Estimates of certain large deviation probabilities for controlled semi-martingales

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Large deviation estimates for controlled semi-martingales

(1.1) $$dX_t = \beta(X_t)dt + \lambda(X_t)dW_t, \quad X_0 = x \in R^N,$$

W_t : M- dim. $\mathcal{F}_t$ B.M., $\lambda(x)$: $R^N \mapsto N \otimes M$, $\beta(x)$: $R^N \mapsto R^N$

(1.2) $$J(\kappa) := \varliminf_{T\to\infty} \frac{1}{T} \inf_{h.} \log P\left(\frac{1}{T} F_T(X_., h_.) \leq \kappa\right).$$

$$F_T(X_., h_.) = F_0 + \int_0^T f(X_s, h_s)ds + \int_0^T \varphi(X_s, h_s)^* dW_s$$

F_0: $\mathcal{F}_0$ - m'ble r.v., h_s: $\mathcal{F}_t$ - prog. m'ble, R^m-valued, $m, N \leq M$

$$f(x,h) := -\frac{1}{2}h^* S(x)h + h^* g(x) + U(x), \quad \varphi(x,h) = \delta(x)h,$$

$$S(x) : R^N \mapsto R^m \otimes R^m, \ g(x) : R^N \mapsto R^m, \ \delta(x) : R^N \mapsto R^M \otimes R^m,$$

Robust version of large deviation estimates

$$\left.\frac{dP^{\zeta}}{dP}\right|_{\mathcal{F}_T} := e^{\int_0^T \zeta_s^* dW_s - \frac{1}{2}\int_0^T |\zeta_s|^2 ds}$$

$$W_t^{\zeta} := W_t - \int_0^t \zeta_s ds \ : \ \text{B.M. under } P^{\zeta}$$

$$dX_t = \{\beta(X_t) + \lambda(X_t)\zeta_t\}dt + \lambda(X_t)dW_t^{\zeta}.$$

(1.3)

$$J_1(\kappa) := \varliminf_{T\to\infty} \frac{1}{T} \inf_{h_\cdot} \sup_{\zeta_\cdot} \log P^{\zeta}\left(\frac{1}{T}\{F_T(X_\cdot, h_\cdot) + \frac{\mu}{2}\int_0^T |\zeta_s|^2 ds\} \le \kappa\right).$$

Motivated examples ”Market model”

Riskless asset:

(1.4) $$dS^0(t) = r(X_t)S^0(t)dt, \quad S^0(0) = s^0.$$

Risky assets:

(1.5) $$\begin{cases} dS^i(t) = S^i(t)\{\alpha^i(X_t)dt + \sum_{k=1}^{n+m} \sigma_k^i(X_t)dW_t^k\}, \\ S^i(0) = s^i, \ \ i = 1, ..., m \end{cases}$$

Factors:

(1.6) $$\begin{cases} dX_t = \beta(X_t)dt + \lambda(X_t)dW_t, \\ X(0) = x \in R^n, \end{cases}$$

Total wealth:

$$V_t = \sum_{i=0}^{m} N_t^i S_t^i$$

N_t^i : Number of the shares

$h_t^i = \frac{N_t^i S_t^i}{V_t}$: Portfolio proportion $\quad i = 0, 1, 2, \ldots, m.$

$h_t = (h_t^1, \ldots, h_t^m)$

$$\frac{dV_t}{V_t} = r(X_t)dt + h(t)^*(\alpha(X_t) - r(X_t)\mathbf{1})dt + h(t)^*\sigma(X_t)dW_t,$$

$$\log V_T = \log V_0$$

$$+\int_0^T \{-\frac{1}{2}h_s^*\sigma\sigma^*(X_s)h_s + h_s^*\widehat{\alpha}(X_s) + r(X_s)\}dt + \int_0^T h_s^*\sigma(X_s)dW_s,$$

$$\widehat{\alpha}(x) = \alpha(x) - r(x)\mathbf{1}.$$

Asymptotics of the minimizing probability :

$$(1.7) \qquad J_0(\kappa) := \varliminf_{T\to\infty} \frac{1}{T} \inf_{h\in\mathcal{H}(T)} \log P(\frac{1}{T} \log V_T(h) \le \kappa).$$

concerns the problem of down-side risk minimization for the given target growth rate κ.

Setting $n+m=M$, $n=N$, and

$$f(x,h) = -\frac{1}{2} h^* \sigma\sigma^*(x) h + h^* \widehat{\alpha}(x) + r(x), \quad \varphi(x,h) = \sigma^*(x) h,$$

we arrive at the above problem (1.2) with $S(x) = \sigma\sigma^*(x) = \delta^*\delta(x)$.

Complete market case: $n = 0$, $m = N = M$,

Assume that SDE

$$dX_t^i = \{\alpha^i(X_t) - \frac{1}{2}(\sigma\sigma^*(X_t))^{ii}\} + \sum_{j=1}^m \sigma_j^i(X_t)dW_t^j, \qquad X_0^i = x^i$$

is given, and set $X_t^i = \log S_t^i$.

The solution to this SDE is regarded as "factors" and $S_t^i = e^{X_t^i}$ satisfies (1.5) with $s^i = e^{x^i}$. Namely,

$$\beta(x)^i = \alpha(x)^i - \frac{1}{2}(\sigma\sigma^*)^{ii}(x), \qquad \lambda(x) = \sigma(x),$$

Downside risk minimization

(1.8)

$$J_0(\kappa) = -\inf_{k\in(\chi_0'(-\infty),\kappa]} \sup_{\theta<0}\{\theta k - \chi_0(\theta)\}, \ \ \chi_0'(-\infty) < \kappa < \chi'(0-)$$

where

$$\chi_0(\theta) = \varliminf_{T\to\infty} \frac{1}{T} \inf_{h_\cdot} \log E[e^{\theta \log V_T(h)}], \qquad \theta < 0. \tag{1.9}$$

Duality relationship (1.8) between downside risk minimization (1.7) and risk-sensitive portfolio optimization over large time (1.9) is shown for several models. In the proofs, analysis of the H-J-B equation of ergodic type plays a crucial role.

cf. Hata - N. - Sheu '10, AAP; Hata-Sekine '10, AMO; N. '11, QF ; Hata '11 APFM; N. '12 AAP

Upside chance maximization

$$(1.10)\qquad J_u(\kappa) = \sup_{h_\cdot} \underline{\lim}_{T\to\infty} \frac{1}{T} \log P(\frac{1}{T}\log V_T(h) \geq \kappa)$$

$$(1.11)\qquad \chi_+(\theta) := \sup_{h\in\mathcal{A}} \overline{\lim}_{T\to\infty} \frac{1}{T} \log E[e^{\theta \log V_T(h)}], \quad 0 < \theta < 1$$

The arguments are rather similar to the proof of the large deviation principle without control parameter. Indeed, "Gärtner-Ellis" theorem holds. Namely, If $\chi_+(\theta)$ is differentiable with respect to $\theta < \theta^*$ and $\lim_{\theta\to\theta^*} \chi'_+(\theta) = \infty$ for some $0 < \theta^* \leq 1$, then, for $\chi'_+(0+) < \kappa < \infty$, one can obtain

$$J_u(\kappa) = -\inf_{k\in[\kappa,\infty)} \sup_{\theta\in[0,\theta^*)} \{\theta k - \chi_+(\theta)\} = -\sup_{\theta\in[0,\theta^*)} \{\theta\kappa - \chi_+(\theta)\}$$

cf. Pham '03; Hata-Sekine '05, Hata-Iida '06; Sekine '06 Hata-Sekine '10; Knispel '12, etc,..

Relationship between downside risk minimization and asymptotic arbitrage

Note that

$$\inf_h P(\frac{1}{T}\log V_T(h) \le \kappa) = 1 - \sup_h P(\frac{1}{T}\log\ V_T(h) > \kappa)$$

and that we are looking at the asymptotic behavior for $\chi'(-\infty) < \kappa < \chi'(0-)$,

$$\inf_h P(\frac{1}{T}\log V_T(h) \le \kappa) \sim e^{-TJ_0(\kappa)}$$

which is close to 0, and hence $\sup_h P(\frac{1}{T}\log\ V_T(h) > \kappa)$ is close to 1. Thus, it implies the relationship between downside risk minimization and "asymptotic arbitrage" discussed by Föllmer-Schachermayer .

Large deviation estimates for controlled semi-martingales

$$dX_t = \beta(X_t)dt + \lambda(X_t)dW_t, \quad X_0 = x \in R^N, \tag{1.1}$$

$$J(\kappa) := \underline{\lim}_{T\to\infty} \frac{1}{T} \inf_{h.} \log P\left(\frac{1}{T}F_T(X_., h_.) \le \kappa\right). \tag{1.2}$$

$$F_T(X_., h_.) = F_0 + \int_0^T f(X_s, h_s)ds + \int_0^T \varphi(X_s, h_s)^* dW_s$$

$$f(x,h) := -\frac{1}{2}h^* S(x)h + h^* g(x) + U(x), \quad \varphi(x,h) = \delta(x)h,$$

$$S(x) : R^N \mapsto R^m \otimes R^m, \;\; g(x) : R^N \mapsto R^m, \;\; \delta(x) : R^N \mapsto R^M \otimes R^m,$$

Risk-sensitive control and its H-J-B equation

Assume that $F_0 = 0$ and consider

$$\hat{\chi}(\theta) = \underline{\lim}_{T\to\infty} \frac{1}{T} \inf_{h\in\mathcal{A}(T)} J(x;h;T), \quad \theta < 0, \tag{2.1}$$

where

$$J(x;h;T) = \log E[e^{\theta\{\int_0^T f(X_s,h_s)ds + \int_0^T \varphi(X_s,h_s)^* dW_s\}}], \tag{2.2}$$

and h ranges over the set $\mathcal{A}(T)$ of all admissible investment strategies defined by

$$\begin{aligned}
\mathcal{A}(T) &= \{h \in \mathcal{H}(T); E[e^{\theta\int_0^T h_s^*\delta^*(X_s)dW_s - \frac{\theta^2}{2}\int_0^T h_s^*\delta^*\delta(X_s)h_s ds}] = 1\}.\\
\mathcal{H}(T) &= \{h; h : [0,T]\times R^N \mapsto R^m;\ \text{Borel},\ |h(t,x)| \le C(1+|x|^l),\\
&\qquad h(t,X_t)\ \text{is progressively m'ble}\}
\end{aligned}$$

Then, we shall see that (2.1) could be considered the dual problem to (1.2).

Assumptions

(2.3)

$\lambda,\ \beta,\ S,\ g,\ \delta$ are smooth and globally Lipschitz, U is smooth and bounded below

$|U(x)|,\ |DU| \leq M_1|x|^2 + M_2$

(2.4) $$c_0\delta^*\delta(x) \leq S(x) \leq c_1\delta^*\delta(x), \quad x \in R^N,\ c_0,\ c_1 > 0$$

(2.5) $$\delta^*\delta(x) \geq c_\delta I, \quad c_\delta > 0$$

(2.6) $$c_2|\xi|^2 \leq \xi^*\lambda\lambda^*(x)\xi \leq c_3|\xi|^2,\ c_2,\ c_3 > 0,\ \xi \in R^n,$$

Note that, when setting

$$Q_\theta := S(x) - \theta\delta^*\delta(x), \quad \theta < 0,$$

Q_θ satisfies

$$(c_0 - \theta)\delta^*\delta(x) \le Q_\theta(x) \le (c_1 - \theta)\delta^*\delta(x) \tag{2.7}$$

and

$$\theta Q_\theta^{-1}(x) \le \frac{\theta}{c_1 - \theta}(\delta^*\delta(x))^{-1}, \quad \frac{\theta}{c_0 - \theta}(\delta^*\delta(x))^{-1} \le \theta Q_\theta^{-1}(x) \tag{2.8}$$

Moreover,

$$\frac{c_0}{c_0 - \theta} I \le I + \theta\delta Q_\theta^{-1}\delta^* \le I \tag{2.9}$$

holds. Indeed, (2.7) follows directly from (2.4) and thus (2.8) is obtained from (2.7). The lefthand side of (2.9) is seen since

$$\frac{\theta}{c_0 - \theta} I \le \frac{\theta}{c_0 - \theta}\delta(\delta^*\delta)^{-1}\delta^* \le \theta\delta Q_\theta^{-1}\delta^*,$$

which follows from (2.8). The right hand side of (2.9) is obvious.

Risk-sensitive control

(2.10)

$$v_*(t,x;T) = \inf_{h_\cdot \in \mathcal{A}(T-t)} \log E[e^{\theta\{\int_0^{T-t} f(X_s,h_s)ds + \int_0^{T-t}\varphi(X_s,h_s)^* dW_s\}}].$$

Under

$$P^h(A) = E[e^{\theta \int_0^T h_s^* \delta^*(X_s)dW_s - \frac{\theta^2}{2}\int_0^T h_s^*\delta^*\delta(X_s)h_s ds} : A],$$

X_t satisfies

$$dX_t = \{\beta(X_t) + \theta\lambda\delta(X_t)h_t\}dt + \lambda(X_t)dW_t^h, \quad X_0 = x$$

with B. M. W_t^h defined by

$$W_t^h := W_t - \gamma \int_0^t \delta(X_s)h_s ds$$

(2.11) $\quad v_*(t,x;T) = \inf_{h_\cdot \in \mathcal{A}(T)} \log E^h[e^{\theta \int_0^{T-t}\{f(X_s,h_s) + \frac{\theta}{2}h_s^*\delta^*\delta(X_s)h_s\}ds}]$

The H-J-B equation :

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 v] + \frac{1}{2}(Dv)^*\lambda\lambda^* Dv \\ \quad + \inf_h\{[\beta + \theta\lambda\delta h]^* Dv + \theta f(x,h) + \frac{\theta^2}{2}h^*\delta^*\delta(x)h\} = 0, \\ v(T,x) = 0 \end{cases}$$

which is written as

$$(2.12) \qquad \begin{cases} \frac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 v] + \beta_\theta^* Dv + \frac{1}{2}(Dv)^*\lambda N_\theta \lambda^* Dv \\ \qquad + \frac{\theta}{2}g^* Q_\theta^{-1} g + \theta U = 0, \\ v(T,x) = 0, \end{cases}$$

where

$$\beta_\theta = \beta + \theta\lambda\delta Q_\theta^{-1} g, \quad N_\theta = I + \theta\delta Q_\theta^{-1}\delta^*, \quad Q_\theta = S - \theta\delta\delta^*.$$

Note that

$$(c_0 - \theta)\delta^*\delta(x) \leq Q_\theta(x) \leq (c_1 - \theta)\delta^*\delta(x) \tag{2.7}$$

and that

$$\frac{c_0}{c_0 - \theta} I \leq N_\theta = I + \theta\delta Q_\theta^{-1}\delta^* \leq I \tag{2.9}$$

Analytical result

Under the assumptions (2.3) - (2.6) H-J-B equation (2.12) has a solution such that

$$v(t,x) \le K_0$$

$$v,\ \frac{\partial v}{\partial t},\ \frac{\partial v}{\partial x_i},\ \frac{\partial^2 v}{\partial x_i \partial x_j}, \in L^p(0,T;L^p_{loc}(R^n))$$

$$\frac{\partial v}{\partial t} \ge -C$$

$$\frac{\partial^2 v}{\partial^2 t},\ \frac{\partial^2 v}{\partial x_i \partial t},\ \frac{\partial^3 v}{\partial x_i \partial x_j \partial x_k},\ \frac{\partial^3 v}{\partial x_i \partial x_j \partial t} \in L^p(0,T;L^p_{loc}(R^n))$$

$$|Dv|_r^2 + \frac{(c_0-\theta)(1+c)}{c_0 c_2}\left(\frac{\partial v}{\partial t} + C\right) \le c'(|DN_\theta|_{2r}^2 + |N_\theta|_{2r}^2 + |D(\lambda\lambda^*)|_{2r}^2 + |D\beta_\theta|_{2r}$$

$$+|\beta_\theta|_{2r}^2 + |\theta U|_{2r} + |\theta DU|_{2r} + |\theta g Q_\theta^{-1} g|_{2r} + |\theta D(g Q_\theta^{-1} g)|_{2r} + 1)$$

$$x \in B_r,\ \ t \in [0,T)$$

cf. Bensoussan-Frehse-N '98 AMO, N. '96, '03 SICON, Then, we have the following verification theorem.

Proposition 1 *Let* $v(t,x;T)$ *be a solution to (2.12). Then,*

$$\widehat{h}(t,x) := Q_\theta^{-1}(\delta^* \lambda^* Dv(t,x) + g(x))$$

$\widehat{h}_t^{(T)} \equiv \widehat{h}_t^{(\theta,T)} := \widehat{h}(t,X_t)$ *is the optimal strategy:*

$$\begin{aligned} v(0,x;T) &= \log E[e^{\theta\{\int_0^T f(X_s,\widehat{h}_s^{(T)})ds + \int_0^T \varphi(X_s,\widehat{h}_s^{(T)})^* dW_s\}}] \\ &= \inf_{h_\cdot \in \mathcal{A}(T)} \log E[e^{\theta\{\int_0^T f(X_s,h_s)ds + \int_0^T \varphi(X_s,h_s)^* dW_s}] \end{aligned}$$

The Related classical stochastic control problem

$$
(2.12) \qquad \begin{cases} \frac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 v] + \beta_\theta^* Dv + \frac{1}{2}(Dv)^*\lambda N_\theta \lambda^* Dv \\ \qquad\qquad + \frac{\theta}{2} g^* Q_\theta^{-1} g + \theta U = 0, \\ v(T,x) = 0, \end{cases}
$$

Noting that

$$
\theta Q_\theta^{-1} = (\delta^*\delta)^{-1}\delta^* N_\theta \delta(\delta^*\delta)^{-1} - (\delta^*\delta)^{-1}
$$

(2.12) is rewritten as follows

$$
\begin{cases}
\frac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 v] + \{\beta - \lambda\delta(\delta^*\delta)^{-1} g\}^* Dv \\
\qquad + \frac{1}{2}[\lambda^* Dv + \delta(\delta^*\delta)^{-1} g]^* N_\theta [\lambda^* Dv + \delta(\delta^*\delta)^{-1} g] \\
\qquad\qquad\qquad\qquad - \frac{1}{2} g^*(\delta^*\delta)^{-1} g + \theta U = 0, \\
v(T, x) = 0,
\end{cases}
$$

which is the H-J-B equation of the stochastic control problem:

$$
\bar{v}_*(0, x; T) = \sup_{z_\cdot} E[\int_0^T \Phi(Y_s, z_s) ds]
$$

subject to

$$
dY_t = \lambda(Y_t) dB_t + \{G(Y_t) + \lambda(Y_t) z_t\} dt, \quad Y_0 = x,
$$

$$
G(y) = \beta(y) - \lambda\delta(\delta^*\delta)^{-1} g
$$

$$
\Phi(y, z; \theta) = -\frac{1}{2} z^* N_\theta^{-1} z + g^*(\delta^*\delta)^{-1}\delta^*(y) z - \frac{1}{2} g^*(\delta^*\delta)^{-1} g(y) + \theta U(y).
$$

Convexity of $v(0, x; T)$ with respect to θ

$\Phi(y, z; \theta)$ is seen to be a linear function of θ. Indeed, we can see that $\frac{\partial^2 N_\theta^{-1}}{\partial \theta^2} = 0$. Further, the verification theorem also holds in the above stochastic control problem, and hence convexity of $v(0, x; T)$ with respect to θ follows since $\Phi(y, z; \theta)$ is a linear function of θ. Further, we shall later see the convexity of $\chi(\theta)$, which is the solution to the H-J-B equation of ergodic type, follows from that of $v(0, x; T)$.

H-J-B equation of ergodic type

H-J-B equation of parabolic type:

$$(2.12)\qquad \begin{cases} \frac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 v] + \beta_\theta^* Dv + \frac{1}{2}(Dv)^*\lambda N_\theta \lambda^* Dv \\ \qquad\qquad + \frac{\theta}{2} g^* Q_\theta^{-1} g + \theta U = 0, \\ v(T,x) = 0, \end{cases}$$

Now let us consider the infinite horizon counterpart of H-J-B equation (2.12), which is called H-J-B equation of ergodic type:

$$\chi(\theta) = \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 w] + \beta_\theta^* Dw + \frac{1}{2}(Dw)^*\lambda N_\theta \lambda^* Dw$$

$$+ \frac{\theta}{2} g^* Q_\theta^{-1} g + \theta U,$$

Proposition 2 *i) Assume that*

$$\lim_{r\to\infty}\inf_{|x|\geq r}\{g^*(\delta^*\delta)^{-1}g(x)+U(x)\}=\infty \tag{3.2}$$

besides the above assumptions. Then, we have a solution $(\chi(\theta), w)$ *of :*

$$\begin{aligned}\chi(\theta) = \frac{1}{2}tr[\lambda\lambda^* D^2 w] + \beta_\theta^* Dw + \frac{1}{2}(Dw)^*\lambda N_\theta \lambda^* Dw \\ +\frac{\theta}{2} g^* Q_\theta^{-1} g + \theta U,\end{aligned} \tag{3.3}$$

such that $w(x) \to -\infty$ *as* $|x| \to \infty$. *Moreover, such solution* (χ, w) *is unique up to additive constants with respect to* w.

ii) The solution w *satisfies the following estimate*

$$|\nabla w(x)|^2 \leq C_w |x|^2 + C_w', \quad C_w, C_w' > 0 \tag{3.4}$$

iii) If we moreover assume

$$c_4|x|^2 - c_5 \leq \frac{1}{c_1 - \theta} g^*(\delta^*\delta)^{-1} g(x) + U(x), \quad c_4, c_5 > 0, \tag{3.5}$$

then, we have

$$w(x) \leq -c_w|x|^2 + c'_w \tag{3.6}$$

Here we note that c_w is a positive constant such that

$$2c_3 c_w^2 + c_\beta c_w < \frac{-c_4\theta}{4} \tag{3.7}$$

and c_β is the one such that $|\beta_\theta(x)| \leq c_\beta|x| + C$

i) cf. Bensoussan-Frehse '92, Reine Angew. Math., N. '12 AAP
ii) and iii) cf. Proposition 3.2 in N. '12 AAP

Large time asymptotics of the solution

Since $w(x) + \chi(T-t)$ is a sub-solution to (2.12) for a specified solution $w(x)$ to H-J-B equation (3.3) such that $w(x) \leq 0$ we have

$$v(t,x;T) \geq w(x) + \chi(T-t), \quad T \geq t > -\infty \tag{3.8}$$

Proposition 3 *Under the assumptions of Proposition 2 iii),*

$$v(t,x;T) - (w(x) + \chi(T-t)) \leq c_E|x|^2 + c'_E \tag{3.9}$$

where c_E, and c'_E are positive constants independent of t.

Now set

$$u(0,x;T) := v(0,x;T) - \{w(x) + \chi(\theta)T\}$$

Theorem 1 *Under the assumptions of Proposition 3, as $T \to \infty$, $u(0,x;T)$ converges to a constant $c_\infty \in R$ uniformly on each compact set.*

Corollary 1 *Under the assumptions of Theorem 1 we have*

$$\lim_{T\to\infty} \frac{v(0,x;T)}{T} = \chi(\theta),$$

where $(\chi(\theta), w(x))$ *is the solution to H-J-B equation of ergodic type:*

$$\chi(\theta) = \frac{1}{2} tr[\lambda\lambda^* D^2 w] + \beta_\theta^* Dw + \frac{1}{2}(Dw)^*\lambda N_\theta \lambda^* Dw \tag{3.3}$$
$$+\frac{\theta}{2} g^* Q_\theta^{-1} g + \theta U,$$

Further, $\chi(\theta)$ *is convex.*

- Theorem 1 is shown by using the methods of Ichihara and Sheu (cf. their preprint and N. ' 12).

Ergodicity of the optimal diffusion

H-J-B equation (2.12) is rewritten as

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 v] + G^* Dv \\ \quad + \frac{1}{2}[\lambda^* Dv + \delta(\delta^*\delta)^{-1}g]^* N_\theta [\lambda^* Dv + \delta(\delta^*\delta)^{-1}g] \\ \qquad\qquad - \frac{1}{2}g^*(\delta^*\delta)^{-1}g + \theta U = 0, \\ v(T,x) = 0 \end{cases}$$

which is the H-J-B equation of the stochastic control problem:

$$\bar{v}_*(0,x;T;\theta) = \sup_{z_\cdot} E[\int_0^T \Phi(Y_s, z_s)ds]$$

subject to

$$dY_t = \lambda(Y_t)dB_t + \{G(Y_t) + \lambda(Y_t)z_t\}dt, \quad Y_0 = x,$$

$$\Phi(y,z;\theta) = -\frac{1}{2}z^* N_\theta^{-1} z + g^*(\delta^*\delta)^{-1}\delta^*(y)z - \frac{1}{2}g^*(\delta^*\delta)^{-1}g(y) + \theta U(y).$$

Thus the verification theorem for this stochastic control problem implies that

$$\chi(\theta) = \lim_{T\to\infty} \frac{1}{T} v(0, x; T) = \lim_{T\to\infty} \frac{1}{T} \sup_{z_\cdot} E[\int_0^T \Phi(Y_s, z_s) ds]$$

Since

$$\begin{aligned} &\tfrac{1}{2}[\lambda^* Dw + \delta(\delta^*\delta)^{-1} g]^* N_\theta [\lambda^* Dw + \delta(\delta^*\delta)^{-1} g] \\ &\quad = \sup_{z\in R^M}\{-\tfrac{1}{2} z^* N_\theta^{-1} z + z^*(\delta(\delta^*\delta)^{-1} g + \lambda^* Dw)\} \end{aligned}$$

the generator of the optimal diffusion process for the problem on infinite time horizon is seen to be

$$\begin{aligned} L^w \psi &:= \tfrac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\psi] + \{G + \lambda N_\theta(\delta(\delta^*\delta)^{-1} g + \lambda^* Dw)\}^* D\psi \\ &= \tfrac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\psi] + \beta_\theta^* D\psi + (Dw)^* \lambda N_\theta \lambda^* D\psi \end{aligned}$$

because of

$$N_\theta = I + \theta\delta Q_\theta^{-1}\delta^*$$

$$G + \lambda N_\theta \delta(\delta^*\delta)^{-1} g = \beta + \theta\lambda\delta Q_\theta^{-1} g \equiv \beta_\theta.$$

Note that

$$\chi(\theta) = L^w w - \frac{1}{2}(Dw)^* \lambda N_\theta \lambda^* Dw + \frac{\theta}{2} g Q_\theta^{-1} g + \theta U$$

Then, by setting $\bar{w} = -w$, we have

$$L^w \bar{w} = -\chi(\theta) - \frac{1}{2}(Dw)^* N_\theta \lambda^* Dw + \frac{\theta}{2} g Q_\theta^{-1} g + \theta U \to -\infty$$

as $|x| \to \infty$. Moreover, $\bar{w}(x) \to \infty, |x| \to \infty$ and we see that L^w is ergodic. The optimal diffusion is governed by:

$$d\bar{X}_t = \lambda(\bar{X}_t) dW_t + \{\beta_\theta + \lambda N_\theta \lambda^* Dw\}(\bar{X}_t) dt$$

Further, under the assumptions of Theorem 1, for each $\theta_1 \leq \theta \leq \theta_0$ there exist positive constants $k > 0$ and $C > 0$ independent of T and $\theta \in [\theta_1, \theta_0]$ such that

$$E[e^{k\bar{w}(\bar{X}_T)}] \leq C \tag{3.10}$$

Differentiability of H-J-B equation

$(EE)'$

$$\bar{\chi}'(\theta) = L^w \bar{w}' - U - \frac{1}{2}(\lambda^* D\bar{w} - \delta(\delta^*\delta)^{-1}g)^* \frac{\partial N_\theta}{\partial \theta}(\lambda^* D\bar{w} - \delta(\delta^*\delta)^{-1}g),$$

where $\bar{w}' = \frac{\partial \bar{w}}{\partial \theta} \equiv -\frac{\partial w}{\partial \theta}$ (cf. Lemma 6.4 in N. '12, AAP),

$$\frac{\partial N_\theta}{\partial \theta} = \delta Q_\theta^{-1}\delta^* + \theta(\delta Q_\theta^{-1}\delta^*)^2 > 0$$

To obtain (EE)', we first need to see the unique existence of the solution $(u, \gamma(\theta))$ of the Poisson equation :

(4.1) $$-\gamma(\theta) = L^w u(x) - f(x)$$

Noting that $\frac{c_0}{c_0-\theta} I \le N_\theta$, we see that for sufficiently large $R_0 > 0$

$$K(x;\bar{w}) \equiv -L^w \bar{w}(x) > \frac{c_0}{(c_0-\theta)\bar{w}} (D\bar{w})^* \lambda\lambda^* D\bar{w}(x), \quad x \in B_{R_0}^c.$$

This condition ensures the unique existence of the solution to (4.1) (cf. Bensoussan's book, or N. '12, AAP). Indeed, by setting

$$F_{\bar{w}} = \{u \in W_{loc}^{2,p};\ \mathrm{esssup}_{x \in B_{R_0}^c} \frac{|u(x)|}{\bar{w}(x)} < \infty\}$$

and

$$F_K = \{f \in L_{loc}^\infty;\ \mathrm{esssup}_{x \in B_{R_0}^c} \frac{|f(x)|}{K(x;\bar{w})} < \infty\},$$

we see that for $f \in F_K$, (4.1) has a solution $(u, \gamma(\theta))$ such that $u \in F_{\bar{w}}$,

$$\gamma(\theta) = \int f(x) m_\theta(dx) \tag{4.2}$$

(cf. Corollary 5.1 in N. '12, AAP).

When setting

$$f(x) = U + \frac{1}{2}(\lambda^* D\bar{w} - \delta(\delta^*\delta)^{-1}g)^* \frac{\partial N_\theta}{\partial \theta}(\lambda^* D\bar{w} - \delta(\delta^*\delta)^{-1}g)$$

equation (4.1) is the one that the derivative of the solution $(\bar{\chi}, \bar{w})$ of H-J-B equation of ergodic type should satisfy and through further analysis we can see that $\bar{\chi}(\theta)$ is differentiable and the solution satisfies (4.1)

Duality theorem

Theorem 2 *For* $\kappa \in (\chi'(-\infty), \chi'(0-))$*, we have*

$$\varliminf_{T\to\infty} \frac{1}{T} \inf_{h\in\mathcal{A}(T)} \log P\left(\frac{1}{T}F_T(X_\cdot, h_\cdot) \le \kappa\right) = -\inf_{k\in(\chi'(-\infty),\kappa]} I(k) = -I(\kappa)$$

$$I(k) := \sup_{\theta<0}\{\theta k - \chi(\theta)\}$$

Moreover, for $\theta(\kappa)$ *such that* $\chi'(\theta(\kappa)) = \kappa \in (\chi'(-\infty), \chi'(0-))$ *take a strategy* $\hat{h}_t^{(\theta(\kappa),T)}$*. Then,*

$$\lim_{T\to\infty} \frac{1}{T} \log P\left(\frac{1}{T}F_T(X_\cdot, h^{(\theta(\kappa),T)}) \le \kappa\right) = -\inf_{k\in(\chi'(-\infty),\kappa]} I(k) = -I(\kappa)$$

The "equivalent" stochastic differential game

(5.1)

$$\bar{J}(0,x;T) = \inf_{h_\cdot} \sup_{\nu_\cdot} E^{h,\nu}[\theta\{\int_0^T f(X_s,h_s)ds + \int_0^T \varphi(X_s,h_s)^* dW_s\}$$

$$-\frac{1}{2}\int_0^T |\nu_s + \theta\delta(\tilde{X}_s)h_s|^2 ds],$$

where X_t is a solution to the stochastic differential equation

(5.2)

$$dX_t = \{\beta(X_t) + \lambda(X_t)(\nu_t + \theta\delta(X_t)h_t)\}dt + \lambda(X_t)d\tilde{W}_t, \quad X_0 = x,$$

$P^{h,\nu}$ is a probability mesure on $(\Omega, \mathcal{F})$:

$$P^{h,\nu}(A) = E[e^{\int_0^T (\nu_s + \theta\delta(X_s)h_s)^* dW_s - \frac{1}{2}\int_0^T |\nu_s+\theta\delta(X_s)h_s|^2 ds};\ A]$$

$$\tilde{W}_t = W_t - \int_0^t (\nu_s + \theta\delta(X_s)h_s)ds. \quad P^{h,\nu} - \text{B.M.}$$

$$E[e^{\int_0^T (\nu_s + \theta\delta(X_s)h_s)^* dW_s - \frac{1}{2}\int_0^T |\nu_s+\theta\delta(X_s)h_s|^2 ds}] = 1$$

$$E^{h,\nu}[\theta\{\int_0^T f(X_s,h_s)ds + \int_0^T \varphi(X_s,h_s)^* dW_s\} - \frac{1}{2}\int_0^T |\nu_s + \theta\delta(X_s)h_s|^2 ds]$$

$$= \; E^{h,\nu}[\textstyle\int_0^T \{-\frac{\theta}{2}h_s^* Q_\theta(X_s)h_s + \theta h_s^* g(X_s) + \theta U(X_s)\}ds - \frac{1}{2}\int_0^T |\nu_s|^2 ds]$$

$$\equiv \; E^{h,\nu}[\textstyle\int_0^T \Xi(X_s,h_s,\nu_s;\theta)ds]$$

$$\Xi(x,h,\nu;\theta) = -\frac{\theta}{2}h^* Q_\theta(x)h + \theta h^* g(x) + \theta U(x) - \frac{1}{2}|\nu|^2$$

Thus, (5.1) is written as

$$\bar{J}(0,x;T) = \inf_{h_\cdot}\sup_{\nu_\cdot} E^{h,\nu}[\int_0^T \Xi(X_s,h_s,\nu_s;\theta)ds] \tag{5.3}$$

whose H-J-B equation (Isaacs Equation) is seen to be

$$\frac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 v] + \sup_\nu \inf_h [\{\beta + \lambda(\nu + \theta\delta h)\}^* Dv + \Xi(x,h,\nu;\theta)] = 0.$$

$$\sup_{\nu}[\nu^*\lambda^*(x)Dv - \frac{1}{2}|\nu|^2] = \frac{1}{2}(Dv)^*\lambda\lambda^*(x)Dv,$$

$$\inf_{h}[\theta\{\lambda\delta h\}^*Dv - \frac{\theta}{2}h^*Q_\theta h + \theta h^* g] = \frac{\theta}{2}(\delta^*\lambda^*Dv + g)^*Q_\theta^{-1}(\delta^*\lambda^*Dv + g),$$

lead to the same H-J-B equation as the risk-sensitive control:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 v] + \beta_\theta^* Dv + \frac{1}{2}(Dv)^*\lambda N_\theta \lambda^* Dv + \frac{\theta}{2}g^* Q_\theta^{-1} g + \theta U = 0$$

The H-J-B equation of ergodic type

$$\chi(\theta) = \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 w] + \sup_{\nu} \inf_{h} [\{\beta + \lambda(\nu + \theta\delta h)\}^* Dw + \Xi(x, h, \nu; \theta)].$$

which is written as

$$\chi(\theta) = L^w w + \Xi(x, \tilde{h}, \tilde{\nu}; \theta),$$

where

$$\tilde{h}(x) = Q_\theta^{-1}(\delta^* \lambda^* Dw(x) + g(x)), \;\; \tilde{\nu}(x) = \lambda^* Dw(x).$$

Outline of the proof of Theorem 2

Note that

(6.1)
$$\chi(\theta) = L^w w + \Xi(x, \tilde{h}, \tilde{\nu}; \theta)$$

(6.1)′
$$\chi'(\theta) = L^w w' + U + \frac{1}{2}(\lambda^* Dw + \delta(\delta^*\delta)^{-1}g)^* \frac{\partial N_\theta}{\partial\theta}(\lambda^* Dw + \delta(\delta^*\delta)^{-1}g)$$

(6.2)
$$\chi(\theta) - \theta\chi'(\theta) = L^w(w - \theta w') - \frac{1}{2}(N_\theta\lambda^* Dw + \theta\delta Q_\theta^{-1}g)^*(N_\theta\lambda^* Dw + \theta\delta Q_\theta^{-1}g)$$

(6.3)
$$(N_\theta\lambda^* Dw + \theta\delta Q_\theta^{-1}g)^*(N_\theta\lambda^* Dw + \theta\delta Q_\theta^{-1}g) = |\tilde{\nu} + \theta\delta^*\tilde{h}|^2$$

Introduce a probability measure $\tilde{P}$ defined by

$$\left.\frac{d\tilde{P}}{dP}\right|_{\mathcal{F}_T} = e^{\int_0^T \{\tilde{\nu}(X_s) + \theta\delta(X_s)\tilde{h}(X_s)\}^* dW_s - \frac{1}{2}\int_0^T |\tilde{\nu}(X_s) + \theta\delta(X_s)\tilde{h}(X_s)|^2 ds}.$$

$$M_t^\theta = \int_0^t \{\tilde{\nu}(X_s) + \theta\delta(X_s)\tilde{h}(X_s)\}^* d\tilde{W}_s$$

$$\begin{aligned} A_1 &= \{-\tfrac{1}{2}\langle M^\theta\rangle_T \geq (\chi(\theta) - \theta\chi'(\theta) - \epsilon)T\} \\ A_2 &= \{\int_0^T f(X_s, h_s)ds + \int_0^T \varphi(X_s, h_s)^* dW_s \leq \kappa T\} \\ A_3 &= \{-M_T^\theta \geq -\epsilon T\} \end{aligned}$$

$$P(\int_0^T f(X_s, h_s)ds + \int_0^T \varphi(X_s, h_s)^* dW_s \leq \kappa T) = \tilde{E}[e^{-\tilde{M}_T^\theta - 1/2\langle \tilde{M}^\theta\rangle_T};\ A_2]$$

$$\geq e^{(\chi(\theta)-\theta\chi'(\theta)-2\epsilon)T}\tilde{P}(A_1 \cap A_2 \cap A_3)$$

$$\geq e^{(\chi(\theta)-\theta\chi'(\theta)-2\epsilon)T}\{1 - \tilde{P}(A_1^c) - \tilde{P}(A_2^c) - \tilde{P}(A_3^c)\}$$

$\tilde{P}(A_1^c) < \epsilon$ for sufficiently large T

$\tilde{P}(A_3^c) < \epsilon$ for sufficiently large T

$$
\begin{aligned}
& \theta \int_0^T f(X_s, h_s) ds + \theta \int_0^T \varphi(X_s, h_s)^* dW_s \\
= \ & \int_0^T \Xi(X_s, \tilde{h}_s, \tilde{\nu}_s; \theta) ds + \theta \int_0^T \tilde{h}_s^* \delta(X_s)^* d\tilde{W}_s + \frac{1}{2} \int_0^T |\tilde{\nu}_s + \theta \delta(X_s) \tilde{h}_s|^2 ds \\
& + \theta \int_0^t (h_s - \tilde{h}_s)^* \delta(X_s)^* d\tilde{W}_s - \frac{\theta}{2} \int_0^T (h_s - \tilde{h}_s)^* S(X_s)(h_s - \tilde{h}_s) ds, \\
= \ & \theta \{ \chi'(\theta) T + w'(X_0) - w'(X_T) + \int_0^T \{ (Dw')^* \lambda(X_s) + \tilde{h}_s^* \delta(X_s)^* \} d\tilde{W}_s \} \\
& \quad + \theta \{ \int_0^T (h_s - \tilde{h}_s)^* (\delta(X_s)^* - S^{1/2}(X_s)) d\tilde{W}_s + (M_T^h - \frac{1}{2} \langle M^h \rangle_T) \}
\end{aligned}
$$

Take κ and ϵ such that $\chi'(-\infty) < \kappa - \epsilon < \chi'(0-)$, and θ_ϵ such that

$$
\sup_{\theta < 0} \{ \theta(\kappa - \epsilon) - \chi(\theta) \} = \theta_\epsilon (\kappa - \epsilon) - \chi(\theta_\epsilon)
$$

namely, $\chi'(\theta_\epsilon) = \kappa - \epsilon$. Then,

$$
\tilde{P}(A_2^c) < \epsilon \quad \text{for sufficiently large } T
$$

Hence we have

$$P(\int_0^T f(X_s, h_s)ds + \int_0^T \varphi(X_s, h_s)^* dW_s \leq \kappa T)$$

$$\geq e^{-T \sup_{\theta<0}\{\theta(\kappa-\epsilon)-\chi(\theta)\}-2\epsilon T}\{1 - \tilde{P}(A_1^c) - \tilde{P}(A_2^c) - \tilde{P}(A_3^c)\}$$

Robust version of large deviation estimates

$$P^{\zeta}: \quad \left.\frac{dP^{\zeta}}{dP}\right|_{\mathcal{F}_T} := e^{\int_0^T \zeta_s^* dW_s - \frac{1}{2}\int_0^T |\zeta_s|^2 ds}$$

$$dX_t = \lambda(X_t) dW_t^{\zeta} + \{\beta(X_t) + \lambda(X_t)\zeta_t\} dt.$$

$$\begin{aligned} J_1(\kappa) &:= \underline{\lim}_{T\to\infty} \frac{1}{T} \inf_{h.} \sup_{\zeta.} \log P^{\zeta}\left(\frac{1}{T}\{F_T(X.,h.) + \frac{\mu}{2}\int_0^T |\zeta_s|^2 ds\} \le \kappa\right) \\ &= -I_1(\kappa). \end{aligned}$$

$$I_1(\kappa) := \sup_{\theta<0}\{\theta\kappa - \chi_1(\theta)\}$$

$$\chi_1(\theta) = \underline{\lim}_{T\to\infty} \frac{1}{T} \inf_{h.} \sup_{\zeta.} \log E^{\zeta}[e^{\theta\{F_T(X.,h.) + \frac{\mu}{2}\int_0^T |\zeta_s|^2 ds\}}].$$

Formulation of the game

Lower value function

$$u_*(0,x;T) := \inf_{h\in\Delta_{\mathcal{H}}} \sup_{\zeta\in\mathcal{Z}} \log E^{\zeta}[e^{\theta\{F_T(X_\cdot,h_\cdot)+\frac{\mu}{2}\int_0^T|\zeta_s|^2ds\}}]$$

$\mathcal{Z} = \{\zeta_t; \zeta_t$ is prog. m'ble, $E[e^{\int_0^T \zeta_s^* dW_s - \frac{1}{2}\int_0^T|\zeta_s|^2 ds}] = 1\}$

$\Delta_{\mathcal{H}} = \{h_t; h_t = h(t, X_t, \zeta_t)$ is prog. m'ble, $h(t,x,\zeta) \in \mathbf{H}$,

$$E[e^{\int_0^T\{\zeta_s+\theta\delta(X_s)h_s\}^* dW_s - \frac{1}{2}\int_0^T|\zeta_s+\theta\delta(X_s)h_s|^2 ds}] = 1\},$$

$\mathbf{H}$: the totality of Borel functions $h(t,x,\zeta) : [0,T]\times R^N \times R^M \mapsto R^m$ such that $|h(t,x,\zeta)| \le C(1+|x|+|\zeta|)$

Risk sensitive stochastic differential game

$$\left.\frac{dP^{\zeta,h}}{dP^{\zeta}}\right|_{\mathcal{F}_T} = e^{\theta\int_0^T \varphi(X_s,h_s)^* dW_s^{\zeta} - \frac{\theta^2}{2}\int_0^T |\varphi(X_s,h_s)|^2 ds}$$

$$W^{\zeta,h} = W^{\zeta} - \theta\int_0^t \varphi(X_s,h_s)ds$$

$$dX_t = \lambda(X_t)dW_t^{\zeta,h} + (\beta(X_t) + \lambda(X_t)\zeta_t + \theta\delta(X_t)h_t)dt$$

$$\begin{aligned} u_*(0,x;T) &= \inf_{h\in\Delta_{\mathcal{H}}}\sup_{\zeta\in\mathcal{Z}} \log E^{\zeta}[e^{\theta\{F_T(X_\cdot,h_\cdot)+\frac{\mu}{2}\int_0^T |\zeta|^2 ds\}}] \\ &= \inf_{h\in\Delta_{\mathcal{H}}}\sup_{\zeta\in\mathcal{Z}} \log E^{\zeta,h}[e^{\theta\int_0^T \eta(X_s,h_s,\zeta_s)}] \end{aligned}$$

$$\begin{aligned} \eta(x,h,\zeta) &= f(x,h) + h^*\delta(x)^*\zeta + \frac{\mu}{2}|\zeta|^2 + \frac{\theta}{2}|\delta(x)h|^2 \\ &= -\frac{1}{2}h^*Q_\theta h + h^*(\delta\zeta + g) + U + \frac{\mu}{2}|\zeta|^2 \end{aligned}$$

Lower Isaacs equation

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 u] + \beta^* Du + \frac{1}{2}(Du)^*\lambda\lambda^* Du + H_-(x, Du) = 0 \\ u(T,x) = 0, \end{cases}$$

$$\begin{aligned} H_-(x,p) &= \sup_{\zeta\in R^M} \inf_{h\in R^m} \Lambda(x,p,\zeta,h) \\ &\equiv \sup_{\zeta\in R^M} \inf_{h\in R^m} [\{\zeta + \theta\delta(x)h\}^*\lambda(x)^* p + \eta(x,\zeta,h)] \end{aligned}$$

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 u] + \beta_\theta^* Du + \frac{1}{2}(Du)^*\lambda N_\theta \lambda^* Du \\ \qquad + \frac{\theta}{2} g^* Q_\theta^{-1} g + \theta U \\ \qquad - \frac{1}{2\theta\mu}(N_\theta\lambda^* Du + \theta\delta Q_\theta^{-1} g)^* R_{\theta,\mu}^{-1}(N_\theta\lambda^* Du + \theta\delta Q_\theta^{-1} g) = 0 \\ u(T,x) = 0, \end{cases}$$

$$\beta_\theta = \beta + \theta\lambda\delta Q_\theta^{-1} g, \quad N_\theta = I + \theta\delta Q_\theta^{-1}\delta^*, \quad R_{\theta,\mu} = I + \frac{1}{\mu}\delta Q_\theta^{-1}\delta^*$$

Rewriting as follows is useful to study the equation:

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 u] + (\beta - \lambda\delta(\delta^*\delta)^{-1}g)^* Du \\ \quad +g^*(\delta^*\delta)^{-1}\delta^* R_{\theta,\mu}^{-1} N_\theta \lambda^* Du - \frac{1-\theta\mu}{2\theta\mu}(Du)^*\lambda N_\theta R_{\theta,\mu}^{-1}\lambda^* Du \\ \quad +\frac{1}{2}g^*(\delta^*\delta)^{-1}\delta^*(N_\theta - I)R_{\theta,\mu}^{-1}\delta(\delta^*\delta)^{-1}g + \theta U = 0 \\ u(T,x) = 0, \end{cases}$$

$$R_{\theta,\mu}^{-1} N_\theta = N_\theta R_{\theta,\mu}^{-1} = (1-\theta\mu)R_{\theta,\mu}^{-1} + \theta\mu I_M$$

$$\frac{\mu c_0}{1+\mu(c_0-\theta)} I_M \leq R_{\theta,\mu}^{-1} N_\theta \leq I_M, \quad \theta < 0, \ \mu > 0$$

$$\delta^*(N_\theta - I)R_{\theta,\mu}^{-1}\delta = \theta\mu\delta^*(I - R_{\theta,\mu}^{-1})\delta = \theta\delta^*\delta Q_\theta^{-1}\delta^* R_{\theta,\mu}^{-1}\delta < -cI_m$$

Optimal feedback

u : sol. to the Lower Isaacs equation

$$\widehat{h}(t,x,\zeta) = \arg \inf_{h \in R^m} \Lambda(x, Du(t,x), \zeta, h)$$

$$\widehat{\zeta}(t,x) = \arg \sup_{\zeta \in R^M} \Lambda(x, Du(t,x), \zeta, \widehat{h}(t,x,\zeta))$$

Thus

$$\begin{aligned} H_-(x, Du) &= \sup_{\zeta \in R^M} \inf_{h \in R^m} \Lambda(x, Du(t,x), \zeta, h) \\ &= \Lambda(x, Du(t,x), \widehat{\zeta}(t,x), \widehat{h}(t,x,\widehat{\zeta}(t,x))) \end{aligned}$$

• It can be seen that

$$\begin{aligned} \widehat{\zeta}(t,x) &= -\frac{1}{\theta\mu} R_{\theta,\mu}^{-1}(N_\theta \lambda^* Du + \theta\delta Q_\theta^{-1} g) \\ \widehat{h}(t,x,\zeta) &= Q_\theta^{-1}(g + \delta^*\zeta + \delta^*\lambda^* Du) \end{aligned}$$

- The verification theorem holds for $\hat{h}(t, X_t, \hat{\zeta}_t)$ and $\hat{\zeta}_t = \hat{\zeta}(t, X_t)$,

$$u_*(0, x; T) = \inf_{h \in \Delta_{\mathcal{H}}} \sup_{\zeta \in \mathcal{Z}} \log E^{\zeta,h}[e^{\theta \int_0^T \eta(X_s, h_s, \zeta_s)}]$$

$$J(\zeta, h(\zeta)) := \log E^{\zeta,h(\zeta)}[e^{\theta \int_0^T \eta(X_s, h(s, X_s, \zeta_s), \zeta_s)}]$$

$$h(\zeta) = h(t, x, \zeta),$$

We can show that

$$J(\zeta, \hat{h}(\zeta)) \le J(\hat{\zeta}, \hat{h}(\hat{\zeta})) \le J(\hat{\zeta}, h(\hat{\zeta}))$$

and hence

$$J(\hat{\zeta}, \hat{h}(\hat{\zeta})) = \inf_{h \in \Delta_{\mathcal{H}}} \sup_{\zeta \in \mathcal{Z}} \log E^{\zeta,h}[e^{\theta \int_0^T \eta(X_s, h_s, \zeta_s)}]$$

H-J-B equation rewritten

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 u] + (\beta - \lambda\delta(\delta^*\delta)^{-1}g)^* Du \\ -\frac{1-\theta\mu}{2\theta\mu}\{\lambda^* Du - \frac{\theta\mu}{1-\theta\mu}\delta(\delta^*\delta)^{-1}g\}^* N_\theta R_{\theta\mu}^{-1}\{\lambda^* Du - \frac{\theta\mu}{1-\theta\mu}\delta(\delta^*\delta)^{-1}g\} \\ +\frac{\theta\mu}{2(1-\theta\mu)}g^*(\delta^*\delta)^{-1}g + \theta U = 0 \\ u(T,x) = 0, \end{cases}$$

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 u] + (\beta - \lambda\delta(\delta^*\delta)^{-1}g)^* Du \\ +\sup_{z\in R^M}\{\frac{\theta\mu}{2(1-\theta\mu)}z^* R_{\theta,\mu} N_\theta^{-1} z + z^*(\lambda^* Du - \frac{\theta\mu}{1-\theta\mu}\delta(\delta^*\delta)^{-1}g)\} \\ +\frac{\theta\mu}{2(1-\theta\mu)}g^*(\delta^*\delta)^{-1}g + \theta U = 0 \\ u(T,x) = 0, \end{cases}$$

The related classical stochastic control problem

$$dX_t = \lambda(X_t)dW_t + \{G(X_t) + \lambda(X_t)Z_t\}dt$$

$$u_*(0, x; T) = \sup_{Z_\cdot} E[\int_0^T \Phi(X_s, Z_s)ds]$$

$$G(x) = \beta - \lambda\delta(\delta^*\delta)^{-1}g$$

$$\begin{aligned}\Phi(x, z; \theta) \;&=\; \tfrac{\theta\mu}{2(1-\theta)} z^* R_{\theta,\mu} N_\theta^{-1} z - \tfrac{\theta\mu}{1-\theta\mu} z^*\delta(\delta^*\delta)^{-1}g \\ &\quad + \tfrac{\theta\mu}{2(1-\theta\mu)} g^*(\delta^*\delta)^{-1}g + \theta U\end{aligned}$$

- $\Phi(x, z)$ is a convex function of θ, which implies the convexity of value function $u_*(0, x; T)$.

- Convexity of $\chi_1(\theta) = \lim_{T\to\infty} \frac{1}{T} u_*(0, x; T)$ is also obtained in a similar way to the above.

Ergodic type equation

$$\chi_1(\theta) = \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 w] + \beta^* Dw + \frac{1}{2}(Dw)^*\lambda\lambda^* Dw + H_-(x, Dw)$$

$$\begin{aligned} H_-(x, Dw) &= \sup_{\zeta\in R^M} \inf_{h\in R^m} \Lambda(x, Dw(x), \zeta, h) \\ &= \Lambda(x, Dw(x), \tilde\zeta(x), \tilde h(x, \tilde\zeta(x))) \end{aligned}$$

$$\begin{aligned} \tilde\zeta(x) &= -\tfrac{1}{\theta\mu} R_{\theta,\mu}^{-1}(N_\theta\lambda^* Dw + \theta\delta Q_\theta^{-1} g) \\ \tilde h(x,\zeta) &= Q_\theta^{-1}(g + \delta^*\zeta + \delta^*\lambda^* Dw) \end{aligned}$$

$$\begin{aligned} \chi_1(\theta) = \tfrac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 w] &+ \beta_\theta^* Dw + \tfrac{1}{2}(Dw)^*\lambda N_\theta\lambda^* Dw \\ &+ \tfrac{\theta}{2} g^* Q_\theta^{-1} g + \theta U \\ &- \tfrac{1}{2\theta\mu}(N_\theta\lambda^* Dw + \theta\delta Q_\theta^{-1} g)^* R_{\theta,\mu}^{-1}(N_\theta\lambda^* Dw + \theta\delta Q_\theta^{-1} g) \end{aligned}$$

Duality theorem

Theorem 3 *For $\kappa \in (\chi_1'(-\infty), \chi_1'(0-))$, we have*

$$\underline{\lim}_{T\to\infty} \frac{1}{T} \inf_{h\in\Delta_{\mathcal{H}}} \sup_{\zeta\in\mathcal{Z}} \log P^{\zeta}\left(\frac{1}{T}\{F_T(X_\cdot, h_\cdot) + \frac{\mu}{2}\int_0^T |\zeta_s|^2 ds\} \le \kappa\right) = -I_1(\kappa)$$

$$I_1(k) := \sup_{\theta<0}\{\theta k - \chi_1(\theta)\}$$

Moreover, for $\theta(\kappa)$ such that $\chi_1'(\theta(\kappa)) = \kappa \in (\chi_1'(-\infty), \chi_1'(0-))$ take a strategy $\hat{h}^{(\theta(\kappa))}(t, x, \tilde{\zeta})$. Then,

$$\lim_{T\to\infty} \frac{1}{T} \log P^{\tilde{\zeta}}\left(\frac{1}{T}\{F_T(X_\cdot, \hat{h}^{(\theta(\kappa))}(\cdot, X_\cdot, \tilde{\zeta}_\cdot)) + \frac{\mu}{2}\int_0^T |\tilde{\zeta}_s|^2 ds\} \le \kappa\right) = -I_1(\kappa)$$

The "equivalent" stochastic differential game

(7.1)

$$\bar{J}(0,x;T) = \inf_{h_\cdot} \sup_{\zeta_\cdot \nu_\cdot} E^{\zeta,h,\nu}[\theta\{\int_0^T f(X_s,h_s)ds + \int_0^T \varphi(X_s,h_s)^* dW_s\}$$

$$+\frac{\mu}{2}\int_0^T |\zeta_s|^2 ds - \frac{1}{2}\int_0^T |\nu_s + \theta\delta(\tilde{X}_s)h_s|^2 ds],$$

where X_t is a solution to the stochastic differential equation

$$dX_t = \{\beta(X_t) + \lambda(X_t)(\zeta_t + \nu_t + \theta\delta(X_t)h_t)\}dt + \lambda(X_t)d\tilde{W}_t, \qquad X_0 = x,$$

$P^{\zeta,h,\nu}$ is a probability mesure:

$$P^{\zeta,h,\nu}(A) = E^{\zeta}[e^{\int_0^T (\nu_s + \theta\delta(X_s)h_s)^* dW_s^{\zeta} - \frac{1}{2}\int_0^T |\nu_s + \theta\delta(X_s)h_s|^2 ds}; A]$$

$$\tilde{W}_t = W_t^{\zeta} - \int_0^t (\nu_s + \theta\delta(X_s)h_s)ds.$$

$$
\begin{aligned}
& E^{\zeta,h,\nu}[\theta\{\textstyle\int_0^T f(X_s,h_s)ds + \int_0^T \varphi(X_s,h_s)^* dW_s + \frac{\mu}{2}\int_0^T |\zeta_s|^2 ds\} \\
& \qquad\qquad -\tfrac{1}{2}\textstyle\int_0^T |\nu_s + \theta\delta(X_s)h_s|^2 ds] \\
= \; & E^{\zeta,h,\nu}[\textstyle\int_0^T \{-\frac{\theta}{2}h_s^* Q_\theta(X_s)h_s + \theta h_s^*(g(X_s)+\zeta_s) + \theta U(X_s) + \frac{\theta\mu}{2}|\zeta_s|^2\}ds \\
& \qquad\qquad -\tfrac{1}{2}\textstyle\int_0^T |\nu_s|^2 ds] \\
\equiv \; & E^{\zeta,h,\nu}[\textstyle\int_0^T \Xi_1(X_s,h_s,\zeta_s,\nu_s;\theta)ds]
\end{aligned}
$$

$$
\Xi_1(x,h,\nu;\theta) = -\frac{\theta}{2}h^* Q_\theta(x)h + \theta h^*(g(x)+\zeta) + \theta U(x) + \frac{\theta\mu}{2}|\zeta|^2 - \frac{1}{2}|\nu|^2
$$

Thus, (7.1) is written as

$$
\bar{J}(0,x;T) = \inf_{h_\cdot}\sup_{\nu_\cdot,\zeta_\cdot} E^{\zeta,h,\nu}[\int_0^T \Xi_1(X_s,h_s,\zeta_s,\nu_s;\theta)ds]
$$

whose H-J-B equation is seen to be

$$\frac{\partial u}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 u] + \sup_{\zeta,\nu}\inf_h[\{\beta + \lambda(\nu + \zeta + \theta\delta h)\}^* Du + \Xi_1(x,h,\zeta,\nu;\theta)] = 0.$$

It reads

$$\frac{\partial u}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 u] + \beta_\theta^* Du + \frac{1}{2}(Du)^*\lambda N_\theta \lambda^* Du + \frac{\theta}{2} g^* Q_\theta^{-1} g + \theta U$$
$$-\frac{1}{2\theta\mu}(N_\theta\lambda^* Du + \theta\delta Q_\theta^{-1} g)^* R_{\theta,\mu}^{-1}(N_\theta \lambda^* Du + \theta\delta Q_\theta^{-1} g) = 0$$

Ergodic type equation

$$\chi_1(\theta) = \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 w]$$

$$+\sup_{\nu\in R^n,\zeta\in R^M}\inf_{h\in R^m}[\{\beta+\lambda\zeta+\lambda(\nu+\theta\delta h)\}^* Dw + \Xi_1(x,h,\zeta,\nu)]$$

can be written as

$$\chi_1(\theta) = L_1^w w + \Xi_1(x,\tilde{h},\tilde{\zeta},\tilde{\nu}) \tag{7.2}$$

where

$$\begin{aligned} L_1^w\psi &= \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\psi] + (\beta - \lambda\delta(\delta^*\delta)^{-1}g)^* D\psi \\ &+ g^*(\delta^*\delta)^{-1}\delta^* R_{\theta,\mu}^{-1} N_\theta \lambda^* D\psi - \frac{1-\theta\mu}{\theta\mu}(Dw)^*\lambda N_\theta R_{\theta,\mu}^{-1}\lambda^* D\psi \end{aligned}$$

$$\begin{aligned} \tilde{h} &= Q_\theta^{-1}(g+\delta^*\tilde{\zeta}+\delta^*\lambda^* Dw) \\ \tilde{\zeta} &= -\frac{1}{\theta\mu}R_{\theta,\mu}^{-1}(N_\theta\lambda^* Dw + \theta\delta Q_\theta^{-1}g) \\ \tilde{\nu} &= \lambda^* Dw \end{aligned}$$

$$\begin{aligned}\chi_1'(\theta) &= L_1^w w' + g^*(\delta^*\delta)^{-1}\delta^*\tfrac{\partial R_{\theta,\mu}^{-1}N_\theta}{\partial\theta}\lambda^* Dw \\ &+\tfrac{1}{2\theta^2\mu}(Dw)^*\lambda N_\theta R_{\theta,\mu}^{-1}\lambda^* Dw - \tfrac{1-\theta\mu}{2\theta\mu}(Dw)^*\lambda\tfrac{\partial N_\theta R_{\theta,\mu}^{-1}}{\partial\theta}\lambda^* Dw \\ &+\tfrac{1}{2}g^*(\delta^*\delta)^{-1}\delta^*(\tfrac{\partial N_\theta R_{\theta,\mu}^{-1}}{\partial\theta} - \tfrac{\partial R_{\theta,\mu}^{-1}}{\partial\theta})\delta(\delta^*\delta)^{-1}g + U\end{aligned}$$

$$\chi_1(\theta) - \theta\chi_1'(\theta) = L_1^w(w-\theta w')$$

$$-\tfrac{1}{2}\{N_\theta R_{\theta,\mu}^{-1}\lambda^* Dw + \theta R_{\theta,\mu}^{-1}\delta Q_\theta^{-1}g\}^*\{N_\theta R_{\theta,\mu}^{-1}\lambda^* Dw + \theta R_{\theta,\mu}^{-1}\delta Q_\theta^{-1}g\}$$

is seen to be

(7.3) $$\chi_1(\theta) - \theta\chi_1'(\theta) = L_1^w(w-\theta w') - \frac{1}{2}|\tilde\nu + \theta\delta\tilde h|^2$$

- μ is the certainty level of β:

$$\| \tilde{\zeta} \|_{L^{\infty}_{loc}} = O(\frac{1}{\mu})$$

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Linear Gaussian case

$$\beta(x) = Bx + b, \quad g(x) = Ax + a, \quad U(x) = \frac{1}{2}x^* V x + m,$$

$\lambda, \delta, S, A, B, V$: const. matrices

$$u(t,x) = \frac{1}{2}x^* P(t)x + q(t)^* x + l(t), \quad \text{sol. of H-J-B eq.}$$

$$\dot{P}(t) - \tfrac{1-\theta\mu}{\theta\mu} P(t)\lambda N_\theta R_{\theta,\mu}^{-1}\lambda^* P(t) + K_1^* P(t) + P(t)K_1 - C^*C + \theta V = 0$$

$$P(T) = 0$$

$$\begin{aligned} K_1 &= B - (1-\theta\mu)\lambda(I - R_{\theta,\mu}^{-1})\delta(\delta^*\delta)^{-1}A \\ C^*C &= -\theta\mu A^*(\delta^*\delta)^{-1}\delta^*(I - R_{\theta,\mu}^{-1})\delta(\delta^*\delta)^{-1}A \end{aligned}$$

$$\dot{q}(t) + (K_1^* - \tfrac{1-\theta\mu}{\theta\mu} P(t)\lambda N_\theta R_{\theta,\mu}^{-1}\lambda^*)q(t) + P(t)b$$

$$-\theta A^*(\delta^*\delta)^{-1}\delta^* R_{\theta,\mu}^{-1}\delta Q_\theta^{-1} a - \tfrac{1-\theta\mu}{\mu} P(t)\lambda R_{\theta,\mu}\delta Q_\theta^{-1} a = 0$$

$$q(T) = 0$$

$$\dot{l}(t) + \tfrac{1}{2}\mathrm{tr}[\lambda\lambda^* P(t)] + \tfrac{1}{2} q(t)\lambda N_\theta \lambda^* q(t)$$

$$+(b + \theta\lambda\delta Q_\theta^{-1} a)^* q(t) + \tfrac{\theta}{2} a^* Q_\theta^{-1} a + \theta m$$

$$-\tfrac{1}{2\theta\mu}(N_\theta\lambda^* q(t) + \theta\delta Q_\theta^{-1} a)^* R_{\theta,\mu}^{-1}(N_\theta\lambda^* q(t) + \theta\delta Q_\theta^{-1} a) = 0$$

$$l(T) = 0$$

In this case we can relax the assumption of uniform ellipticity for $\lambda\lambda^*$. Indeed, if B is stable, then,

$$P(t;T) \to \overline{P}, \;\; T \to \infty$$

$\overline{P}$ is a solution to the stationary equation such that $K_1 - \frac{1-\theta\mu}{\theta\mu}\lambda N_\theta R_{\theta,\mu}^{-1}\lambda^*\overline{P}$ is stable.