Two Price Economy in Continuous Time

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Happy Birthday Freddy

- It is fitting that your birthday corresponds with another.

- This is the 21st birthday of Mathematical Finance, at least the Journal and possibly the discipline.

Freddy Personal Remarks

- I do not remember the date or year I met Freddy but will not forget the occasion.

- I had just taken a big step of moving to the Finance department of the Maryland Business School from a carreer in Statistics and Economics and was new to Finance.

- I was at the time well aware of Freddy's status as an intellectual giant in Mathematical Finance.

- The place I met Freddy was a conference in Montreal. Freddy probably does not remember, but I gave a talk after which Freddy came up to me and said, in almost a whisper, (Freddy is never loud), you know your stuff.

- Needless to say I was pleased and reassured.

- I can only presume that Freddy has been a supporter of my efforts. Recently he organized a Financial Mathematics meeting in Mumbai, where I grew up, and asked if I would participate.

- Of course I answered yes in seconds, and took the opportunity to show Freddy and Rita the precise location where I grew up.

The Intellectual Debt I

- For years my screen saver read out my debt to Freddy's leadership.

- It stated, in a business school , that
  - Asset Prices are Semimartingales, Investment is a Stochastic Integral and Brownian Motion must be Time Changed.

- The first two are clear, for the third I asked myself that since all semimartingales are a time changed Brownian motion, what was the nature of the time change.

- Presuming the time change to be an increasing process with a martingale component, I was led to conclude that it must be discontinuous.

- This is because there are no continuous increasing processes with a martingale component.

- I have generally, since, modeled stock prices as purely discontinuous.

- The compensated jump martingale being, in any case, a richer, more flexible and far more interesting object than Brownian motion for a modeller.

- I recognize that for a theorist the opposite may be the case as one may have more theorems.

The Intellectual Debt II

- The second debt to Freddy's work was the definition of risk measures, but for me more exactly, the definition of Acceptable Risks.

- I maintain that classical economics in its pursuit of its defense of free market capitalism neglected the main purpose of capital.

- They built models with no financial primitives, just preferences, endowment and technology and then went on to conclude that finance, and by implication capital, are irrelevant to modern economies and to capitalism itself.
  - It is strange that Finance departments begin their education from this perspective of financial irrelevance.

- I shall argue today that Acceptable Risks as defined by Freddy are the missing financial primitive in modeling a modern financial economy.

A Synthesis

- As a synthesis I will explain how local law invariance as discussed by Hans, bid ask spreads as discussed by Walter and Yuri, nonlinear expectations as discussed by Shige, all come together in a truly financial economy where the Walrasian auctioneer trades to "Freddy" acceptability.

- Many questions remain and I list a few
  - Are bid and ask prices semimartingales.
  - What are the precise conditional acceptability sets induced by a nonlinear PIDE in a presumed Markovian context.
  - Is Freddy's capital requirement in need of revision, just to save capitalism.

* For me, if we ask for X, capital c such that $c - X \in \mathcal{A}$ then

$$\begin{aligned} c &= \sup_{Q \in \mathcal{M}} E^Q[X] \\ &> E^{Q_{RN}}[X] \end{aligned}$$

* and capital exceeding price, may just be too much.

* I say give credit for the bid and set

$$c = \sup_{Q \in \mathcal{M}} E^Q[X] - \inf_{Q \in \mathcal{M}} E^Q[X].$$

– Should we campaign against the absurdity of DVA that requires one to take profit from an enhanced perceived inability to meet one's obligations.

Motivation

- In classical economic theory the law of one price prevails and market participants trade freely in both directions at the same price.

- Furthermore, these prices are determined by market clearing equating aggregate demand to supply or excess demand to zero.

- Recently, M. (Annals of Finance 2012) presented an equilibrium model in which both the law of one price and market clearing simultaneously fail.

- The law of one price is replaced by a two price economy and market participants continue to trade freely with the market but the terms of trade now depend on the trade direction.

- The starting of this paper is the equilibrium pricing rule that prevails in such a two price economy.

Rationale for the two prices

- The failure of market clearing occurs on account of a gap between the events that can occur and the events that can be contracted.

- The latter is a much smaller set of events.

- As a result unexpected events can cause endowments to disappear, making the clearing of precommitted demands impossible.

- In such situations markets must be supported by a financial system that approves trades by participants and covers any subsequent losses.

The Market Structure

- All market participants are modeled as selling their endowments to the financial system for a conservative valuation.

- They then spend the proceeds of this sale to meet their demands by purchasing from the financial system at an inflated valuation.

- The financial system in turn sets the spread between its conservative purchase price and its inflated sale price with a view to making trades acceptable.

The market as a passive auctioneer

- The financial system is not an optimizing agent but passively sets the terms at which market participants may trade.

- The financial system may be viewed as the Walrasian auctioneer operating in a world in which market clearing is not attainable.

- Therefore, instead of determining the market clearing price, the auctioneer, now subject to potential losses, determines the two prices of a two price economy with a view to making such loss exposures acceptable.

Avoiding Game Theoretic Considerations

- The purpose here is to develop the continuous time theory for such two price economies.

- The two prices may be termed bid and ask prices for some precision and brevity but they should not be confused with the literature relating bid-ask spreads to transactions costs, the modeling of illiquidity, the effects of asymmetric information or other frictions involved in modeling the financial industry (see Freixas and Rochet (2008)).

- There is a large literature both empirical and theoretical studying bid ask spreads by focusing on the costs, incentives, objectives and constraints of liquidity providers seen as rational agents operating as market makers in exchange traded securities.

- Modeling the optimal behavior of rational agents introduces interesting game theoretic considerations into the analysis.

- In contrast the approach taken here is to model passively the Walrasian auctioneer with a limited interest in attempting to clear markets

Risk Acceptability and Nonlinear Pricing

- The two prices of a two price economy are determined in a non market-clearing equilibrium with a view to making loss exposures acceptable.

- Acceptability is itself defined as a positive expectation under a family of test measures or scenarios.

- As a result the bid price is the infimum of test valuations and the ask price is the supremum of such valuations.

- On the space of random variables, the bid price functional is then a concave functional while the ask price functional is convex.

- Economically packaged risks are more attractive as they embody potential diversification benefits while the linearity of arbitrage pricing disappears given the absence of the law of one price.

Dynamic Nonlinear Pricing

- Given that the bid and ask price functionals are respectively concave and convex their dynamic counterparts are of necessity examples of nonlinear expectation operators.

- Nonlinear expectation operators are a fast developing field of mathematical analysis (see Peng (2004), Rosazza Gianin (2006)).

- These connections were noted in M., Pistorius and Schoutens (2011), and M. and Schoutens (2012b) by relating to Cohen and Elliott (2010).

- Cohen and Elliott (2010) develop nonlinear expectations as solutions to backward stochastic difference equations in the context of a finite state discrete time Markov chain.

- Nonlinear expectation operators provide us with dynamically consistent nonlinear pricing rules as discussed in Jobert and Rogers (2008) and Bion-Nadal (2009).

Nonlinear Pricing in Continuous Time

- Encouraged by the work of Peng (2006) in developing nonlinear $\mathcal{G}$-expectations (see section 3 of Peng (2006)) we propose here a continuous time nonlinear $\mathcal{G}$-expectation operator for the continuous time modeling of two price economies.

- We simultaneously model both a linear expectation operator and two nonlinear operators for the bid and the ask.

- The linear expectation operator serves the purpose of a traditional risk neutral valuation operator except that all trades occur at the nonlinear prices.

- However, we maintain some of the advantages of a linear operator by preserving linearity on comonotone risks.

Two Price and Risk Management

- From a financial and risk management perspective the contribution here is to provide operational algorithms for the computation of risk sensitive bid and ask prices as functionals on the space of random variables.

- The Basel system has sought such procedures for years building ad-hoc approaches in the interim.

- Further as argued in M. (2012) for two price economies marking to market must be interpreted as marking to two price markets with assets marked to bid and liabilities marked to ask.

- It is then insufficient to just have available linear risk neutral valuation operators, one needs the nonlinear two price operators to mark the books.

- Additionally capital reserves reflect the asset shaves and liability add ons built into the bid and ask functionals relative to the expectation functional.

- In this regard all three operators, the nonlinear bid and ask and the linear expectation are employed.

- We deliver all three operators with the property that under the linear expectation the bid price is a submartingale while the ask price is a super martingale.

Outline of the Presentation

- First we review the two price economy and its bid and ask price functionals in a static one period context.

- The discrete time dynamic construction with its links to nonlinear expectations for finite state Markov chains is summarized next.

- We then introduce the continuous time bid and ask price functionals as nonlinear $\mathcal{G}$-expectations in the context of a Hunt (1966) process.

- Illustrative valuations are then conducted and presented.

- The methods developed are applied to the valuation of a derivatives book.

The Static Two Price Economy

- Much has been written on modeling the mathematical representation of consumers, producers, firms, financial institutions, financial intermediaries and other market participants.

- They are all generally seen as optimizing agents with various approaches taken to represent their objectives and constraints.

- But what about the Walrasian auctioneer or the market itself?

- Technically in the Arrow Debreu theory the market is modeled as a non-optimizing agent that merely seeks to set prices with a view to ensuring market clearing.

- The two price economy focuses attention on the Walrasian auctioneer or the market itself as another agent with whom all must trade.

- This implicit agent, however, by virtue of being the counterparty for all trades, is too powerful and does not optimize.

- This auctioneer or more generally the market merely defines passively the terms of trade for all participants, remaining interested in market clearing.

- The difference between classical economic theory and the theory of a two price economy is that market clearing though an objective for the market seen as a passive agent is in fact unattainable.

- Were clearing possible with positive excess supplies for all items in all states, the law of one price would return.

- Recognizing that markets cannot always clear, the interest shifts to making excess supplies acceptable, though not necessarily nonnegative.

- The market tries to get excess supplies to belong to some small prespecified cone containing the nonnegative random variables.

- This is done with a view to minimizing loss exposures.

- The size of this cone serves as a financial primitive in defining the two price economy.

- The larger the size of this cone the greater is the set of approved trading opportunities and the larger is the size of the real economy.

- On the contrary when this cone contracts, the real economy shrinks, the market approves of fewer transactions and economic activity is reduced.

The Cone of Market Acceptable Risks

- Consider now an economy trading state contingent claims on a classical probability space $(\Omega, \mathcal{F}, P)$.

- In addition to endowments, preferences, technology and firm objectives we now have to define the set of acceptable aggregate excess supplies.

- This set is by construction a convex cone of random variables $\mathcal{A}$ containing the nonnegative random variables.

- Artzner, Delbaen, Eber and Heath (1999) show that all such sets are defined by a convex set of probability measures $\mathcal{M}$ with the defining condition being

$$X \in \mathcal{A} \Longleftrightarrow E^{Q}[X] \geq 0 , \quad (Q \in \mathcal{M}).$$

The set of probability measures $\mathcal{M}$ has been called the set of test measures or scenarios that test for and approve the acceptability of a random variable.

- In fact the Federal Reserve Board now requires major banks with more than 50 billion in assets to conduct such stress tests annually (FRB Press Release, November 22 2011) with a view to ascertaining capital adequacy.

The Cone of Acceptable Counterparty Risks

- In a two price economy the market targets the acceptability of excess supplies ($X \in \mathcal{A}$) defined in this way for some set of test or scenario measures $\mathcal{M}$.

- The market's interest lies in keeping $\mathcal{A}$ small and therefore $\mathcal{M}$ is large.

- However in trading with economic agents, all of whom must trade with the market, the market is more lenient and is willing to define a larger set of acceptability, $\mathcal{B}$, with a related much smaller set of test measures $\mathcal{N}$.

- Indeed it is possible that even with this generous definition of acceptability offered to individual market participants the aggregate excess supply may nonetheless enter the required smaller set $\mathcal{A}$.

- By way of contrast with classical economic theory as opposed to the two price economy one notes that classically $\mathcal{B}$ is a very generous half space with $\mathcal{N} = \{Q\}$ for the risk neutral measure Q

- and $\mathcal{A}$ is the cone of nonnegative random variables with $\mathcal{M}$ being the set of all probability measures.

The Two Static Pricing Functionals

- When the market offers individual market participants the cone $\mathcal{B}$ of acceptability it is shown in M. (2012) and easily observed that the price system offered by the market is now a two price system with bid price $b(X)$ and ask price $a(X)$ defined by

$$
\begin{aligned}
b(X) &= \inf_{Q \in \mathcal{N}} E^{Q}[X] \\
a(X) &= \sup_{Q \in \mathcal{N}} E^{Q}[X].
\end{aligned}
$$

  These Equations and define the price system offered in equilibrium to all market participants by the market as the counterparty for all trades.

- We note at this point that by construction the bid pricing functional will be a concave functional on the space of random variables while the ask price functional will be a convex functional.

- They are then both nonlinear pricing operators and it is these properties that will later take us to nonlinear expectation operators.

- Furthermore, one only needs to learn how to construct the bid pricing functional as the ask price is always the negative of the bid price of $-X$.

Law Invariant Cones

- The next step in the operational development of two price economies comes in the construction of the set of approving probability measures.

- The realization here is not to give up completely on classical theory and its selection of a risk neutral equilibrium pricing operator, but to ensure that the cone of a two price economy is strictly contained in the classical half space.

- We therefore begin by selecting a classical risk neutral equilibrium pricing measure Q^* as an element of $\mathcal{N}$.

- Next we consider the possibility of defining acceptability of a random variable X completely in terms of the probability law of X under Q^*.

- Acceptability must then be defined with just the distribution function $F_X(x)$ of X under Q^* as an input or the definition of acceptability is law invariant in the sense of Kusuoka (2001).

- Such a definition based only on the probability law may be objectionable from the perspective of human agents who may wish to consider how the random variable enters the portfolio of risks being held.

- However, we are modeling here the market or Walrasian auctioneer with the single minded interest of eventual clearing suitably modified for two price economies.

- There is no portfolio to refer to or preferences to formulate.

- With these qualifying remarks we proceed to define acceptability just in terms of the risk neutral distribution function.

Comonotone Risks

- The next item to be addressed is the preservation of some linearity in the pricing functionals.

- They are nonlinear by construction but we may ask for linearity for some set of risks.

- In this regard we note that two random variables X, Y are said to be comonotone if

  $(X(\omega_1)-X(\omega_2))\,(Y(\omega_1)-Y(\omega_2)) \geq 0$ almost surely.

  Comonotone variables always move together in the same direction, or one is in fact an increasing function of the other.

- Preserving linearity for comonotone variables is a useful reduction in the complexity of the pricing operator and we can ask that

$$b(X+Y) = b(X) + b(Y)$$

  for X, Y comonotone.

Concave Distortions

- Assuming both law invariance and linearity for comonotone risks yields by Kusuoka (2001) a representation for all such functionals as a distorted expectation.

- More specifically there must then exist a concave distribution function $\Psi(u)$ for $0 \leq u \leq 1$, with $\Psi(0) = 0$, $\Psi(1) = 1$ such that for all X we have

$$b(X) = \int_{-\infty}^{\infty} x d\Psi(F_X(x)).$$

- Such distorted expectations were proposed as models for bid prices in Cherny and M. (2010).

- A distorted expectation is an expectation under a change of measure via

$$b(X) = \int_{-\infty}^{\infty} x\Psi'(F_X(x))dF_X(x)$$

with the measure change $\Psi'(F_X(x))$ depending on X and hence the nonlinearity.

- With a view to reweighting losses and discounting gains whereby Ψ' tends to infinity and zero as u tends to zero or unity, Cherny and M. (2009) proposed the distortion termed minmaxvar and defined by

$$\Psi(u) = 1 - (1 - u^{\frac{1}{1+\gamma}})^{1+\gamma}.$$

  The computations conducted in this paper employ this distortion.

- It is critical to note that when there is no distortion being applied and $\Psi'(u) = 1$ we recover the expectation and the bid equals the ask.

- With a distortion the reweighting upwards of losses and downwards of gains forces the bid price to fall below the expectation.

- Similar considerations force the ask price to be above the expectation.

Distortions and Choquet Capacities

- One may relate to any such distortion Ψ a Choquet capacity $c(A)$ (Choquet (1954)) defined via
$$c(A) = \Psi(Q^*(A))$$
for every $A \in \mathcal{F}$.

- It is shown in the appendix that c defined this way is a Choquet capacity.

- One may also define a Choquet capacity ν on $\mathbb{R}$ by
$$\nu(A) = \Psi\left(Q^*\left(X \in A\right)\right).$$

- The distorted expectation for the bid price is the Choquet type integral
$$-\int_{-\infty}^{0} \nu\left(X \leq y\right) dy + \int_{0}^{\infty} [1 - \nu\left(X \leq y\right)] dy.$$

Given the wide use of Choquet capacities in numerous contexts, it is noteworthy to observe that the bid pricing functional proposed under law invariance and linearity under comonotonicity is a Choquet integral.

The Test Measures for Distortions

- Cherny and M. (2010) show that the set of measures supporting acceptability consists of all distribution functions on the unit interval dominated pointwise above by the distortion.

- The connection with Choquet capacities provides an alternative demonstration of the set of supporting measures $\mathcal{N}$.

Some Static Applications

- M. (2009) employs the static bid price to define capital requirements and monitor leverage.

- M. (2010) determines option hedges for complicated claims written on many underliers with a view to minimizing the ask price of the unhedged risk.

- Eberlein and M. (2012) use these methods to determine capital requirements for the major US banks at the end of 2008 along with determining the value of the limited liability option to put losses back into the economy.

- Carr, M. and Vicente Alvarez (2011) advocate capital requirements as the difference of ask and bid prices.

- Eberlein, M. and Schoutens (2012) relate this capital requirement to risk weighted assets as defined in the Basel accords.

- Cherny and M. (2010) also estimate stress levels of distortions from market bid and ask price quotes of put and call options.

- M. and Schoutens (2011a) study clientele effects on optimal debt in the absence of tax advantages to debt via an application of two price economy accounting.

- M. and Schoutens (2011b) apply the static two price theory to the valuation of contingent capital notes.

- M. (2011) models risk weighted assets with these methods for pricing contingent capital notes.

- M. and Schoutens (2012a) study the equlibrium of two price economies trading structured notes.

- Eberlein, Gehrig and M. (2012) show how valuing liabilities at ask prices mitigates the level of profits associated with debt valuation adjustments (DVA).

The Two Price Economy in Discrete Time

- Consider now a discrete time economy with the uncertainty evolution described by a finite state Markov chain.

- For computational purposes and model calibrations one may employ Markov chain approximations to more general processes as described in Mijatović and Pistorius (2009).

- Following Cohen and Elliott (2010) we may view the Markov chain $(X_t, t = 1, \cdots, T)$ as taking values in the unit vectors of N-dimensional space $\mathbb{R}^N$, i.e.

$$X_t \in \{e_1, e_2, \cdots, e_N\},$$

with $e_i = (0, 0, .., 0, 1, 0, ..., 0)' \in \mathbb{R}^N$.

- The price of a stock S_t for example could then be modeled as ,

$$S_t = (e^{x_1}, e^{x_2}, \ldots, e^{x_N})\, X_t,$$

where the $x_i' s$ are the N possible values for the logarithm of the stock price at each time step.

- The chain is described by T transition probability matrices that could be time dependent.

- Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \le t \le T}, Q^*\right)$ be the filtered probability space generated by some risk neutral process X.

- Let C be a terminal cash flow known at time T.

- The set of all terminal cash flows to be valued may be taken to be a subset $\mathcal{C}$, $\mathcal{C} \subset L^2\left(\mathcal{F}_T\right)$.

- Anticipating the nonlinearity of bid and ask pricing operators we follow Cohen and Elliott (2010) in first defining a system of dynamically consistent nonlinear expectation operators.

Nonlinear Conditional Expectation Operators

- A nonlinear, dynamically consistent system of conditional expectations is a set of operators

$$\mathcal{E}(.|\mathcal{F}_t) : L^2(\mathcal{F}_T) \rightarrow L^2(\mathcal{F}_t)$$

satisfying the following four properties.

  - For any $C, C' \in \mathcal{C}\mathcal{E}(C|\mathcal{F}_t) \geq \mathcal{E}(C'|\mathcal{F}_t) \quad Q^*-a.s.$
  - whenever $C \geq C' \quad Q^* - a.s.$ with equality iff $C = C' \quad Q^* - a.s.$
  - $\mathcal{E}(C|\mathcal{F}_t) = C \quad Q^* - a.s.$ for any $\mathcal{F}_t$-measurable C.
  - $\mathcal{E}(\mathcal{E}(C|\mathcal{F}_t)|\mathcal{F}_s) = \mathcal{E}(C|\mathcal{F}_s) \quad Q^* - a.s.$ for any $s \leq t$.

– For any $A \in \mathcal{F}_t, \quad \mathbf{1}_A \mathcal{E}\left(C|\mathcal{F}_t\right) = \mathcal{E}\left(\mathbf{1}_A C|\mathcal{F}_t\right)$ $Q^* - a.s.$

- The dynamically consistent system of bid and ask prices will be respectively concave and convex systems of nonlinear expectation operators. Furthermore they are dynamically translation invariant in the sense that for any $C \in \mathcal{C}$ and any $q \in \mathcal{F}_t$

$$\mathcal{E}\left(C + q|\mathcal{F}_t\right) = \mathcal{E}\left(C|\mathcal{F}_t\right) + q.$$

NLE and BSDE

- The construction of such dynamically translation invariant nonlinear expectations on a finite state Markov chain is linked to the solution of backward stochastic difference equations by Theorem 5.1 of Cohen and Elliott (2010).

- We denote a nonlinear expectation by $\mathcal{E}$ while a classical linear expectation is denoted by E.

- To describe these equations and their solution for a finite state Markov chain we introduce the martingale difference process

$$M_t = X_t - E\left[X_t|\mathcal{F}_{t-1}\right] \in \mathbb{R}^N.$$

- A backward stochastic difference equation ($BSDE$) for our purposes is defined by a real-valued driver

$F(\omega, u, Y_u, Z_u)$ where Y is a real-valued stochastic process adapted to the Markov chain, Z is an $\mathbb{R}^N$-valued stochastic process, and F is a progressively measurable map

$$F : \Omega \times \{0, \ldots, T\} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$$

which is essentially bounded.

- A $BSDE$ based on M with driver F and terminal value C is an equation of the form

$$Y_t - \sum_{t \le u < T} F(\omega, u, Y_u, Z_u) + \sum_{t \le u < T} Z_u^{'} M_{u+1} = C,$$

where C is an essentially bounded $\mathcal{F}_T$-measurable random variable, with Y and Z the unknowns.

- In difference form we may write

$$Y_t - F(\omega, t, Y_t, Z_t) + Z_t^{'} M_{t+1} = Y_{t+1}$$

and taking $\mathcal{F}_t$-conditional linear expectations we see that

$$Y_t = E\left[Y_{t+1}|\mathcal{F}_t\right] + F(\omega, t, Y_t, Z_t),$$

and so we solve for Y_t backwards by evaluating the conditional expectation of Y_{t+1} and adding the penalty given by the driver.

- By Theorem 5.1 of Cohen and Elliott (2010) for the construction of a nonlinear dynamically consistent and translation invariant conditional expectation the driver is independent of Y_t and is itself the nonlinear expectation of the zero mean one step ahead risk or

$$F\left(\omega, t, Y_t, Z_t\right) = \mathcal{E}\left(Z_t^{'} M_{t+1}|\mathcal{F}_t\right).$$

Distortion Based Drivers

- We define $Z_t^{'}$ by

$$Z_t^{'} M_{t+1} = Y_{t+1} - E\left[Y_{t+1} | \mathcal{F}_t\right].$$

- For computing bid prices, denoted Y_t^b, the driver is

$$\begin{aligned} F_b(\omega, t, Y_t^b, Z_t) &= b(Z_t^{'} M_{t+1}) \\ &= b\left(Y_{t+1}^b - E\left[Y_{t+1}^b | \mathcal{F}_t\right]\right), \end{aligned}$$

- while for ask prices Y_t^a, the driver is

$$\begin{aligned} F_a(\omega, t, Y_t^a, Z_t) &= a\left(Z_t^{'} M_{t+1}\right) \\ &= a\left(Y_{t+1}^a - E\left[Y_{t+1}^a | \mathcal{F}_t\right]\right). \end{aligned}$$

  The functions b, a are one step ahead distorted expectations.

Bid and Ask as Sub and Super Martingales

- We may observe from this construction, recalling that bid prices lie below expectations while ask prices are above them, that the bid price process satisfies

$$\begin{aligned} Y_t^b &= E\left[Y_{t+1}^b | \mathcal{F}_t\right] + b\left(Y_{t+1}^b - E\left[Y_{t+1}^b | \mathcal{F}_t\right]\right) \\ &\leq E\left[Y_{t+1}^b | \mathcal{F}_t\right] \end{aligned}$$

whereas for the ask price process we have

$$\begin{aligned} Y_t^a &= E\left[Y_{t+1}^a | \mathcal{F}_t\right] + a\left(Y_{t+1}^a - E\left[Y_{t+1}^a | \mathcal{F}_t\right]\right) \\ &\geq E\left[Y_{t+1}^a | \mathcal{F}_t\right]. \end{aligned}$$

- Hence dynamically consistent bid prices are submartingales while ask prices are supermartingales. This property is preserved in the continuous time formulation.

Discrete Time Applications

- M., Pistorius and Schoutens (2011) price a variety of structured products in a context where transition probabilities are estimated to meet marginal densities extracted from option prices.

- M. and Schoutens (2012b) investigate the effect of the discrete tenor on such pricing sequences.

- M. (2010) implements dynamic hedging modified to minimize capital requirements defined as the difference between dynamically consistent ask and bid price sequences as advocated in Carr, M. and Vicente Alvarez (2011).

- M., Wang and Heckman (2011) apply these methods to the pricing of insurance loss liabilities, the determination of capital minimizing reinsurance attachment points and the financial hedging of securitized insurance loss exposures.

Continuous time modeling of bid and ask price functionals

- The static and discrete time models for two price economies, as useful as they are in a variety of contexts, fall short of providing valuations for claims delivered at arbitrary time points in the future.

- It is like an option pricing theory constrained to maturities being an integer multiple of some tenor.

- The objective now is to extend the theory of two price economies to continuous time.

- This leads us naturally to dynamically consistent nonlinear pricing in continuous time.

- Fortunately much progress has already been made here in the construction of $\mathcal{G}$-expectations by Peng (2006). Our task reduces to describing the $\mathcal{G}$ in our application of $\mathcal{G}$-expectation.

The Continuous Time Plan

- We proceed in stages.
- First we introduce the context in which we work by reviewing the construction of expectations that are to be lifted to $\mathcal{G}$-expectations.
- Next we present the general approach of $\mathcal{G}$-expectations that we will follow.
- A presentation of two particular nonlinear $\mathcal{G}$-operators follows.
- The operators are related to the distortions employed in the static and discrete time cases.
- Finally we study the Doob-Meyer decomposition of the bid price under the linear expectation operator and shows that bid prices are submartingales.

The underlying uncertainty and expectation operator

- The underlying uncertainty is given by a pure jump Lévy process $(X_t, 0 \leq t \leq T)$.

- More generally one could take an underlying Hunt (1966) process.

- The applications made use of such a process by allowing the parameters of the jump compensator to depend mildly on the current level of the process.

- However, for the theoretical discussion such a dependence is not necessary.

- The pure jump Lévy process is specified by the drift term α and the Lévy measure $k(y)dy$ defined for $y \neq 0$.

- An example that we shall work with is the variance gamma process (M. and Seneta (1990), M. Carr and Chang (1998)) for which the Lévy density is given in CGMY format (Carr, Geman, M. and Yor (2002)) by

$$k(y) = \frac{C}{|y|} \left(\exp\left(-G|y|\right) \mathbf{1}_{y<0} + \exp\left(-M|y|\right) \mathbf{1}_{y>0} \right).$$

- In general the Lévy measure is not a finite measure but satisfies

$$\int_{-\infty}^{\infty} \left(y^2 \wedge 1 \right) k(y) dy < \infty.$$

- We shall work with processes satisfying the stronger condition

$$\int_{-\infty}^{\infty} y^2 k(y) dy < \infty.$$

- In such cases the infinitesimal generator $\mathcal{L}$ of the process is given by

$$
\begin{aligned}
& \mathcal{L}(u) \\
= \; & \alpha u_x(x,t) \\
& + \int_{-\infty}^{\infty} \left(u(x+y,t) - u(x,t) - u_x(x,t)y \right) k(y) dy.
\end{aligned}
$$

Claim Valuation Functional

- Now let $u(x,t)$ be the time zero financial value when $X(0) = x$, of a claim paying $\phi(X_t)$ at time t.

- The function $u(x,t)$ for a constant interest rate of r, is the solution of the partial integro-differential equation

$$u_t = \mathcal{L}(u) - ru$$

subject to the boundary condition $u(x,0) = \phi(x)$.

- More formally

$$u(x,t) = E\left[e^{-rt}\phi(X_t)|X_0 = x\right],$$

with X_t the driving Lévy process.

- This linear expectation equation is what we shall generalize to a nonlinear partial integro-differential equation that will yield the bid and ask pricing functionals of our two price economy in continuous time.

The $\mathcal{G}$-expectation approach

- The infinitesimal generator $\mathcal{L}$ is a linear operator on u.

- Peng (2006) created $\mathcal{G}$-expectations defined as non-linear expectations that are unique viscosity solutions to nonlinear equations of the form

$$u_t = \mathcal{G}(u) \tag{1}$$

  for the boundary condition $u(x, 0) = \phi(x)$.

- The result is

$$u(x,t) = \mathcal{E}\left(\phi(X_t) | X_0 = x\right),$$

  where $\mathcal{E}$ is a dynamically consistent nonlinear expectation operator.

- The operator $\mathcal{G}$ is now a nonlinear operator. For the definition of $\mathcal{G}$-Brownian motion the specific operator $\mathcal{G}$ is given by

$$\mathcal{G}(a) = \frac{1}{2}\left(a^{+} - \sigma_0^2 a^{-}\right),\ 0 \le \sigma_0 \le 1,$$

  where $a^{+} = \max(a, 0)$ and $a^{-} = \max(-a, 0)$ and one solves the equation

$$u_t = \mathcal{G}\left(u_{xx}\right).$$

Nonlinear Bid Ask Operators

- The way to get nonlinear bid and ask price functionals described in the following section is to replace the linear operator $\mathcal{L}$ by a suitable nonlinear operator $\mathcal{G}$ and then to solve

  $$u_t = \mathcal{G}(u) - ru$$

  for $u(x, 0) = \phi(x)$. The solution of this equation is a financial bid price

  $$u(x, t) = b\left(\phi\left(X_t\right) | X_0 = x\right).$$

Distortions for G-expectations

- Concave distortions are applied to distribution functions of random variables exaggerating their low states and discounting their high states.

- The role of the Lévy measure in the expression for $\mathcal{L}(u)$ is not unlike that of a probability as it is the limit of probabilities.

- However, the Lévy measure does not integrate to unity whereas distortions operate on the unit interval.

- Our first suggestion is to rewrite the integral expression in $\mathcal{L}(u)$ with the objective of forcing a probability measure into view.

- We may equivalently write for the integral in $\mathcal{L}$ the expression

$$\int_{-\infty}^{\infty} (u(x+y,t) - u(x,t) - u_x(x,t)y) \times \frac{\int_{-\infty}^{\infty} y^2 k(y)dy}{y^2} \times g(y)dy$$

where we now write

$$g(y) = \frac{y^2 k(y)}{\int_{-\infty}^{\infty} y^2 k(y)dy}.$$

- The function $g(y)$ is positive, integrates to unity and thus is a probability density.

- In the altered expression $g(y)$ is employed to compute the expectation of the variable which is random in y for a given x, t and defined by

$$\begin{aligned} & Y_{x,t}(y) \\ = \; & (u(x+y,t) - u(x,t) - u_x(x,t)y) \times \\ & \frac{\int_{-\infty}^{\infty} y^2 k(y)dy}{y^2}. \end{aligned}$$

- We may equivalently define the distribution function

$$F_{Y_{x,t}}(v) \;=\; \int_{A(x,t,v)} g(y)dy$$
$$A(x,t,v) \;=\; \left\{y|Y_{x,t}(y) \leq v\right\}$$

and then write the integral as

$$\int_{-\infty}^{\infty} v dF_{Y_{x,t}}(v).$$

- Now we consider the distorted expectation

$$\int_{-\infty}^{\infty} v d\Psi(F_{Y_{x,t}}(v))$$

which agrees with the integral

$$-\int_{-\infty}^{0} \Psi\left(P^g(Y_{x,t} \leq v)\right) dv + \int_{0}^{\infty} \left[1 - \Psi\left(P^g(Y_{x,t} \leq v)\right)\right] dv,$$

where P^g indicates that we evaluate probability under the quadratic variation scaled density $g(y)$.

- We define

$$\mathcal{G}_{QV}(u) = \alpha u_x - \int_{-\infty}^{0} \Psi\left(P^g(Y_{x,t} \leq v)\right) dv + \int_{0}^{\infty} \left[1 - \Psi\left(P^g(Y_{x,t} \leq v)\right)\right] dv$$

and solve then for the nonlinear bid price.

- The ask price is the negative of the bid for the negative cash flow.

- We note that scaling by quadratic variation is a way of ignoring or truncating the small moves.

- Another way to ignore these jumps given that the function being evaluated is of order $O(y^2)$ is to consider the integral

$$\int_{|y|\geq\varepsilon} (u(x+y,t) - u(x,t) - u_x(x,t)y) \times k(y)dy,$$

  where we effectively truncate a small neighbourhood of zero.

- We may now rewrite and force the probability $h(y)$ as

$$\int_{|y|\geq\varepsilon} (u(x+y,t) - u(x,t) - u_x(x,t)y) \times \left(\int_{|y|\geq\varepsilon} k(y)dy\right) h(y)dy,$$

where we now define the density

$$h(y) = \frac{k(y)}{\left(\int_{|y|\geq\varepsilon} k(y)dy\right)} \mathbf{1}_{|y|\geq\varepsilon}.$$

- The random variable in y for fixed x, t is now

$$\begin{aligned}\widetilde{Y}_{x,t} &= (u(x+y,t) - u(x,t) - u_x(x,t)y) \times \\ &\quad \left(\int_{|y|\geq\varepsilon} k(y)dy\right)\end{aligned}$$

and the relevant nonlinear operator denoted $\mathcal{G}_{NL}$ for normalized Lévy is

$$\begin{aligned}\mathcal{G}_{NL}(u) &= \alpha u_x - \int_{-\infty}^{0} \Psi\left(P^h(\widetilde{Y}_{x,t} \leq v)\right) dv \\ &\quad + \int_{0}^{\infty} \left[1 - \Psi\left(P^h(\widetilde{Y}_{x,t} \leq v)\right)\right] dv\end{aligned}$$

- We have thus defined two nonlinear partial integro-differential operators the solutions of which yield nonlinear bid prices for claims written on the terminal value of the Lévy process.

- These are the QV and NL approaches which ignore small jumps and induce a probability to distort.

- In our applications the nonlinear partial integro-differential equations are solved numerically for the value of contracts.

Bid prices as submartingales under the original linear expectation

- Consider, in the case of zero rates and quadratic variation scaling, the bid price at time t for the claim paying $\phi(X_T)$ at time T when the original Lévy process is at X_t.

- This bid price is given by
$$b_t = u(X_t, T - t)$$
where the function $u(x,t)$ solves
$$u_t = \mathcal{G}_{QV}(u)$$
for the boundary condition $u(x,0) = \phi(x)$.

- We observe in the appendix that for all functions $u(.,t)$ we have as a consequence of distortions the inequality
$$\mathcal{G}_{QV}(u(.,t)) \le \mathcal{L}(u(.,t)).$$

- We then develop the Doob-Meyer decomposition of b_t under the original expectation operator as

$$\begin{aligned} & b_t \\ = \; & b_0 + M_t \\ & + \int_0^t ds \left(\mathcal{L}(u(.,T-s))\right)(X_s) \\ & - \mathcal{G}_{QV}(u(.,t))(X_s)). \end{aligned}$$

The domination then establishes the submartingale property for the bid price under the original linear expectation.

- The proof follows on showing that

$$u\left(X_t, T-t\right) = u(X_0, T) + M_t + \int_0^t ds \left[\begin{array}{c} \mathcal{L}(u(.,T-s)(X_s) \\ -\frac{\partial}{\partial t} u(X_s, T-s) \end{array} \right].$$

- Here M_t denotes a martingale. We note some similarity of this equation with Proposition 2.1 in Kunita (1997).

- From these demonstrations one observes that the risk charge on a risk with distribution function $F_{x,t}(v)$ is given by

$$\int_{-\infty}^{\infty} (\Psi(F_{x,t}(v)) - F_{x,t}(v)) dv,$$

  and is strongly influenced by the concavity of the distortion.

Some illustrative valuations

- For numerical computations the underlying uncertainy may easily be taken to be more general than that of a Lévy process and one may entertain Hunt processes (Hunt (1966)).

- In the applications presented in the following, the underlying uncertainty is given by a pure jump Hunt process with space dependent jump compensators for the logarithm of the stock price at x.

- Let the arrival rate for a jump of size $y \in \mathbb{R}\backslash\{0\}$, when the log stock price is x, be $k(x, y)$.

- The function $k(x, y)$ is taken from the class of variance gamma (VG) Lévy measures and we then have

$$k(x, y) = \frac{C_x}{|y|} \left(\begin{array}{c} \exp\left(-G_x|y|\right) \mathbf{1}_{y<0} \\ +\exp\left(-M_x|y|\right) \mathbf{1}_{y>0} \end{array} \right).$$

- For a specification of the dependence of the VG parameters on the space variable the VG model is reparameterized with parameters a, v, q where a is the ratio of positive to negative variation, v is the level of finite variation in the symmetric process with exponential decay at $(G+M)/2$ and q is the quadratic variation of the symmetric process.

- Such a reparameterization allows us to model via a, mean reversion or momentum depending on how the rate of positive variation moves with the level of the stock.

- The behavior of v specifies peakedness of densities. Peakedness is greater at lower levels of v.

- The parameter q captures the behavior of volatility or quadratic variation.

A Specific Parameterization

- For the stock price ratio S/S_0 below .75 or above 1.25 the parameters are assumed to be constant.
- For the ratios of 0.75, 1.0 and 1.25 we specify the value of the three parameters and interpolate linearly in the interval $[0.75, 1.25]$.
- The computations presented are for the parameterization

S/S_0	.75	1	1.25
a	.4	.25	.5
v	.1	.16	.1
q	.02	.0126	.02

- For this parameterization of a Hunt process we present in Figures and the bid, ask and expectation for a $0.9, 1.1$ half year and one year strangle for a stock priced at unity for both the QV and NL pricing models where for NL, $\varepsilon = .001$.

- The distortion employed is minmaxvar at the stress level 0.1.

- We further report in the Figure the bid and ask prices and the common expectation for a Lévy model with a, v, q specification $0.3, 0.2, 0.02$.

The two ways of truncating small jumps are observed to be comparable and the next section considers the valuation of a derivatives book for just the QV specification.

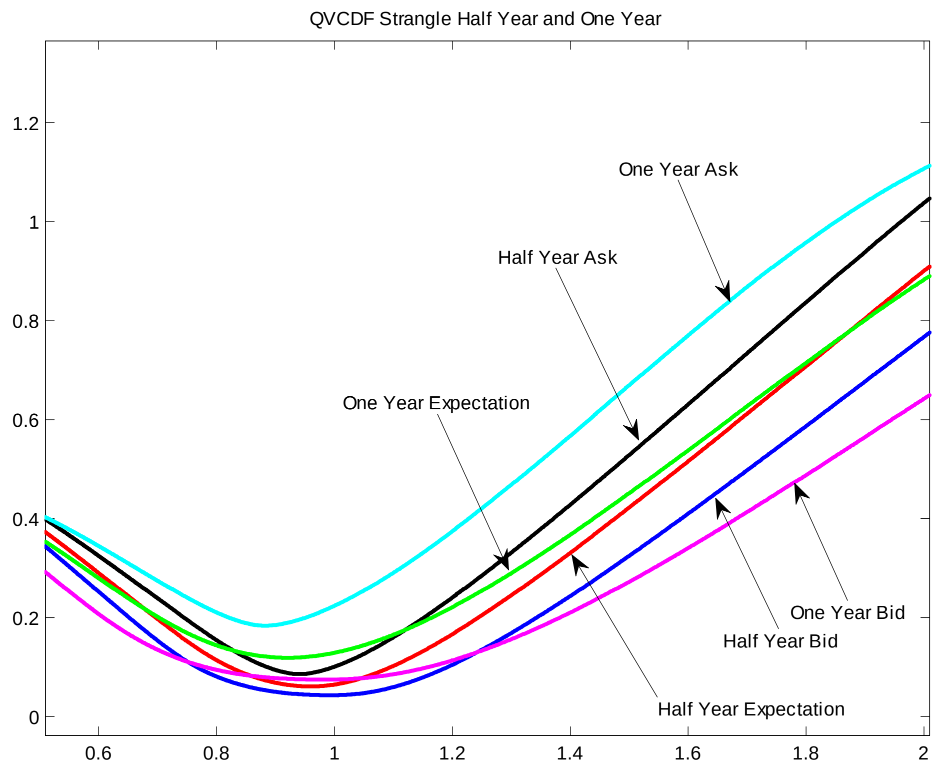

Figure 1: QVCDF Strangle at 6 months and one year

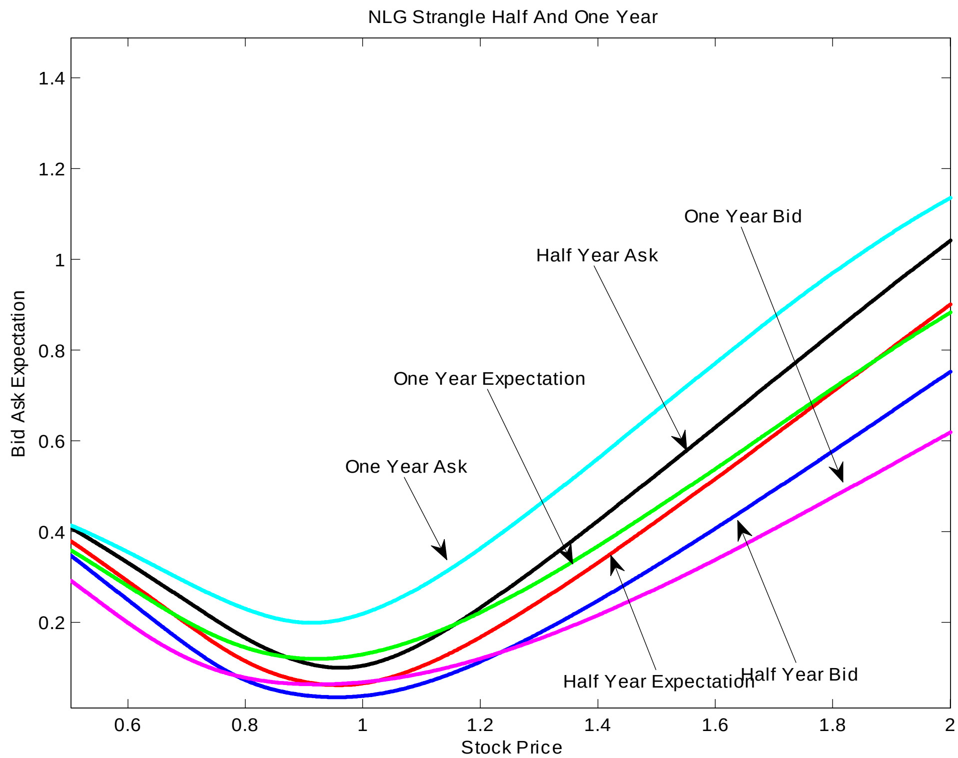

Figure 2: NLG strangle at 6 months and one year.

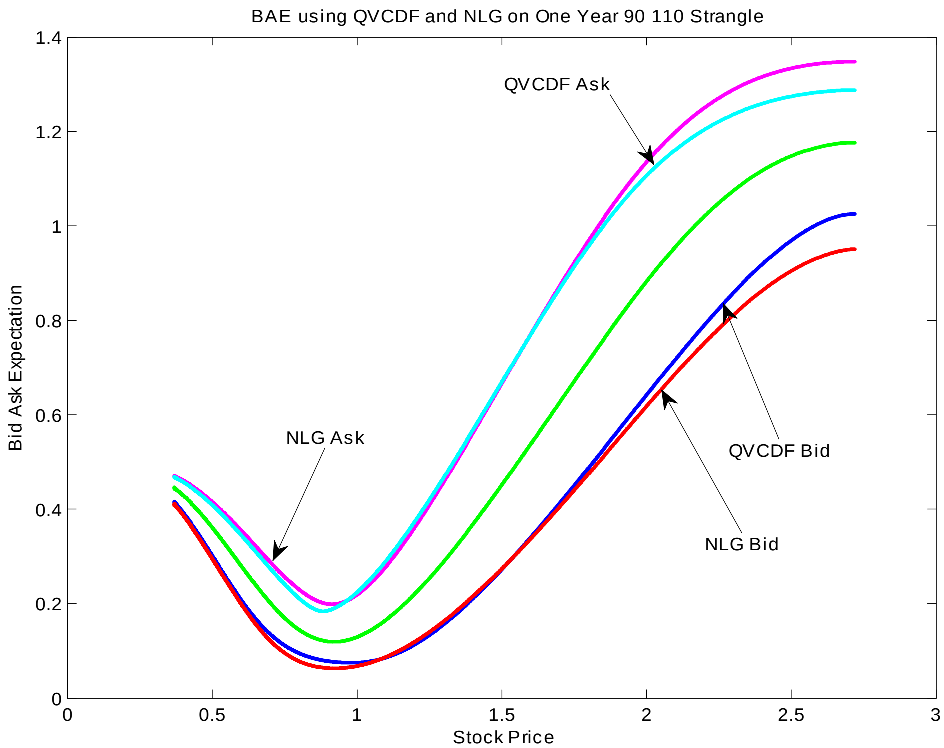

Figure 3: Comparison of QVCDF and NLG pricing on a one year 0.9 ,1.1 strangle at minmaxvar stress 0.1 for a, v, q at 0.3, 0.2, 0.02.

Completely monotone Basis Functions

- We also present in Figure the QV bid, ask and expected values for the basis functions $\exp(-a|x|)$, of the comonotone class considered earlier, for $a = 1$ and 2.

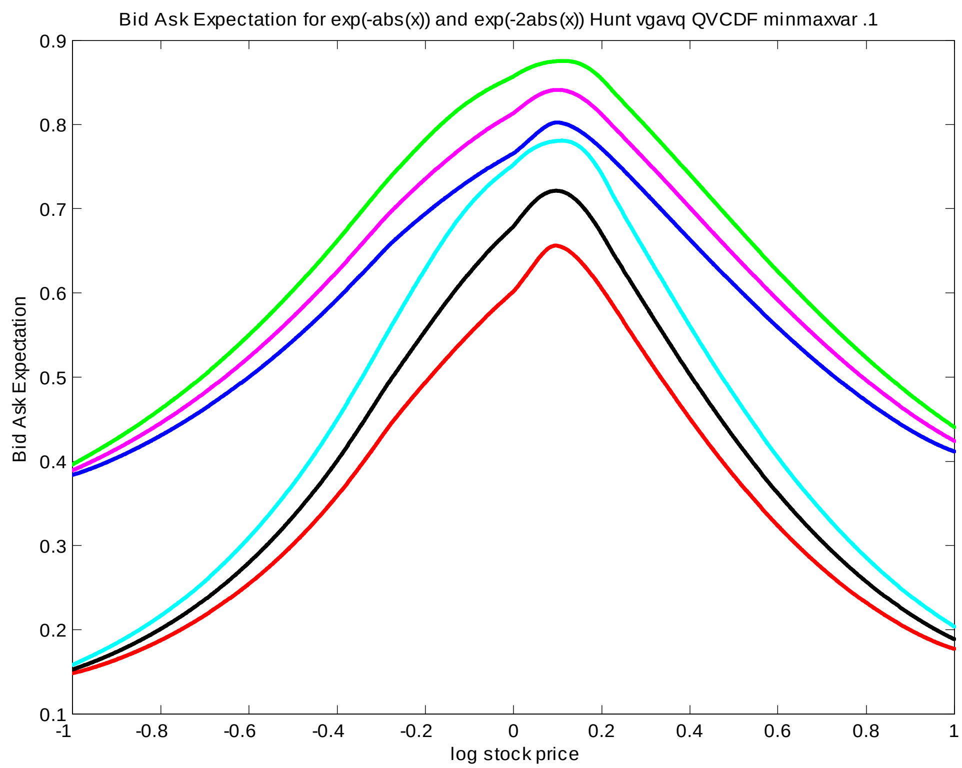

Figure 4: Nonlinear valuations of extreme points of the class of even, completely monotone functions.

Valuing a derivatives book

- Consider a book of derivatives on a single underlier with cash flows $\phi\left(x_i, t_j\right)$ at future dates t_j for $j = 1, \cdots, N$ that may be interpolated to build payout functions $\phi(x, t_j)$ and extrapolated as constant at the value at the nearest neighbour, out of the range of specified points.

- The nonlinear value of a derivatives book cannot be determined as the sum of the nonlinear values of each item as nonlinear values are not additive.

- In implementation we shall use an interpolated grid specification but in general we consider the nonlinear valuation $u(x, t)$ for

$$\sum_{t_j > t} \phi(x, t_j).$$

- To determine this valuation we define

$$v(x,t) = u(x, t_N - t)$$

and set

$$v^N(x,0) = \phi(x, t_N).$$

- We then solve in the interval $0 < s \leq t_N - t_{N-1}$

$$v_s^N = \mathcal{G}\left(v^N\right)$$

and define the solution $v^N(x, t_N - t_{N-1})$.

- We then define

$$v^{N-1}(x,0) = v^N(x, t_N - t_{N-1}) + \phi(x, t_{N-1})$$

and solve in the interval $0 < s \leq t_{N-1} - t_{N-2}$ the function

$$v_s^{N-1} = \mathcal{G}(v^{N-1}),$$

to define $v^{N-1}(x, t_{N-1} - t_{N-2})$.

- We then define

  $$v^{N-2}(x,0) = v^{N-1}(x, t_{N-1} - t_{N-2}) + \phi(x, t_{N-2})$$

  until we have computed $v^1(x, t_1)$ that is the value of the book.

- We then define

$$\begin{aligned} u(x,t) &= v^j(x, t - t_{j-1}), \\ t_{j-1} &\leq t \leq t_j,\ j = 1, \cdots, N. \end{aligned}$$

- By way of an example we take four cash flows at the four maturities, one, three, six and 12 months to meet target greek positions. The targeted greeks are

	1	3	6	12
gamma	.3597	.9248	-.1336	-.3902
vega	-.8316	-.0402	-.8600	-.3296
vanna	-.3612	-.1837	-.0621	.3270
volga	.0889	-.3950	.0720	.2055
skew	-.9292	-.3172	-.4930	.1531

- The cash flows accessed to meet these target greeks are displayed in Figure for the four maturities.

- The cash flows are from positions in 21 strikes for out of the money options at each maturity.

- The strikes range from 80 to 120 in steps of 2 dollars.

- The positions are determined to match the target greeks.

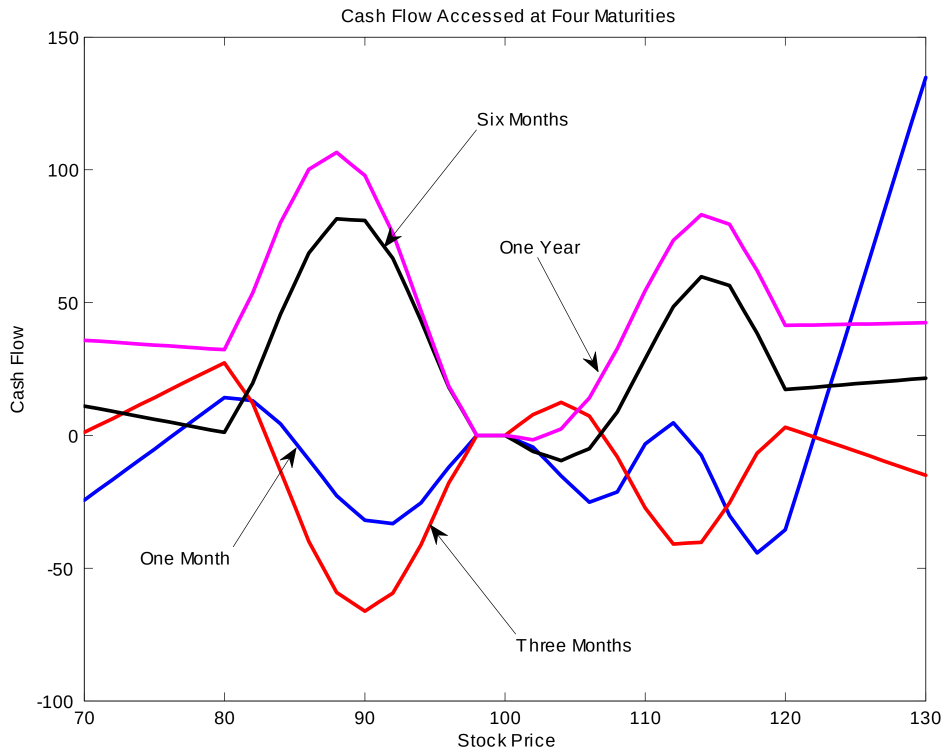

Figure 5: Cash flows accessed at four maturities

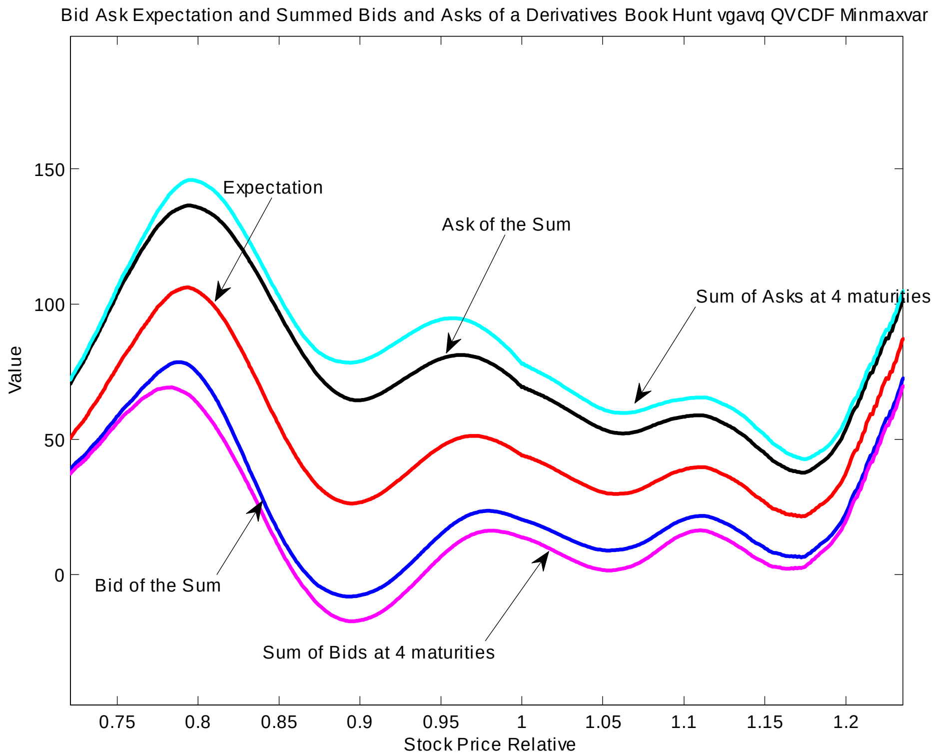

Figure 6: Bid, ask, expectation and the sum of four separate bid and ask prices for each of four maturities.

- For these cash flows we compute the expectation, as well as the bid and ask prices for the four maturities taken together and the sum of the bid and ask prices for the four maturities taken separately. The result is displayed in Figure.

- We see clearly the effect of nonlinear pricing on the bid and the ask with the gap between the ask and bid of the sum and the sum of the bid and ask prices.

Illustrative Calibration

- For monotone payoffs like put and call options one may price under an altered Lévy measure

- One may then approximate a Hunt process by a finite state Markov chain on non-uniform grid and tilt the transition rates where for a local VG evolution we have explicitly that

$$G(y) = \left(\frac{1}{G^2} + \frac{1}{M^2}\right)^{-1} \times \left[\begin{array}{c} \frac{(1+Gy)e^{-Gy}}{G^2} \mathbf{1}_{y>0} \\ + \left(\frac{1}{G^2} + \frac{1-(1+My)e^{-My}}{M^2}\right) \mathbf{1}_{y<0} \end{array} \right]$$

and

$$\widetilde{G}(y) = \left(\frac{1}{G^2} + \frac{1}{M^2}\right)^{-1} \times \left[\begin{array}{c} \left(\frac{(1+My)e^{-My}}{M^2}\right) \mathbf{1}_{y>0} \\ + \left(\frac{1}{M^2} + \frac{1-(1+Gy)e^{-Gy}}{G^2}\right) \mathbf{1}_{y<0} \end{array} \right]$$

- For data on SPX options for April 20 2009, the parameter estimates for a, v, q in the Hunt specification were

S	75	100	125
a	0.2462	0.4965	0.0023
v	0.3717	3.0583	0.0518
q	2.9893	0.1369	0.0122

  The stress parameter was 20 basis points. Figures and present the graphs for the fit to Bid and Ask prices.

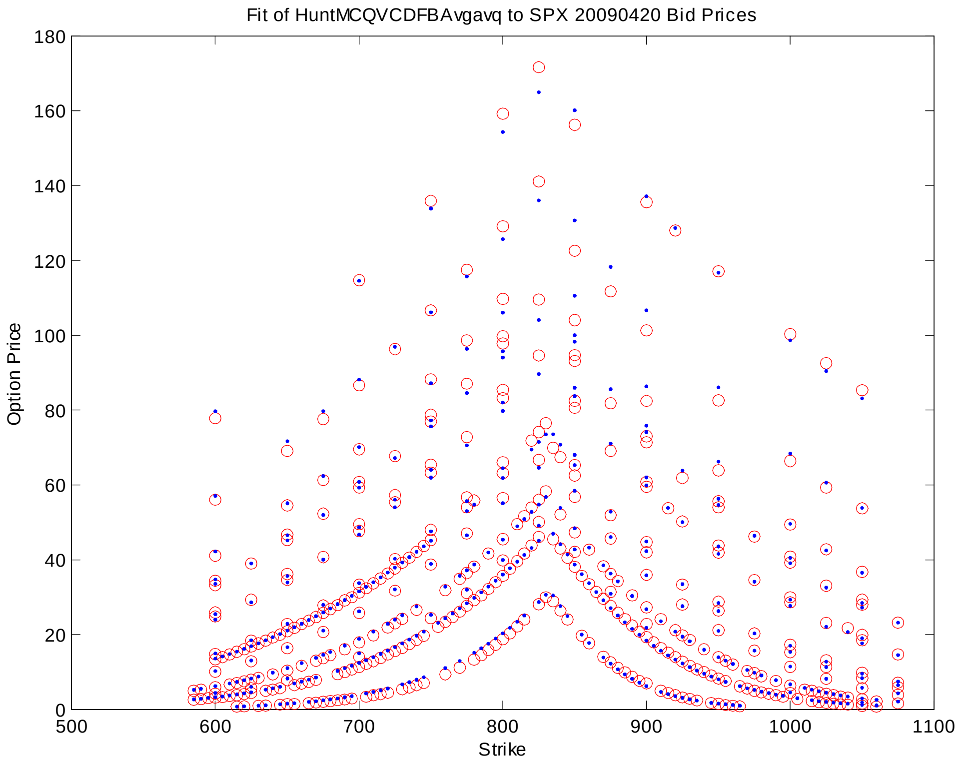

Figure 7: Graph of the Hunt process fit with VG a, v, q parameterization to SPX Bid prices for April 20 2009.

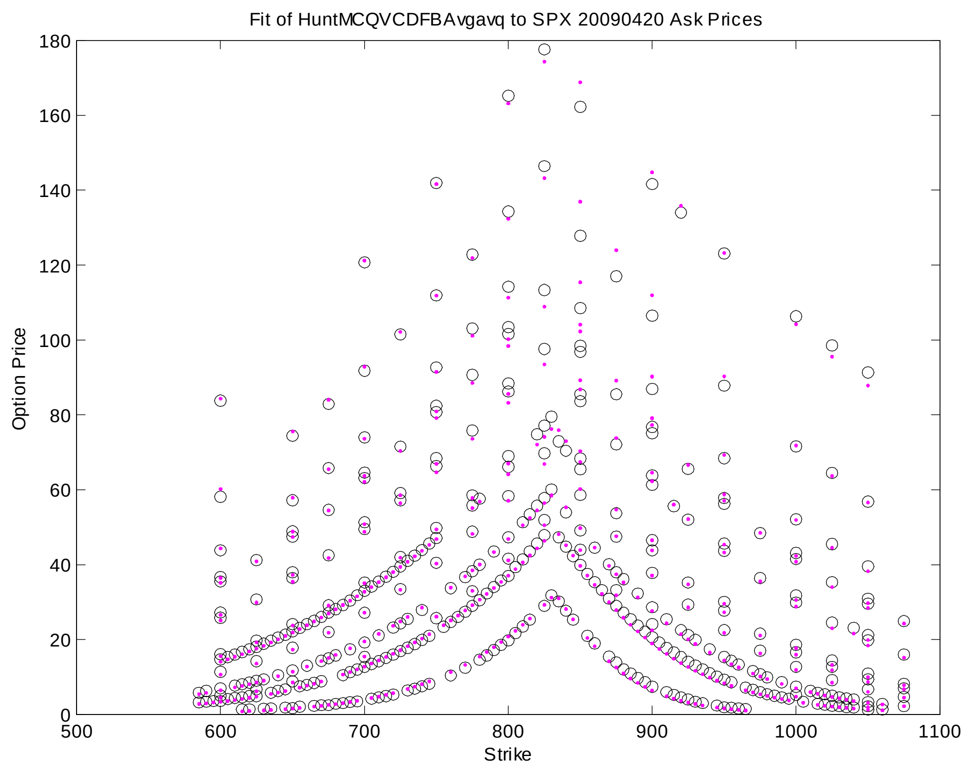

Figure 8: Graph of the Hunt process fit with VG a, v, q parameterization to SPX Ask prices for April 20 2009.