -\sup\{m \in \mathbb{R} \mid X \in \mathcal{A}^m := \mathcal{A} + m\}\end{aligned}$$

  is *cash additive.*

- But if we abandon cash additivity there is no reason why we should insist on having:

$$\mathcal{A}^m := \mathcal{A} + m$$

Acceptability Indices and associated acceptability family

Drapeau Kupper (2011)

Let $(\mathcal{A}^m)$, $m \in \mathbb{R}$, be a collection of subsets $\mathcal{A}^m \subseteq L^\infty$ such that

1. $\mathcal{A}^m$ is convex, for any m
2. $\mathcal{A}^m \downarrow$ with respect to m
3. $\mathcal{A}^m$ is monotone $(Y \geq X \in \mathcal{A}^m \Rightarrow Y \in \mathcal{A}^m)$, for any m

Then

$$\Phi(X) := \sup\{m \in \mathbb{R} \,|X \in \mathcal{A}^m\}$$

is MON ($\uparrow$) and Quasi-Concave.
Viceversa, to any MON ($\uparrow$) and Quasi-Concave map $\Phi : L^\infty \to \mathbb{R}$ we may associate the acceptance set at level m:

$$\mathcal{A}^m_\Phi := \{X \in L^\infty | \Phi(X) \leq m\}$$

and $(\mathcal{A}^m_\Phi)$ is an acceptability family.

Example of quasi-convex risk measures based on acceptance sets

- We will show that the above simple construction is very useful, as it can be used in practice to build several risk measures (as well as performance indices) based on given reference families of acceptance sets.

- We will generalize the notion of the $V@R$ by considering a risk prudent agent who is willing to accept greater losses only with smaller probabilities. To this end we introduce in the definition of $V@R$ a function Λ that describes - via the corresponding acceptance sets - the balance between the amount of the loss and its probability.

Dual representation of (static) quasi-convex cash-subadditive risk measures

Theorem (Cerreia-Maccheroni-Marinacci-Montrucchio, 2011)

A function $\rho : L^\infty \to \overline{\mathbb{R}}$ is QCO cash-subadditive MON (↓) if and only if

$$\begin{aligned}\rho(X) &= \max_{Q \in ba_+(1)} R(E_Q[-X], Q),\\ R(m, Q) &= \inf\{\rho(\xi) \mid \xi \in L^\infty \text{ and } E_Q[-\xi] = m\}\end{aligned}$$

where $R : \mathbb{R} \times ba_+(1) \to \overline{\mathbb{R}}$ and $R(m, Q)$ is the reserve amount required today, under the scenario Q, to cover an expected loss m in the future.

This result follows from well known quasi-convex duality: Penot-Volle 90, Volle 98.

- It is well known that the representation results for risk measures may be used in decision theory for the robust approach to model uncertainty

Dual representation

Under continuity assumptions, in the convex case:

$$\rho(X) = \sup_{Q \in \mathcal{P}} \{E_Q[-X] - \alpha(Q)\}$$

in the quasi-convex case

$$\rho(X) = \sup_{Q \in \mathcal{P}} \{R(E_Q[-X], Q)\}$$

where $\mathcal{P} := \{Q \text{ probability s.t. } Q \ll P\}$.

- Cautious approach:

$$R\left(E_Q\left[-X\right], Q\right) \geq E_Q\left[-X\right] - \rho^*(-Q).$$

- Notice that a lsc convex ρ may be represented by many penalty functions α, but only the "minimal" penalty $\alpha = \rho^*$ is lsc and convex.

Complete duality

In other terms, the Fenchel Coniugacy is complete:

- there is one to one correspondence in the class of proper convex lsc functions

$$\rho \longleftrightarrow \rho^*$$

- Which is the corresponding complete duality in the quasi-convex case?
- The answer for real valued maps $\rho : E \to \overline{\mathbb{R}}$ is known: it is essentially the class of evenly quasi convex functions.
- We will address this problem in the conditional setting.

Evenly convex sets

Let E be a topological vector space

Definition

(Fenchel, 1952) A set $C \subseteq E$ is Evenly Convex if it is the intersection of open half spaces or equivalently if for every $y \notin C$ there exists a continuous linear functional μ such that

$$\mu(y) < \mu(\xi) \quad \forall \xi \in C.$$

- Both open convex sets and closed convex sets are evenly convex.
- Let $0 \in C$. C is evenly convex iff $C = C^{00}$.

[Martinex-Legaz (83-...), Rodriguez (01), Borwein Lewis (92)].

Evenly convex functions

Definition

A function $\rho : E \to \overline{\mathbb{R}}$ is

1. Quasiconvex if all the lower level sets

$$\{X \in E \mid \rho(X) \leq c\}, \; c \in \mathbb{R}, \text{ are convex;}$$

2. Evenly Quasiconvex if all the lower level sets $\{X \in E \mid \rho(X) \leq c\}$, $c \in \mathbb{R}$, are evenly convex.

Indirect utility functions are evenly quasi convex.

[Crouzeix (77), Martinez-Legaz (91), Singer (97)]

Simple properties about evenly convex maps

- If $\rho : E \to \overline{\mathbb{R}}$ is l.s.c. and quasi-convex then it is evenly quasi-convex.
- If $\rho : E \to \overline{\mathbb{R}}$ is u.s.c. and quasi-convex then it is evenly quasi-convex.
- If $\rho : E \to \overline{\mathbb{R}}$ then

$$\rho_{eqc}(X) = \inf \{ m \in \mathbb{R} \mid X \in eqc \{ Y \mid \rho(Y) \leq m \} \} .$$

Why evenly quasiconvexity?

Definition

There is a complete duality between a class $\mathcal{R}$ of maps

$$R : \mathbb{R} \times \mathcal{P} \to \overline{\mathbb{R}}$$

and a class $\mathcal{L}$ of functions

$$\rho : E \to \overline{\mathbb{R}}$$

if for every $\rho \in \mathcal{L}$ there exists a unique $R \in \mathcal{R}$ such that

$$\rho(X) = \sup_{Q \in \mathcal{P}} R(E_Q[-X], Q)$$

and viceversa.

Literature in the STATIC case ($\rho : E \to \mathbb{R}$)

- Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2009) provide a complete duality, under fairly general conditions, for monotone (↑) Evenly Quasiconcave real valued maps, hence covering both cases of maps $\rho : E \to \overline{\mathbb{R}}$ that are:

  monotone (↓), q.co. and l.s.c.
  monotone (↓), q.co. and u.s.c.

- Drapeau and Kupper (2010) provide a similar solution, under weaker assumptions on the vector space E, for maps $\rho : E \to \overline{\mathbb{R}}$ that are:

  monotone (↓), q.co. and l.s.c.

On the conditional setting: vector space approach

In the conditional (or dynamic $\mathcal{F}_s \subseteq \mathcal{F}_t$, $s < t$) setting we consider:

$$\mathcal{G} \subseteq \mathcal{F}$$

$$\rho : L_{\mathcal{F}} \longrightarrow L_{\mathcal{G}}$$

$L_{\mathcal{F}}$ is a Topological Vector Space of $\mathcal{F}$-measurable r.v. on $(\Omega, \mathcal{F}, P)$
$L_{\mathcal{G}}$ is a Topological Vector Space of $\mathcal{G}$-measurable r.v. on $(\Omega, \mathcal{G}, P)$

Example: the capital requirement

Let $C_T \subset E$ be a convex set and for $m \in \mathbb{R}$ denote by $v_t(m, \omega)$ the price at time t of m euros at time T. The function $v_t(m, \cdot)$ will be in general $\mathcal{G}$ measurable (e.g. $v_t(m, \omega) = D_t(\omega)m$).

$$\rho_{C_T, v_t}(X)(\omega) = ess \inf_{Y \in L^0_{\mathcal{G}}} \{v_t(Y, \omega) \mid X + Y \in C_T\}.$$

Under suitable conditions on v_t, the map ρ_{C_T, v_t} is a cash subadditive quasi-convex (in general not convex) risk measure.

Example: the Conditional Certainty Equivalent

Consider a Stochastic Dynamic Utility (SDU) $u(x, t, \omega)$

$$u : \mathbb{R} \times [0, \infty) \times \Omega \to \mathbb{R} \cup \{-\infty\}$$

Definition

Let u be a SDU and X be a $\mathcal{F}_t$ measurable random variable. For each $s \in [0, t]$, the backward Conditional Certainty Equivalent $\rho_{s,t}(X)$ of X is the $\mathcal{F}_s$ measurable random variable solution of the equation:

$$u(\rho_{s,t}(X), s, \omega) = E\left[u(X, t, \omega) | \mathcal{F}_s\right].$$

This valuation operator $\rho_{s,t}(X) = u^{-1}\left(E\left[u(X, t, \omega) | \mathcal{F}_s\right], s, \omega\right)$ is the natural generalization to the dynamic and stochastic environment of the classical definition of the certainty equivalent, as given in Pratt 1964. Even if $u(., t, \omega)$ is concave $\rho_{s,t}$ is not a concave functional, but it is conditionally quasiconcave.

Assumptions in the vector space case

The following representation theorem holds under the assumptions that:

- $$L^{\infty} \subseteq L_{\mathcal{F}} \subseteq L^{0}.$$
- The dual space $L_{\mathcal{F}}^{*} \subseteq L^{1}$
- The map ρ is regular:

  (REG) $\forall A \in \mathcal{G},\ \rho(X_1 \mathbf{1}_A + X_2 \mathbf{1}_A^C) = \rho(X_1)\mathbf{1}_A + \rho(X_2)\mathbf{1}_A^C$

The dual representation of conditional quasiconvex maps

Vector space approach

Theorem

If $\rho : L_{\mathcal{F}} \to L_{\mathcal{G}}$ is MON ($\downarrow$), QCO, REG and either $\sigma(L_{\mathcal{F}}, L^*_{\mathcal{F}})$-LSC or $\sigma(L_{\mathcal{F}}, L^*_{\mathcal{F}})$-USC then

$$\rho(X) = ess \sup_{Q \in L^*_{\mathcal{F}} \cap \mathcal{P}} R(E_Q[-X|\mathcal{G}], Q)$$

where

$$R(Y, Q) := ess \inf_{\xi \in L_{\mathcal{F}}} \{\rho(\xi) \mid E_Q[-\xi|\mathcal{G}] \geq_Q Y\}, \;\; Y \in L_{\mathcal{G}}$$

$$\mathcal{P} =: \left\{ \frac{dQ}{dP} \mid Q << P \text{ and } Q \text{ probability} \right\}$$

Exactly the same representation of the real valued case, but with conditional expectations.

Two more steps for the complete duality

In order to obtain the Complete Duality in the conditional setting we need:

- to embed the theory in L^0-Modules
- to extend the notion of an evenly convex set to the conditional setting

On L^0 Modules

$$L^0(\Omega, \mathcal{G}, \mathbb{P}; \mathbb{R}) = L^0 \qquad L^0(\Omega, \mathcal{G}, \mathbb{P}; \overline{\mathbb{R}}) = \overline{L}^0$$

$$L^p_{\mathcal{G}}(\mathcal{F}) = L^0(\mathcal{G}) L^p(\mathcal{F}) = \{YX \mid Y \in L^0(\mathcal{G}),\ X \in L^p(\mathcal{F})\}$$

- At time t every F_t-measurable ($\mathcal{G} = \mathcal{F}_t$) random variable will be known. Every $Y \in L^0 = L^0(\Omega, \mathcal{G}, \mathbb{P})$ will act as a 'constant 'when computing the risk of a position.

[Guo (1992-2012) "Random Locally Convex Modules"]
[Filipovic, Kupper and Vogelpoth (2009), (2010) "Locally L^0-convex Modules"]

Definition (Topological L^0-module)

We say that (E, τ) is a topological L^0-module if E is a L^0-module and τ is a topology on E such that the module operation
(i) $(E, \tau) \times (E, \tau) \to (E, \tau)$, $(x_1, x_2) \mapsto x_1 + x_2$,
(ii) $(L^0, \tau_0) \times (E, \tau) \to (E, \tau)$, $(\gamma, x_2) \mapsto \gamma x_2$
are continuous w.r.t. the corresponding product topology.

Two selections for the topology on L^0:

- Guo: τ_0 is the topology on L^0 of the convergence in probability
- FKV: τ_0 is a uniform topology on L^0 (here (L^0, τ_0) is only a topological ring)

Seminorms valued in L^0

Definition

A map $\|\cdot\| : E \to L^0_+$ is a L^0-seminorm on E if

(i) $\|\gamma x\| = |\gamma|\|x\|$ for all $\gamma \in L^0$ and $x \in E$,

(ii) $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$ for all $x_1, x_2 \in E$.
The L^0-seminorm $\|\cdot\|$ becomes a L^0-norm if in addition

(iii) $\|x\| = 0$ implies $x = 0$.

- $\mathcal{Z}$ will be a family of L^0-seminorms satisfying in addition the property:

$$\sup\{\|x\| \mid \|x\| \in \mathcal{Z}\} = 0 \text{ iff } x = 0,$$

Separation theorems holds

Definition (L^0-module associated to $\mathcal{Z}$)

We say that $(E, \mathcal{Z}, \tau)$ is a L^0-module associated to $\mathcal{Z}$ if:

1. $\mathcal{Z}$ is a family of L^0-seminorms,
2. (E, τ) is a topological L^0-module,
3. A net $\{x_\alpha\}$ converge to x in (E, τ) iff $\|x_\alpha - x\|$ converge to 0 in (L^0, τ_0) for each $\|\cdot\| \in \mathcal{Z}$.

- In both setting (Guo and FKV) it was shown that appropriate versions of (H-B) separation theorems holds
- In both setting, the dual of the L^0-module $L^p_{\mathcal{G}}(\mathcal{F})$ can be identified with the L^0-module $L^q_{\mathcal{G}}(\mathcal{F})$.

On the L^0-Module $L^p_{\mathcal{G}}(\mathcal{F})$ (FKV 2009)

For every $p \geq 1$ let:

$$L^p_{\mathcal{G}}(\mathcal{F}) =: \{X \in L^0(\Omega, \mathcal{F}, \mathbb{P}) \mid \|X|\mathcal{G}\|_p \in L^0(\Omega, \mathcal{G}, \mathbb{P})\}$$

where $\|\cdot|\mathcal{G}\|_p : \bar{L}^0_{\mathcal{G}}(\mathcal{F}) \to \bar{L}^0_+(\mathcal{G})$

$$\|X|\mathcal{G}\|_p =: \begin{cases} \lim_{n\to\infty} E[|X|^p \wedge n|\mathcal{G}]^{\frac{1}{p}} & \text{if } p < +\infty \\ ess.\inf\{Y \in \bar{L}^0(\mathcal{G}) \mid Y \geq |X|\} & \text{if } p = +\infty \end{cases}$$

Then $(L^p_{\mathcal{G}}(\mathcal{F}), \|\cdot|\mathcal{G}\|_p)$ is an $L^0(\mathcal{G})$-normed module having the product structure:

$$L^p_{\mathcal{G}}(\mathcal{F}) = L^0(\mathcal{G})L^p(\mathcal{F}) = \{YX \mid Y \in L^0(\mathcal{G}),\ X \in L^p(\mathcal{F})\}$$

On the dual elements of $L^p_{\mathcal{G}}(\mathcal{F})$

The dual elements can be identified with conditional expectations

For $p \in [1, +\infty)$, any $L^0(\mathcal{G})$-linear continuous functional

$$\mu : L^p_{\mathcal{G}}(\mathcal{F}) \to L^0(\mathcal{G})$$

can be identified with a random variable $Z \in L^q_{\mathcal{G}}(\mathcal{F})$, $\frac{1}{p} + \frac{1}{q} = 1$, s.t.

$$\mu(\cdot) = E[Z \cdot |\mathcal{G}].$$

Of course the μ are regular:

$$\forall A \in \mathcal{G}, \mu(X_1 \mathbf{1}_A + X_2 \mathbf{1}_A^C) = \mu(X_1)\mathbf{1}_A + \mu(X_2)\mathbf{1}_A^C$$

Define the set of normalized dual elements by:

$$\mathcal{P}^q = \left\{ \frac{dQ}{dP} \in L^q_{\mathcal{G}}(\mathcal{F}) \mid Q \text{ probability}, \; E\left[\frac{dQ}{dP}|\mathcal{G}\right] = 1 \right\}$$

Countable Concatenation Property

From now on $E = L^p_{\mathcal{G}}(\mathcal{F})$ even though most of the results hold in general framework.

(CSet) A subset $\mathcal{C} \subset E$ has the countable concatenation property if for every countable partition $\{A_n\}_n \subseteq \mathcal{G}$ and for every countable collection of elements $\{X_n\}_n \subset \mathcal{C}$ we have $\sum_n \mathbf{1}_{A_n} X_n \in \mathcal{C}$.

Given $\mathcal{C} \subseteq E$, we denote by $\mathcal{C}^{cc}$ the countable concatenation hull of $\mathcal{C}$, namely the smallest set $\mathcal{C}^{cc} \supseteq \mathcal{C}$ which satisfies (CSet)

Remark:

For $p \geq 1$, $(L^p(\mathcal{F}))^{cc} = L^p_{\mathcal{G}}(\mathcal{F})$.

Conditionally Evenly Convex Sets

Some components of C may degenerate to the entire space E: in this case there is no hope to pointwise separate X from C.

Notation:

Fix a set $\mathcal{C} \subseteq E$ and the class $\mathcal{A}(\mathcal{C}) := \{A \in \mathcal{G} \mid \mathcal{C}\mathbf{1}_A = E\mathbf{1}_A\}$. We denote by $A_\mathcal{C}$ the $\mathcal{G}$-measurable maximal element of the class $\mathcal{A}(\mathcal{C})$ and with $\mathcal{D}_\mathcal{C}$ the (P-a.s. unique) complement of $A_\mathcal{C}$. Hence

$$\mathcal{C}\mathbf{1}_{A_\mathcal{C}} = E\mathbf{1}_{A_\mathcal{C}}.$$

(i.e. $\forall X \in E \; \exists \xi \in C : \xi\mathbf{1}_{A_\mathcal{C}} = X\mathbf{1}_{A_\mathcal{C}}$).

Definition

A set $\mathcal{C}$ is conditionally evenly convex if there exist $\mathcal{L} \subseteq L^q_\mathcal{G}(\mathcal{F})$ such that

$$\mathcal{C} = \bigcap_{\mu \in \mathcal{L}} \{X \in L^p_\mathcal{G}(\mathcal{F}) \mid \mu(X) < Y_\mu \text{ on } D_\mathcal{C}\} \quad \text{for some } Y_\mu \in L^0(\mathcal{G}). \quad (1)$$

Conditionally Evenly Convex Sets

Definition

For $X \in L^p_{\mathcal{G}}(\mathcal{F})$ and $\mathcal{C} \subset L^p_{\mathcal{G}}(\mathcal{F})$, we say that X is *outside* $\mathcal{C}$ if $\mathbf{1}_A\{X\} \cap \mathbf{1}_A\mathcal{C} = \emptyset$ for every $A \in \mathcal{G}$ with $A \subseteq D_{\mathcal{C}}$ and $\mathbb{P}(A) > 0$.

Theorem

Let $\mathcal{C} \subseteq L^p_{\mathcal{G}}(\mathcal{F})$. The following statements are equivalent:

1. *$\mathcal{C}$ is conditionally evenly convex.*
2. *$\mathcal{C}$ satisfies (CSet) and for every X outside $\mathcal{C}$ there exists a $\mu \in L^q_{\mathcal{G}}(\mathcal{F})$ such that*

$$\mu(X) > \mu(\xi) \text{ on } D_{\mathcal{C}},\ \forall \xi \in \mathcal{C}.$$

Bipolar Theorem

Definition

For $\mathcal{C} \subseteq L^p_{\mathcal{G}}(\mathcal{F})$ we define the polar and bipolar sets as follows

$$\mathcal{C}^\circ := \left\{\mu \in L^q_{\mathcal{G}}(\mathcal{F}) \mid \mu(X) < 1 \text{ on } D_{\mathcal{C}} \text{ for all } X \in \mathcal{C}\right\},$$
$$\mathcal{C}^{\circ\circ} := \left\{X \in L^p_{\mathcal{G}}(\mathcal{F}) \mid \mu(X) < 1 \text{ on } D_{\mathcal{C}} \text{ for all } \mu \in \mathcal{C}^\circ\right\}.$$

Theorem

For any $\mathcal{C} \subseteq L^p_{\mathcal{G}}(\mathcal{F})$ such that $0 \in \mathcal{C}$ we have:

1. $\mathcal{C}^\circ = \left\{\mu \in L^q_{\mathcal{G}}(\mathcal{F}) \mid \mu(X) < 1 \text{ on } D_{\mathcal{C}} \text{ for all } X \in \mathcal{C}^{cc}\right\}$
2. *The bipolar $\mathcal{C}^{\circ\circ}$ is a conditional evenly convex set containing $\mathcal{C}$.*
3. *The set $\mathcal{C}$ is conditional evenly convex if and only if $\mathcal{C} = \mathcal{C}^{\circ\circ}$.*

Bipolar Theorem for cones

- $\mathcal{C} \subset L^p_{\mathcal{G}}(\mathcal{F})$ is convex and closed then is conditional evenly convex.
- Suppose that the set $\mathcal{C} \subseteq L^p_{\mathcal{G}}(\mathcal{F})$ is a L^0-cone, i.e. $\alpha X \in \mathcal{C}$ for every $X \in \mathcal{C}$ and $\alpha \in L^0_{++}$. In this case:

$$\begin{aligned} \mathcal{C}^\circ := \quad & \left\{\mu \in L^q_{\mathcal{G}}(\mathcal{F}) \mid \mu(X) \leq 0 \text{ on } D_{\mathcal{C}} \text{ for all } X \in \mathcal{C}\right\}, \\ \mathcal{C}^{\circ\circ} := \quad & \left\{X \in L^p_{\mathcal{G}}(\mathcal{F}) \mid \mu(X) \leq 0 \text{ on } D_{\mathcal{C}} \text{ for all } \mu \in \mathcal{C}^\circ\right\}. \end{aligned}$$

- Under the same assumption of the Bipolar Theorem, any conditional evenly convex L^0-cone containing the origin is closed.

Conditional Risk Measures

Consider a map

$$\rho : L^p_{\mathcal{G}}(\mathcal{F}) \to \overline{L}^0(\mathcal{G})$$

Note: If $\rho(\xi)1_A = +\infty 1_A$, $A \in G$, $P(A) > 0$
then: $\{\xi \in L^p_{\mathcal{G}}(\mathcal{F}) \mid \rho(\xi) \leq Y\} = \varnothing$, $Y \in L^0(\mathcal{G})$.

Hence we introduce the maximal $\mathcal{G}$ measurable set T_ρ such that

$$\begin{array}{lll} \rho(\xi) = +\infty & \text{on } \Gamma_\rho & \text{for every } \xi \in L^p_{\mathcal{G}}(\mathcal{F}), \\ \rho(\zeta) < +\infty & \text{on } T_\rho & \text{for some } \zeta \in L^p_{\mathcal{G}}(\mathcal{F}) \end{array}$$

and for any $Y \in L^0(\mathcal{G})$ define

$$U^Y_\rho := \{\xi \in L^p_{\mathcal{G}}(\mathcal{F}) \mid \rho(\xi)\mathbf{1}_{T_\rho} \leq Y\}.$$

Conditional risk maps

Definition

Let $X_1, X_2 \in L^p_{\mathcal{G}}(\mathcal{F})$. The map $\rho : L^p_{\mathcal{G}}(\mathcal{F}) \to \bar{L}^0(\mathcal{G})$ is:

- (MON↓) monotone if $X_1 \leq X_2 \Longrightarrow \rho(X_1) \geq \rho(X_2)$
- (REG) regular if $\forall A \in \mathcal{G}$, $\rho(X_1 \mathbf{1}_A + X_2 \mathbf{1}_A^C) = \rho(X_1)\mathbf{1}_A + \rho(X_2)\mathbf{1}_A^C$
- (QCO) quasi-convex if the sets U_ρ^Y are $L^0(\mathcal{G})$-convex $\forall Y \in L^0(\mathcal{G})$. Equivalently for all $\mathcal{G}$-measurable r.v. Λ, $0 \leq \Lambda \leq 1$,

$$\rho(\Lambda X + (1-\Lambda)Y) \leq \rho(X) \vee \rho(Y).$$

- (EVQ) evenly quasi-convex if the sets U_ρ^Y are evenly $L^0(\mathcal{G})$-convex $\forall Y \in L^0(\mathcal{G})$.
- (LSC) lower semicontinuous if the sets U_ρ^Y are closed $\forall Y \in L^0(\mathcal{G})$.

Remark:

(QCO)+(LSC) imply (EVQ)

Our main result: complete duality for modules of L^p type

By applying the separation theorem in $L^0(\mathcal{G})$-normed module (FKV) or (GUO) - which directly provides the existence of a dual element in terms of a conditional expectation - and the idea of the proof in the static case (as in CMMM09) and the results on conditional evenly convex sets we get:

Theorem

The map $\rho : L^p_{\mathcal{G}}(\mathcal{F}) \to \bar{L}^0(\mathcal{G})$ is an evenly quasi-convex regular risk measure - i.e. it satisfies MON(↓), REG and EVQ - if and only if

$$\rho(X) = ess \sup_{Q \in \mathcal{P}^q} R\left(E\left[-\frac{dQ}{dP}X|\mathcal{G}\right], Q\right)$$

with

$$R(Y, Q) = ess \inf_{\xi \in L^p_{\mathcal{G}}(\mathcal{F})} \left\{\rho(\xi) \mid E\left[-\frac{dQ}{dP}\xi|\mathcal{G}\right] = Y\right\}$$

unique in the class $\mathcal{R}$.

The class $\mathcal{R}$ for the complete duality

Define the class $\mathcal{R}$ of maps $K : L^0(\mathcal{G}) \times \mathcal{P}^q \to \bar{L}^0(\mathcal{G})$ with:

- K is increasing in the first component.
- $K(Y\mathbf{1}_A, Q)\mathbf{1}_A = K(Y, Q)\mathbf{1}_A$ for every $A \in \mathcal{G}$.
- $\inf_{Y \in L^0(\mathcal{G})} K(Y, Q) = \inf_{Y \in L^0(\mathcal{G})} K(Y, Q')$ for every $Q, Q' \in \mathcal{P}^q$.
- K is $\diamond$-evenly $L^0(\mathcal{G})$-quasiconcave: for every $(\bar{Y}, \bar{Q}) \in L^0(\mathcal{G}) \times \mathcal{P}^q$, $A \in \mathcal{G}$ and $\alpha \in L^0(\mathcal{G})$ such that $K(\bar{Y}, \bar{Q}) < \alpha$ on A, there exists $(\bar{V}, \bar{X}) \in L^0_{++}(\mathcal{G}) \times L^p_{\mathcal{G}}(\mathcal{F})$ with
$$\bar{Y}\bar{V} + E\left[\bar{X}\frac{d\bar{Q}}{dP}|\mathcal{G}\right] < Y\bar{V} + E\left[\bar{X}\frac{dQ}{dP}|\mathcal{G}\right] \text{ on } A$$
for every (Y, Q) such that $K(Y, Q) \geq \alpha$ on A.
- the set $\mathcal{K} = \left\{K(E[X\frac{dQ}{dP}|\mathcal{G}], Q) \mid Q \in \mathcal{P}^q\right\}$ is upward directed for every $X \in L^p_{\mathcal{G}}(\mathcal{F})$.
- $K(Y, Q_1)\mathbf{1}_A = K(Y, Q_2)\mathbf{1}_A$, if $\frac{dQ_1}{d\mathbb{P}}\mathbf{1}_A = \frac{dQ_2}{d\mathbb{P}}\mathbf{1}_A$, $Q_i \in \mathcal{P}^q$, and $A \in \mathcal{G}$.

Byproducts...

Adding cash additivity

$$(CA)\ \forall\ X \in L^p_{\mathcal{G}}(\mathcal{F}) \text{ and } \forall\ \alpha \in L^0(\mathcal{G}),\ \rho(X+\alpha) = \rho(X) - \alpha.$$

we recover the following

Corollary

1. *If $Q \in \mathcal{P}^q$ and if ρ is (MON↓), (REG) and (CA) then*

$$R(E_Q(-X|\mathcal{G}), Q) = E_Q(-X|\mathcal{G}) - \rho^*(-Q)$$

where

$$\rho^*(-Q) = \sup_{\xi \in L^p_{\mathcal{G}}(\mathcal{F})} \left\{ E_Q[-\xi|\mathcal{G}] - \rho(\xi) \right\}.$$

2. *Under the same assumptions of the Theorem and if ρ satisfies in addition (CA) then*

$$\rho(X) = \sup_{Q \in \mathcal{P}^q} \left\{ E_Q(-X|\mathcal{G}) - \rho^*(-Q) \right\}.$$

On a class of quasi-convex risk measures

$$\Phi : \mathcal{P}(\mathbb{R}) \to \overline{\mathbb{R}}$$

defined on the convex set $\mathcal{P}(\mathbb{R})$ of distributions on $\mathbb{R}$

built from a family of acceptance sets

(Joint with Marco Maggis and Ilaria Peri)

Idea: .A risk prudent agent is willing to accept greater losses only with smaller probabilities. We introduce in the definition of $V@R$ a function Λ that describes the balance between the amount of the loss and its probability.

Law Invariant Risk Measures

Suppose that a risk measure $\rho : \mathcal{X} \subset L^0 \to \overline{\mathbb{R}}$ is law invariant, i.e.:

$$X \sim_{\mathcal{D}} Y \Rightarrow \rho(X) = \rho(Y).$$

Then we may consider the new map $\Phi : \mathcal{P}(\mathbb{R}) \to \overline{\mathbb{R}}$

$$\Phi(P_X) = \rho(X)$$

defined on the set $\mathcal{P}(\mathbb{R})$ of the distributions on $\mathbb{R}$.

Notations

- $P_X(B) := \mathbb{P}(X^{-1}(B))$ is the distribution of X, $B \in \mathcal{B}_{\mathbb{R}}$
- $\mathcal{P} =: \mathcal{P}(\mathbb{R})$ is the set of the distributions on $\mathbb{R}$.
- $F_X(x) := P_X(-\infty, x]$ is the distribution function of X.

REMARK: If $X \geq Y$ then $F_X \leq F_Y$.

- This suggest to adopt the opposite ($\uparrow$ instead of $\downarrow$) monotonicity property.

Risk Measures on distributions

We consider on $\mathcal{P}$ the following order structure:

$$P \preccurlyeq Q \quad \Leftrightarrow \quad F_P(x) \leq F_Q(x) \quad \text{for all } x \in \mathbb{R}.$$

which is simply the opposite of First Stochastic Dominance.

Definition

A Risk Measure on $\mathcal{P}$ is a map $\Phi : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$:

(Mon↑) monotone: $P \preccurlyeq Q$ implies $\Phi(P) \leq \Phi(Q)$;

(Qconv) quasi-convex: $\Phi(\lambda P + (1-\lambda)Q) \leq \Phi(P) \vee \Phi(Q)$, $\lambda \in [0,1]$.

Translation Invariant Property of RMs on distributions

Let $T_m : \mathcal{P} \to \mathcal{P}$ the translation operator s.t.

$$T_m P_X = P_{X+m} \quad \forall m \in \mathbb{R}$$

hence it maps the distribution $F_X(x)$ into $F_X(x-m)$.

Definition

If $\Phi : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ is a risk measure on $\mathcal{P}$, we say that

(TrI) Φ is translation invariant if $\Phi(T_m P_X) = \Phi(P_X) - m$.

Notice that **(TrI)** of Φ corresponds to *cash additivity* of risk measures defined on random variables.

But **(Qconv)** + **(TrI)** of $\Phi \nRightarrow$ *convexity* (e.g $V@R$).

Additional topological conditions

- We endow $\mathcal{P}(\mathbb{R})$ with the $\sigma(\mathcal{P}(\mathbb{R}), C_b(\mathbb{R}))$ topology, where C_b is the space of bounded continuous function $f : \mathbb{R} \to \mathbb{R}$
- The dual pairing $\langle \cdot, \cdot \rangle : C_b \times \mathcal{P} \to \mathbb{R}$ is given by

$$\langle f, P \rangle = \int f dP$$

Lemma

Let $\Phi : \mathcal{P} \to \overline{\mathbb{R}}$ be (Mon). Then the following are equivalent:
Φ is $\sigma(\mathcal{P}, C_b)$-lower semicontinuous
Φ is continuous from below: $P_n \uparrow P$ implies $\Phi(P_n) \uparrow \Phi(P)$.

We say that $P_n \uparrow P$ whenever $\{P_n\}$ is *increasing* and $F_{P_n}(x) \uparrow F_P(x)$ for every $x \in \mathcal{C}(F_P)$, the set of points in which F_P is continuous.

On a class of risk measures on distributions

We build the maps Φ from a family $\{\mathcal{A}^m\}_{m\in\mathbb{R}}$ of *acceptance sets* of distribution functions.

Definition

Given a family $\{F_m\}_{m\in\mathbb{R}}$ of functions $F_m : \mathbb{R} \to [0,1]$, we consider the associated sets of probability measures

$$\mathcal{A}^m := \{Q \in \mathcal{P} \mid F_Q \leq F_m\}$$

and the associated map $\Phi : \mathcal{P} \to \overline{\mathbb{R}}$ defined by

$$\Phi(P) := -\sup\{m \in \mathbb{R} \mid P \in \mathcal{A}^m\}.$$

Feasible families

A family $\{F_m\}_{m\in\mathbb{R}}$ of functions $F_m : \mathbb{R} \to [0,1]$ is **feasible** if

- For any $P \in \mathcal{P}$ there exists m such that $P \notin \mathcal{A}^m$,
- $F_m(\cdot)$ is *right continuous* (w.r.t. x) $\forall m \in \mathbb{R}$,
- $F_{\cdot}(x)$ is decreasing and left continuous (w.r.t. m) $\forall x \in \mathbb{R}$.

Theorem

If $\{F_m\}_{m\in\mathbb{R}}$ is a feasible family, then

- *$\{\mathcal{A}^m\}_{m\in\mathbb{R}}$ is monotone decreasing and left continuous;*
- *$\mathcal{A}^m$ is convex and $\sigma(\mathcal{P}, C_b)-$closed, for any m.*
- **The associated map $\Phi : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ is well defined, (Mon), (Qconv) and $\sigma(P, C_b)$-l.s.c.**

Example: the Worst Case Risk Measure

As a risk measure on distributions

$$
\begin{aligned}
F_m(x) \quad &: \quad = \mathbf{1}_{[m,+\infty)}(x) \\
\mathcal{A}^m \quad &: \quad = \{Q \in \mathcal{P} \mid F_Q \leq F_m\} = \{Q \in \mathcal{P} \mid Q \preccurlyeq \delta_m\} \\
\Phi_w(P) \quad &: \quad = -\sup\{m \mid P \in \mathcal{A}^m\} \\
&= \quad -\sup\{m \mid P \preccurlyeq \delta_m\} = -\sup\{x \in \mathbb{R} \mid F_P(x) = 0\}
\end{aligned}
$$

If X is a random variable and P_X is its distribution

$$\Phi_w(P_X) = -ess\inf(X) := \rho_w(X)$$

coincide with the worst case risk measure ρ_w.

- As the family $\{F_m\}$ is feasible, Φ_w **is (Mon), (Qconv) and** $\sigma(\mathcal{P}, C_b)-$**l.s.c**. In addition, it also satisfies *(TrI)*.
- Even though $\rho_w : L^0 \to \mathbb{R} \cup \{-\infty\}$ is convex the associated map $\Phi_w : \mathcal{P} \to \mathbb{R} \cup \{-\infty\}$ is not convex, but it is *quasi-convex* and *concave*.

Example: The V@R, as a risk measure on distributions

$$
\begin{aligned}
F_m(x) &:= \lambda \mathbf{1}_{(-\infty,m)}(x) + \mathbf{1}_{[m,+\infty)}(x) \\
\mathcal{A}^m &:= \{Q \in \mathcal{P} \mid F_Q \leq F_m\} \\
\Phi_{V@R_\lambda}(P) &:= -\sup\{m \mid P \in \mathcal{A}^m\}
\end{aligned}
$$

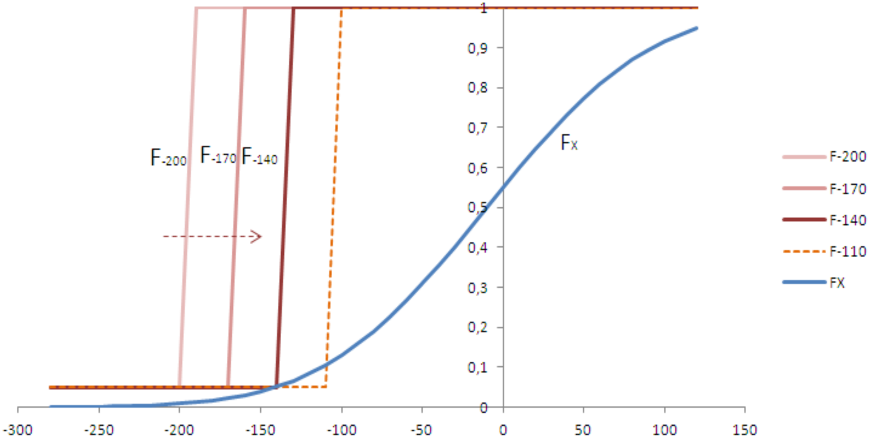
If X is a random variable, P_X its distribution and $q_X^+(\lambda)$ its right quantile

$$
\begin{aligned}
\Phi_{V@R_\lambda}(P_X) &:= -\sup\{m \mid P_X \in \mathcal{A}^m\} \\
&= -\sup\{m \mid \mathbb{P}(X \leq m) \leq \lambda\} \\
&= -q_X^+(\lambda) := V@R_\lambda(X)
\end{aligned}
$$

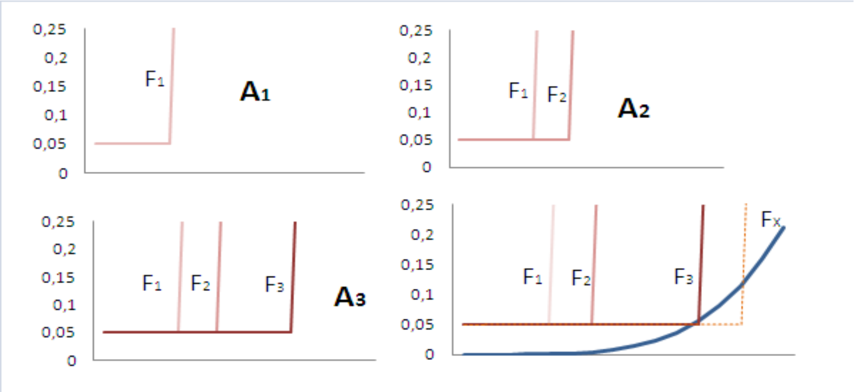
coincide with the $V@R$ **of level** $\lambda \in (0,1)$.

- As the family $\{F_m\}$ is feasible, $\Phi_{V@R_\lambda} : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ **is (Mon), (Qconv),** $\sigma(\mathcal{P}, C_b)$**-l.s.c.**
- $V@R_\lambda : L^0 \to \mathbb{R} \cup \{-\infty\}$ is not **(Qconv)**, as a map on random variables.

In the example $V@R_{0.05}(X) = 140$.



In addition, $\Phi_{V@R_\lambda}$ also satisfies *(TrI)*.



The V@R with Probability/Loss function

The Value at Risk with Prob/Loss function

- We replace the constant λ with the function: $\Lambda : \mathbb{R} \to [0, 1)$.

Define $F_m : \mathbb{R} \to [0, 1]$ by:

$$F_m(x) := \Lambda(x)\mathbf{1}_{(-\infty,m)}(x) + \mathbf{1}_{[m,+\infty)}(x).$$

where Λ is an *increasing* and *right continuous* function.

Definition

The map $\Lambda V@R : \mathcal{P}(\mathbb{R}) \to \overline{\mathbb{R}}$ is defined by

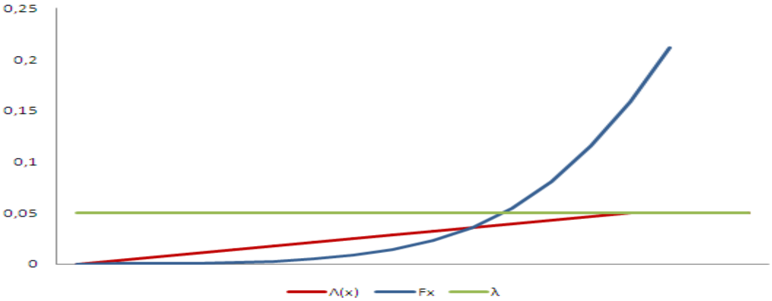
$$\Lambda V@R(P) := -\sup \{m \mid P \in \mathcal{A}^m\}.$$

where $\mathcal{A}^m = \{Q \in \mathcal{P} \mid F_Q \leq F_m\}$.

- **As the family $\{F_m\}_{m\in\mathbb{R}}$ is feasible then the $\Lambda V@R : \mathcal{P} \to \overline{\mathbb{R}}$ is well defined, (Mon), (Qconv) and $\sigma(\mathcal{P}, C_b)$-l.s.c.**

Thus, in case of a random variable X

$$\Lambda V@R(P_X) = -\sup\{m \in \mathbb{R} \mid F_X(x) \leq \Lambda(x),\ \forall x \leq m\}.$$

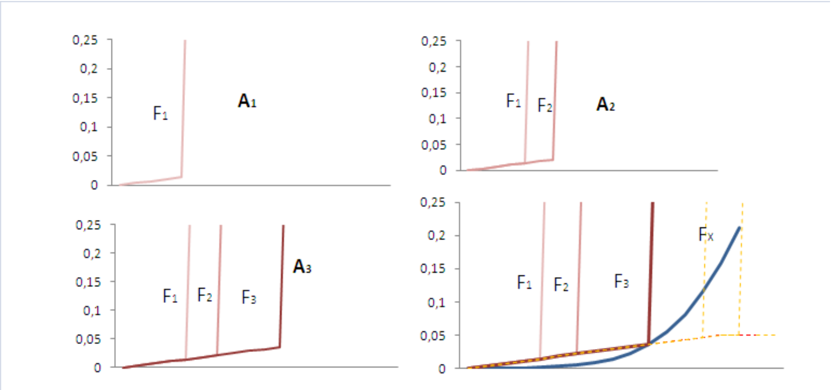


Idea: The risk prudent agent requires smaller probabilities for greater losses. The function Λ describes the balance between the amount of the loss and its probability.

The Value at Risk with Prob/Loss function

Family of acceptance sets

The acceptance sets $\mathcal{A}_m$ will not be anymore the translation of the acceptance set A_0 ($A_m \neq A_0 + m$).



We drop in this way cash additivity *(TrI)*, but we obtain another similar property, which is the counterpart of *(TrI)* for the $\Lambda V@R$:

$$\Lambda V@R(P_{X+\alpha}) = \Lambda^{\alpha} V@R(P_X) - \alpha, \quad \alpha \in \mathbb{R}$$

where $\Lambda^{\alpha}(x) := \Lambda(x+\alpha)$.

Interpretation: If we add a sure positive amount α to a risky position X then the risk decreases of the value α, constrained to lower level of risk aversion described by $\Lambda^{\alpha} \geq \Lambda$.

For any map $\Phi : \mathcal{P} \to \mathbb{R} \cup \{\infty\}$, let $Dom(\Phi) := \{Q \in \mathcal{P} \mid \Phi(Q) < \infty\}$ and Φ^* be the convex conjugate

$$\Phi^*(f) := \sup_{Q \in \mathcal{P}} \left\{ \int f dQ - \Phi(Q) \right\}, \quad f \in C_b.$$

Fenchel-Moreau Theorem: Suppose that $\Phi : \mathcal{P} \to \mathbb{R} \cup \{\infty\}$ is $\sigma(\mathcal{P}, C_b)$-l.s.c. and convex. If $Dom(\Phi) \neq \varnothing$ then $Dom(\Phi^*) \neq \varnothing$ and

$$\Phi(Q) = \sup_{f \in C_b} \left\{ \int f dQ - \Phi^*(f) \right\}.$$

Proposition

The only $\sigma(\mathcal{P}, C_b)$-l.s.c., convex and (TrI) map $\Phi : \mathcal{P} \to \mathbb{R} \cup \{\infty\}$ is $\Phi = +\infty$.

$$
\begin{aligned}
C_b^- \quad : \quad &= \{f \in C_b \mid f \text{ is decreasing}\} \\
&= \left\{ f \in C_b \mid Q, P \in \mathcal{P} \text{ and } Q \preceq P \Rightarrow \int f dQ \le \int f dP \right\}
\end{aligned}
$$

Theorem

Any $\sigma(\mathcal{P}, C_b)-$lsc risk measure $\Phi : \mathcal{P} \to \mathbb{R} \cup \{\infty\}$ can be represented as

$$\Phi(P) = \sup_{f \in C_b^-} R\left(\int f dP, f \right)$$

where $R : \mathbb{R} \times C_b \to \overline{\mathbb{R}}$ is defined by

$$R(t, f) := \inf_{Q \in \mathcal{P}} \left\{ \Phi(Q) \mid \int f dQ \ge t \right\}.$$

Dual representation of RMs defined by a feasible family

Proposition Suppose in addition that for every m, $F_m(x)$ is increasing in x and $\lim_{x\to\infty} F_m(x) = 1$. Then the associated map $\Phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ is a $\sigma(\mathcal{P}(\mathbb{R}), C_b)$-lsc Risk Measure that can be represented as

$$\Phi(P) = \sup_{f \in C_b^-} R^-\left(\int f dP, f\right),$$

with:

$$R^-(t, f) = \inf\left\{m \in \mathbb{R} \mid \gamma(m, f) \geq t\right\},$$

where $\gamma : \mathbb{R} \times C_b(\mathbb{R}) \to \overline{\mathbb{R}}$ is given by:

$$\gamma(m, f) := \int f dF_{-m} + F_{-m}(-\infty) f(-\infty),\ m \in \mathbb{R}.$$

Computation of $\gamma(m, f)$ for the $\Lambda V@R$

It includes the cases of the

- $V@R$, when $\Lambda = \lambda \in \mathbb{R}$
- Worst case risk measure, when $\Lambda = 0$

In this two cases the formula coincides with what already obtained by Drapeau and Kupper (2010).

As $F_m = \Lambda(x)\mathbf{1}_{(-\infty,m)}(x) + \mathbf{1}_{[m,+\infty)}(x)$, we compute explicitly

$$\gamma(m, f) = \int_{-\infty}^{-m} f d\Lambda + (1 - \Lambda(-m))f(-m) + \Lambda(-\infty)f(-\infty)$$

and associated R^-.

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3. *Complete Duality for Quasiconvex Risk Measures on L^0-Modules of the L^p type,*
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5. *Risk Measures on P(R) and Value At Risk with Probability/Loss function*
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THANK YOU FOR YOUR ATTENTION!!!