Brownian Motion under Nonlinear Expectation and related BSDE

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Perspectives in Analysis and Probability

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Robust representation of a coherent risk measure

- Huber Robust Statistics (1981), for finite state case.
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Theorem (Robust Representation of coherent risk measure)

$\hat{\mathbb{E}}[\cdot]$ *is a sublinear expectation iff there exists a family* $\{E_\theta\}_{\theta\in\Theta}$ *of linear expectations s.t.*

$$\hat{\mathbb{E}}[X] = \sup_{\theta\in\Theta} E_\theta[X], \quad \forall X \in \mathcal{H}.$$

Motivated from g-Expectation [P.1994-1997] on Wiener probability space $(\Omega, \mathcal{F}, P)$

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- Then define:

$$\mathbb{E}^g[X] := y(0), \quad \mathbb{E}^g[X|(B(s))_{s\in[0,t]}] := y(t).$$

- [Artzner-Delbean-Eden-Heath1999] Coherent measures of risk, Math. finance.
- [Coquet-Hu-P.-Memin2002], [P. 2005]: A dominated and $\mathcal{F}_t$-dynamic expectation a g-expectation;

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- Serious problem: under volatility uncertainty, it is impossible to find a reference probability measure.

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Knight, 1921

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Knightian uncertainty

The prob. and distr. are unknown— "uncertainty of probability measures".

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- Hansen & Sargent: Robust control method.

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- We often "equivalently" assume:

$$X_1, \cdots, X_n \in \mathcal{H} \implies \varphi(X_1, \cdots, X_n) \in \mathcal{H}, \quad \forall \varphi \in C_{Lip}(\mathbb{R}^n)$$

Daniell's Expectation (1918) $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$

$X \in \mathcal{H} \implies |X| \in \mathcal{H}$

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Theorem (Daniell-Stone Theorem)

- *There exists a probability measure P on $(\Omega, \sigma(\mathcal{H}))$ s.t.*

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G-Expectation and G-Brownian Motion

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- [Denis-Hu-Peng2008] Capacity related to Sublinear Expectations: appl. to G-Brownian Motion Paths.

Uncertainty version of distributions in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$

Definition

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- $$Y \text{ indenp. of } X \iff \hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}].$$

Central Limit Theorem (CLT) under Knightian Uncertainty

Theorem

Let $\{X_i\}_{i=1}^{\infty}$ *in* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ *be i.i.d.:* $X_i \sim X_1$ *and* X_{i+1} *Indep.* $(X_1, \cdots, X_i)$. *Assume:*

$$\hat{\mathbb{E}}[|X_1|^{2+\alpha}] < \infty \quad , \hat{\mathbb{E}}[X_1] = \hat{\mathbb{E}}[-X_1] = 0.$$

Then:

$$\lim_{n\to\infty} \hat{\mathbb{E}}[\varphi(\frac{X_1 + \cdots + X_n}{\sqrt{n}})] = \hat{\mathbb{E}}[\varphi(X)], \ \forall \varphi \in C_b(\mathbb{R}),$$

with $X \sim N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$, *where*

$$\overline{\sigma}^2 = \hat{\mathbb{E}}[X_1^2], \ \ \underline{\sigma}^2 = -\hat{\mathbb{E}}[-X_1^2].$$

Definition

A loss position X in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is normally in uncertainty distribution if

$$aX + b\bar{X} \sim \sqrt{a^2 + b^2}X, \quad \forall a, b \geq 0.$$

where $\bar{X}$ is an independent copy of X.

Normal distributions under Knightian uncertainty

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- $\hat{\mathbb{E}}[X] = \hat{\mathbb{E}}[-X] = 0.$
- $X \overset{d}{=} N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$, where

$$\overline{\sigma}^2 := \hat{\mathbb{E}}[X^2], \quad \underline{\sigma}^2 := -\hat{\mathbb{E}}[-X^2].$$

- **(1)** For each **convex** φ, we have

$$\hat{\mathbb{E}}[\varphi(X)] = \frac{1}{\sqrt{2\pi\overline{\sigma}^2}} \int_{-\infty}^{\infty} \varphi(y) \exp(-\frac{y^2}{2\overline{\sigma}^2}) dy$$

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- **(2)** For each **concave** φ, we have,

$$\hat{\mathbb{E}}[\varphi(X)] = \frac{1}{\sqrt{2\pi\underline{\sigma}^2}} \int_{-\infty}^{\infty} \varphi(y) \exp(-\frac{y^2}{2\underline{\sigma}^2}) dy$$

Remark.

If $\underline{\sigma}^2 = \overline{\sigma}^2$, then $N(0; [\underline{\sigma}^2, \overline{\sigma}^2]) = N(0, \overline{\sigma}^2)$.

Remark.

The larger to $[\underline{\sigma}^2, \overline{\sigma}^2]$ the stronger the uncertainty.

Remark.

But $X \overset{d}{=} N(0; [\underline{\sigma}^2, \overline{\sigma}^2])$ does not simply implies

$$\hat{\mathbb{E}}[\varphi(X)] = \sup_{\sigma \in [\underline{\sigma}^2, \overline{\sigma}^2]} \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} \varphi(x) \exp\{\frac{-x^2}{2\sigma}\} dx$$

G-normal distribution characterized by nonlinear infinitesimal generator

CLT converges in uncertainty distribution to $\mathcal{N}(0,[\underline{\sigma}^2,\overline{\sigma}^2])$:

Theorem

$X \overset{d}{=} N(0,[\underline{\sigma}^2,\overline{\sigma}^2])$ in $(\Omega,\mathcal{H},\hat{\mathbb{E}})$, then for each C_b function φ,

$$\mathcal{S}_t(\varphi)(x) := \hat{\mathbb{E}}[\varphi(x+\sqrt{t}X)], \quad x \in \mathbb{R}, \quad t \geq 0$$

defines a nonlinear semigroup, since: $\mathcal{S}_0[\varphi](x) = \hat{\mathbb{E}}[\varphi(x)] = \varphi(x)$, *and*

$$\begin{aligned}
\mathcal{S}_{t+s}[\varphi](x) &= \hat{\mathbb{E}}[\varphi(x+\sqrt{t+s}X)] \\
&= \hat{\mathbb{E}}[\varphi(\overbrace{x+\sqrt{t}X}+\sqrt{s}\bar{X})] \\
&= \hat{\mathbb{E}}\left[\hat{\mathbb{E}}[\varphi(\overbrace{x+\sqrt{t}y}+\sqrt{s}\bar{X})]_{y=X}\right] \\
&= \hat{\mathbb{E}}\left[(\mathcal{S}_s[\varphi])(x+\sqrt{t}X)\right] = \mathcal{S}_t[\mathcal{S}_s[\varphi]](x).
\end{aligned}$$

$$\mathcal{A}\varphi(x) := \lim_{t\to 0} \frac{\mathcal{S}_t(\varphi)(x) - \varphi(x)}{t} = G(u_{xx}).$$

where

$$G(a) = \hat{\mathbb{E}}[\frac{a}{2}X^2] = \frac{1}{2}(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-)$$

Thus we can solve the PDE

$$\begin{aligned} u_t &= G(\partial^2_{xx} u), \quad t > 0, \quad x \in \mathbb{R} \\ u|_{t=0} &= \varphi. \end{aligned}$$

Law of Large Numbers (LLN), Central Limit Theorem (CLT)

Striking consequence of LLN & CLT

Accumulated independent and identically distributed random variables tends to a normal distributed random variable, whatever the original distribution.

Maximal distribution $M([\underline{\mu}, \overline{\mu}])$ under Knightian uncertainty

Definition

A random variable Y in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is maximally distributed, denoted by $Y \overset{d}{=} M([\underline{\mu}, \overline{\mu}])$, if

$$aY + b\bar{Y} \overset{d}{=} (a+b)Y, \quad a, b \geq 0.$$

where $\bar{Y}$ is an independent copy of Y,

$$\overline{\mu} := \hat{\mathbb{E}}[Y], \quad \underline{\mu} := -\hat{\mathbb{E}}[-Y].$$

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- We can prove that

$$\hat{\mathbb{E}}[\varphi(Y)] = \sup_{y \in [\underline{\mu}, \overline{\mu}]} \varphi(y).$$

Case with mean-uncertainty $\mathbb{E}[\cdot]$

Definition

A pair of random variables (X, Y) in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is $\mathcal{N}([\underline{\mu}, \overline{\mu}], [\underline{\sigma}^2, \overline{\sigma}^2])$-distributed ($(X, Y) \stackrel{d}{=} \mathcal{N}([\underline{\mu}, \overline{\mu}], [\underline{\sigma}^2, \overline{\sigma}^2])$) if

$$(aX + b\bar{X}, a^2 Y + b^2 \bar{Y}) \stackrel{d}{=} (\sqrt{a^2 + b^2} X, (a^2 + b^2) Y), \quad \forall a, b \geq 0.$$

where $(\bar{X}, \bar{Y})$ is an independent copy of (X, Y),

$$\overline{\mu} := \hat{\mathbb{E}}[Y], \ \underline{\mu} := -\hat{\mathbb{E}}[-Y]$$
$$\overline{\sigma}^2 := \hat{\mathbb{E}}[X^2], \ \underline{\sigma}^2 := -\hat{\mathbb{E}}[-X], \ \ (\hat{\mathbb{E}}[X] = \hat{\mathbb{E}}[-X] = 0).$$

Theorem

$(X, Y) \overset{d}{=} \mathcal{N}([\underline{\mu}, \overline{\mu}], [\underline{\sigma}^2, \overline{\sigma}^2])$ in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ iff for each $\varphi \in C_b(\mathbb{R})$ the function

$$u(t, x, y) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X, y + tY)], \quad x \in \mathbb{R}, \quad t \geq 0$$

is the solution of the PDE

$$\begin{aligned} u_t &= G(u_y, u_{xx}), \quad t > 0, \quad x \in \mathbb{R} \\ u|_{t=0} &= \varphi, \end{aligned}$$

where

$$G(p, a) := \hat{\mathbb{E}}[\frac{a}{2}X^2 + pY].$$

Theorem

Let $\{X_i + Y_i\}_{i=1}^{\infty}$ be i.i.d. sequence. We assume furthermore that

$$\hat{\mathbb{E}}[|X_1|^{2+\alpha}] + \hat{\mathbb{E}}[|Y_1|^{1+\alpha}] < \infty, \quad \hat{\mathbb{E}}[X_1] = \hat{\mathbb{E}}[-X_1] = 0.$$

Then, for each $\varphi \in C_b(\mathbb{R})$,

$$\lim_{n\to\infty} \hat{\mathbb{E}}[\varphi(\frac{X_1 + \cdots + X_n}{\sqrt{n}} + \frac{Y_1 + \cdots + Y_n}{n})] = \hat{\mathbb{E}}[\varphi(X + Y)].$$

where (X, Y) is $\mathcal{N}([\underline{\mu}, \overline{\mu}], [\underline{\sigma}^2, \overline{\sigma}^2])$-distributed.

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Brownian Motion $(B_t(\omega))_{t\geq 0}$ in $(\Omega, \mathcal{F}, \hat{\mathbb{E}})$

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- $B_t \overset{d}{=} B_{s+t} - B_s$, for all $s, t \geq 0$
- $\hat{\mathbb{E}}[|B_t|^3] = o(t)$. .

Theorem.

If $(B_t(\omega))_{t\geq 0}$ is a G–**Brownian motion** and $\hat{\mathbb{E}}[B_t] = \hat{\mathbb{E}}[-B_t] \equiv 0$ then:
$B_{t+s} - B_s \overset{d}{=} N(0, [\underline{\sigma}^2 t, \overline{\sigma}^2 t]), \ \forall\ s, t \geq 0$ □

Sketch of Proof.

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$$= \underbrace{\hat{\mathbb{E}}[\frac{1}{2}\varphi_{xx}(x)B_t^2]}_{=G(\varphi_{xx})t,} + o(t), \quad G(a) := \hat{\mathbb{E}}[\frac{B_1^2}{2}a].$$

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- Thus $\partial_t \mathcal{S}_t[\varphi](x)|_{t=0} = G(\varphi_{xx}(x))$: the infinitesimal generator of $(\mathcal{S}_t)_{t\geq 0}$.

□

Construct G-BM on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$

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- For each $X(\omega) = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})$, with $t_i < t_{i+1}$, we set

$$\hat{\mathbb{E}}[X] := \tilde{\mathbb{E}}[\varphi(\sqrt{t_1}\xi_1, \sqrt{t_2 - t_1}\xi_2, \cdots, \sqrt{t_n - t_{n-1}}\xi_n)]$$

where

$$\xi_i \overset{d}{=} N(0, [\underline{\sigma}^2, \overline{\sigma}^2]),\ \ \xi_{i+1} \text{ is indep. of } (\xi_1, \cdots, \xi_i) \text{ under } \tilde{\mathbb{E}}.$$

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- Conditional expectation:

$$\hat{\mathbb{E}}_{t_1}[X] = \tilde{\mathbb{E}}[\varphi(x, \sqrt{t_2 - t_1}\xi_2, \cdots, \sqrt{t_n - t_{n-1}}\xi_n)]_{x = B_{t_1}}$$

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- Note that if $G_1 \leq G_2$ then $L_{G_1}^p(\Omega) \supset L_{G_2}^p(\Omega)$.

Probability Space	Nonlinear Expectation Space
$(\Omega, \mathcal{F}, P)$	$(\Omega, \mathcal{H}, \mathbb{E})$: (sublinear is basic)
Distributions: $X \overset{d}{=} Y$	$X \overset{d}{=} Y$,
Independence: Y indep. of X	Y indep. of X, (non-symm.)
LLN and CLT	LLN + CTL
Normal distributions	G-Normal distributions
Brownian motion $B_t(\omega) = \omega_t$	G-B.M. $B_t(\omega) = \omega_t$,
Qudratic variat. $\langle B \rangle_t = t$	$\langle B \rangle_t$: still a G-Brownian motion
Lévy process	G-Lévy process

Probability v.s. Nonlinear Expectation

Probability Space	**Nonlinear Expectation Space**
Itô's calculus for BM	Itô's calculus for G-BM
SDE $dx_t = b(x_t)dt + \sigma(x_t)dB_t$	$dx_t = \cdots + \beta(x_t)d\langle B\rangle_t$
Diffusion: $\partial_t u - \mathcal{L}u = 0$	$\partial_t u - G(Du, D^2u) = 0$
Markovian pro. and semi-grou	Nonlinear Markovian
Martingales	G-Martingales
$E[X\|\mathcal{F}_t] = E[X] + \int_0^T z_s dB_s$	$\mathbb{E}[X\|\mathcal{F}_t] = \mathbb{E}[X] + \int_0^t z_s dB_s + K_t$
	$K_t \overset{?}{=} \int_0^t \eta_s d\langle B\rangle_s - \int_0^t 2G(\eta_s)ds$

Probability Space	Nonlinear Expectation Space
P-almost surely analysis	$\hat{c}$-quasi surely analysis
	$\hat{c}(A) = \sup_\theta E_{P_\theta}[\mathbf{1}_A]$
$X(\omega)$: P-quasi continuous	$X(\omega)$: $\hat{c}$-quasi surely
$\Longleftrightarrow$ X is $\mathcal{B}(\Omega)$-meas.	continuous $\Longrightarrow$ X is $\mathcal{B}(\Omega)$-meas.

Based on: [Hu, Mingshang & P.]: G-Lévy Processes under Sublinear Expectations, (in arXiv)

Definition

A d-dimensional process $(X_t)_{t\geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a Lévy process:

1. $X_0 = 0$.
2. $X_{t+s} - X_t$ is indep. of $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$, $\forall t, s > 0, \quad t_1, t_2, \cdots, t_n \in [0, t]$.
3. Stationary increments: $X_{t+s} - X_t \overset{d}{=} X_s$.

Pure jump Lévy process on $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$.

Consider a pure jump case: $X_t = X_t^d$

Assumption:

$$\limsup_{t \downarrow 0} \hat{\mathbb{E}}[|X_t|] t^{-1} < \infty.$$

Proposition.

$u(t,x) = \mathcal{S}_t\varphi(x) := \hat{\mathbb{E}}[\varphi(x+X_t)]$ is a semigroup on $\varphi \in C_{b,Lip}(\mathbb{R}^d)$:

- $$\mathcal{S}_{t+s}\varphi(x) = \mathcal{S}_t\mathcal{S}_s\varphi(x), \ \ \mathcal{S}_0\varphi(x) = \varphi(x)$$

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- $$\mathcal{S}_{t+s}\varphi(x) = \mathcal{S}_t\mathcal{S}_s\varphi(x), \;\; \mathcal{S}_0\varphi(x) = \varphi(x)$$
- $$[\partial_t \mathcal{S}_t\varphi]_{t=0}(x) = G_X[\varphi(x+\cdot) - \varphi(x)],$$

□

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- $$[\partial_t\mathcal{S}_t\varphi]_{t=0}(x) = G_X[\varphi(x+\cdot) - \varphi(x)],$$
- G_X is well-defined on
$$\mathcal{L}_0 := \{f \in C_{b,Lip}(\mathbb{R}^d) : f(0) = 0 \text{ and } f(x) = o(|x|)\}$$

□

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\langle B\rangle_s$$
$$- \int_t^T Z_s dB_s - (K_T - K_t).$$

Under a Lipschitz condition of f and g in Y and Z. The existence and uniqueness of the solution (Y, Z, K) is proved, where K is a decreasing G-martingale.

G-Martingale representation

G-martingale M is of the form

$$M_t = M_0 + \bar{M}_t + K_t,$$

$$\bar{M}_t := \int_0^t z_s B_s,$$

$$K_t := \int_0^t \eta_s \langle B \rangle_s - \int_0^t 2G(\eta_s) ds.$$

GBSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\langle B\rangle_s \\ - \int_t^T Z_s dB_s - (K_T - K_t).$$

- f independent of z (and $g = 0$):

$$Y_t^i = \hat{\mathbb{E}}_t^{G_i}[\xi^i + \int_t^T f^i(s, Y_s)ds].$$

Peng [2005,07,10].
BSDE corresponding to (path-depedent) system of PDE:

$$\partial_t u^i + G^i(u^i, Du^i, D^2 u^i) + f^i(t, x, u^1, \cdots, u^k) = 0,$$
$$u^i(x, T) = \varphi^i(x),$$
$$i = 1, \cdots, k.$$

G^i satisfy the dominate condition:

$$G^i(x, y, p, A) - G^i(x, \bar{y}, \bar{p}, \bar{A}) \le c(|y - \bar{y}| + |p - p|) + \hat{G}(A - \bar{A}),$$

Existing results on fully nonlinear BSDEs

- [Soner, Touzi and Zhang, 2BSDE]
- $(Y, Z, K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_H^\kappa}$, $\mathbb{P} \in \mathcal{P}_H^\kappa$, the following BSDE

$$Y_t = \xi + \int_t^T F_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s + (K_T^{\mathbb{P}} - K_t^{\mathbb{P}}), \quad \mathbb{P}\text{-a.s.},$$

  with

-

$$K_t^{\mathbb{P}} = \operatorname*{ess\,inf}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t+,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[K_T^{\mathbb{P}}], \quad \mathbb{P}\text{-a.s.}, \quad \forall \mathbb{P} \in \mathcal{P}_H^\kappa, \; t \in [0, T].$$

A priori estimates

- $(\Omega_T, L_G^1(\Omega_T), \hat{\mathbb{E}})$
- $\Omega_T = C_0([0,T], \mathbb{R})$,
- $\overline{\sigma}^2 = \hat{\mathbb{E}}[B_1^2] \geq -\hat{\mathbb{E}}[-B_1^2] = \underline{\sigma}^2 > 0$.

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t), \quad \text{(GBSDE)}$$

where

$$f(t, \omega, y, z) : [0,T] \times \Omega_T \times \mathbb{R}^2 \to \mathbb{R}$$

Assumption: some $\beta > 1$ such that

(H1) for any y, z, $f(\cdot, \cdot, y, z) \in M_G^\beta(0, T)$,

(H2) $|f(t, \omega, y, z) - f(t, \omega, y', z')| \le L(|y - y'| + |z - z'|)$.

For (Y, Z, K) such that $Y \in S_G^\alpha(0, T)$, $Z \in H_G^\alpha(0, T)$, K: a decreasing G-martingale with $K_0 = 0$ and $K_T \in L_G^\alpha(\Omega_T)$.

An important observation

Lemma 3.4.

Let $X \in S_G^\alpha(0,T)$ for some $\alpha > 1$ and $\alpha^* = \frac{\alpha}{\alpha-1}$. Assume that K^j, $j = 1,2$, are two decreasing G-martingales with $K_0^j = 0$ and $K_T^j \in L_G^{\alpha^*}(\Omega_T)$. Then the process defined by

$$\int_0^t X_s^+ dK_s^1 + \int_0^t X_s^- dK_s^2$$

is also a decreasing G-martingale. □

A typical application of Lemma 3.4

- $-dY_t^i = f(s, Y_s^i, Z_s^i)ds - Z_s^i dB_s - dK_t^i, \quad i = 1, 2$

- $-dY_t^i = f(s, Y_s^i, Z_s^i)ds - Z_s^i dB_s - dK_t^i, \quad i = 1, 2$
- $|\hat{Y}_T|^2 - \int_t^T 2\hat{Y}_s \hat{f}_s ds - \int_t^T |\hat{Z}_s|^2 d\langle B\rangle_s + \int_t^T 2\hat{Y}_s \hat{Z}_s dB_s$

A typical application of Lemma 3.4

- $-dY_t^i = f(s, Y_s^i, Z_s^i)ds - Z_s^i dB_s - dK_t^i, \quad i = 1, 2$
- $|\hat{Y}_T|^2 - \int_t^T 2\hat{Y}_s\hat{f}_s ds - \int_t^T |\hat{Z}_s|^2 d\langle B\rangle_s + \int_t^T 2\hat{Y}_s\hat{Z}_s dB_s$
- $= |\hat{Y}_t|^2 + \int_t^T 2\hat{Y}_s d(K_t^1 - K_t^2)$

A typical application of Lemma 3.4

- $-dY_t^i = f(s, Y_s^i, Z_s^i)ds - Z_s^i dB_s - dK_t^i, \quad i = 1, 2$
- $|\hat{Y}_T|^2 - \int_t^T 2\hat{Y}_s \hat{f}_s ds - \int_t^T |\hat{Z}_s|^2 d\langle B\rangle_s + \int_t^T 2\hat{Y}_s \hat{Z}_s dB_s$
- $= |\hat{Y}_t|^2 + \int_t^T 2\hat{Y}_s d(K_t^1 - K_t^2)$
- $= |\hat{Y}_t|^2 + 2\int_t^T [(\hat{Y}_s)^+ dK_s^1 + (\hat{Y}_s)^- dK_s^2] - 2\int_t^T [(\hat{Y}_s)^- dK_s^1 + (\hat{Y}_s)^+ dK_s^2]$

A typical application of Lemma 3.4

- $-dY_t^i = f(s, Y_s^i, Z_s^i)ds - Z_s^i dB_s - dK_t^i, \quad i = 1, 2$
- $|\hat{Y}_T|^2 - \int_t^T 2\hat{Y}_s\hat{f}_s ds - \int_t^T |\hat{Z}_s|^2 d\langle B\rangle_s + \int_t^T 2\hat{Y}_s\hat{Z}_s dB_s$
- $= |\hat{Y}_t|^2 + \int_t^T 2\hat{Y}_s d(K_t^1 - K_t^2)$
- $= |\hat{Y}_t|^2 + 2\int_t^T [(\hat{Y}_s)^+ dK_s^1 + (\hat{Y}_s)^- dK_s^2] - 2\int_t^T [(\hat{Y}_s)^- dK_s^1 + (\hat{Y}_s)^+ dK_s^2]$
- $\geq |\hat{Y}_t|^2 + 2\int_t^T [(\hat{Y}_s)^+ dK_t^1 + (\hat{Y}_s)^- dK_t^2]$

A typical application of Lemma 3.4

- $-dY_t^i = f(s, Y_s^i, Z_s^i)ds - Z_s^i dB_s - dK_t^i, \quad i = 1, 2$
- $|\hat{Y}_T|^2 - \int_t^T 2\hat{Y}_s \hat{f}_s ds - \int_t^T |\hat{Z}_s|^2 d\langle B\rangle_s + \int_t^T 2\hat{Y}_s \hat{Z}_s dB_s$
- $= |\hat{Y}_t|^2 + \int_t^T 2\hat{Y}_s d(K_t^1 - K_t^2)$
- $= |\hat{Y}_t|^2 + 2\int_t^T [(\hat{Y}_s)^+ dK_s^1 + (\hat{Y}_s)^- dK_s^2] - 2\int_t^T [(\hat{Y}_s)^- dK_s^1 + (\hat{Y}_s)^+ dK_s^2]$
- $\geq |\hat{Y}_t|^2 + 2\int_t^T [(\hat{Y}_s)^+ dK_t^1 + (\hat{Y}_s)^- dK_t^2]$
- Thus

$$|\hat{Y}_t|^2 \leq \hat{\mathbb{E}}_t[|\hat{Y}_T|^2 - \int_t^T 2\hat{Y}_s \hat{f}_s ds - \int_t^T |\hat{Z}_t|^2 d\langle B\rangle_t]$$

Proposition 3.5.

Assume (H1)-(H2) and $(Y, Z, K_T) \in \mathbb{S}^\alpha(0,T) \times \mathbb{H}^\alpha(0,T) \times \mathbb{S}^\alpha(\Omega_T)$ solves

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t),$$

where K is a decreasing process with $K_0 = 0$. Then

$$\hat{\mathbb{E}}[(\int_0^T |Z_s|^2 ds)^{\frac{\alpha}{2}}] \le C_\alpha\{\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t|^\alpha] + (\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t|^\alpha])^{\frac{1}{2}} (\hat{\mathbb{E}}[(\int_0^T |f_s^0| ds)^\alpha])^{\frac{1}{2}}\},$$

$$\hat{\mathbb{E}}[|K_T|^\alpha] \le C_\alpha\{\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t|^\alpha] + \hat{\mathbb{E}}[(\int_0^T |f_s^0 ds)^\alpha]\},$$

$$f_s^0 := |f(s,0,0)| + L^w \varepsilon$$

Proposition 3.7.

We assume (H1) and (H2). Assume that $(Y, Z, K) \in \mathfrak{S}_G^{\alpha}(0, T)$ for some $1 < \alpha < \beta$ is a solution (GBSDE). Then

- There exists a constant $C_\alpha := C(\alpha, T, \underline{\sigma}, L^w) > 0$ such that

$$|Y_t|^\alpha \leq C_\alpha \hat{\mathbb{E}}_t[|\xi|^\alpha + \int_t^T |f_s^0|^\alpha ds],$$

$$\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t|^\alpha] \leq C_\alpha \hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[|\xi|^\alpha + \int_0^T |f_s^0|^\alpha ds]],$$

  where $f_s^0 = |f(s, 0, 0)| + L^w \varepsilon$.

- For any given α' with $\alpha < \alpha' < \beta$, there exists a constant $C_{\alpha,\alpha'}$ depending on α, α', T, $\underline{\sigma}$, L^w such that

$$\hat{\mathbb{E}}[\sup_{t\in[0,T]} |Y_t|^\alpha] \leq C_{\alpha,\alpha'}\{\hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[|\xi|^\alpha]]$$

$$+ (\hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[(\int_0^T f_s^0 ds)^{\alpha'}]])^{\frac{\alpha}{\alpha'}}$$

Proposition 3.8.

Let f_i, $i = 1, 2$, satisfy (H1) and (H2). Assume

$$Y_t^i = \xi^i + \int_t^T f_i(s, Y_s^i, Z_s^i)ds - \int_t^T Z_s^i dB_s - (K_T^i - K_t^i),$$

where $Y^i \in \mathbb{S}^\alpha(0, T)$, $Z^i \in \mathbb{H}^\alpha(0, T)$, K^i is a decreasing process with $K_0^i = 0$ and $K_T^i \in \mathbb{L}^\alpha(\Omega_T)$ for some $\alpha > 1$. Set $\hat{Y}_t = Y_t^1 - Y_t^2$, $\hat{Z}_t = Z_t^1 - Z_t^2$ and $\hat{K}_t = K_t^1 - K_t^2$. Then there exists a constant $C_\alpha := C(\alpha, T, \underline{\sigma}, L^w) > 0$ such that

$$\hat{\mathbb{E}}[(\int_0^T |\hat{Z}_s|^2 ds)^{\frac{\alpha}{2}}] \le C_\alpha \{\|\hat{Y}\|_{\mathbb{S}^\alpha}^\alpha + \|\hat{Y}\|_{\mathbb{S}^\alpha}^{\frac{\alpha}{2}} \sum_{i=1}^2 [||Y^i||_{\mathbb{S}^\alpha}^{\frac{\alpha}{2}} + ||\int_0^T f_s^{i,0} ds||_{\alpha,G}^{\frac{\alpha}{2}}]\},$$

where $f_s^{i,0} = |f_i(s, 0, 0)| + L^w \varepsilon$, $i = 1, 2$. □

Proposition 3.9.

Let $\xi^i \in L_G^\beta(\Omega_T)$ with $\beta > 1$, $i = 1, 2$, and f_i satisfy (H1) and (H2). Assume that $(Y^i, Z^i, K^i) \in \mathfrak{S}_G^\alpha(0, T)$ for some $1 < \alpha < \beta$ are the solutions of equation (GBSDE) to ξ^i and f_i. Then

(i) $|\hat{Y}_t|^\alpha \le C_\alpha \hat{\mathbb{E}}_t[|\hat{\xi}|^\alpha + \int_t^T |\hat{f}_s|^\alpha ds]$, where $\hat{f}_s = |f_1(s, Y_s^2, Z_s^2) - f_2(s, Y_s^2, Z_s^2)| + L_1^w \varepsilon$.

(ii) For any given α' with $\alpha < \alpha' < \beta$, there exists a constant $C_{\alpha,\alpha'}$ depending on α, α', T, $\underline{\sigma}$, L^w such that

$$\hat{\mathbb{E}}[\sup_{t\in[0,T]} |\hat{Y}_t|^\alpha] \le C_{\alpha,\alpha'}\{\hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[|\hat{\xi}|^\alpha]] + (\hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[(\int_0^T \hat{f}_s ds)^{\alpha'}]])^{\frac{\alpha}{\alpha'}} + \hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[(\int_0^T \hat{f}_s ds)^{\alpha'}]]\}.$$

□

Existence and uniqueness of G-BSDEs

$$\partial_t u + G(\partial^2_{xx} u) + h(u, \partial_x u) = 0, \quad u(T, x) = \varphi(x). \qquad \text{(GPDE)}$$

We approximate the solution f by those of equations (GBSDE) with much simpler $\{f_n\}$. More precisely, assume that $\|f_n - f\|_{M_G^\beta} \to 0$ and (Y^n, Z^n, K^n) is the solution of the following G-BSDE

$$Y_t^n = \xi + \int_t^T f_n(s, Y_s^n, Z_s^n)ds - \int_t^T Z_s^n dB_s - (K_T^n - K_t^n).$$

We try to prove that (Y^n, Z^n, K^n) converges to (Y, Z, K) and (Y, Z, K) is the solution of the following G-BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t).$$

Main results:

Theorem

Assume that $\xi \in L_G^{\beta}(\Omega_T)$, $\beta > 1$ *and* f *satisfies (H1) and (H2). Then equation (G-BSDE) has a unique solution* (Y, Z, K)*. Moreover, for any* $1 < \alpha < \beta$ *we have* $Y \in S_G^{\alpha}(0, T)$, $Z \in H_G^{\alpha}(0, T)$ *and* $K_T \in L_G^{\alpha}(\Omega_T)$.

Sketch of Proof of Theorem.

Step 1. $f(t,\omega,y,z) = h(y,z)$, $h \in C_0^\infty(\mathbb{R}^2)$.
Part 1) $\xi = \varphi(B_T - B_{t_1})$: $\exists\ \alpha \in (0,1)$ s.t.,

$$||u||_{C^{1+\alpha/2,2+\alpha}([0,T-\kappa]\times\mathbb{R})} < \infty, \quad \kappa > 0.$$

Itô's formula to $u(t, B_t - B_{t_1})$ on $[t_1, T-\kappa]$, we get

$$u(t,B_t - B_{t_1}) = u(T-\kappa, B_{T-\kappa} - B_{t_1}) + \int_t^{T-\kappa} h(u,\partial_x u)(s, B_s - B_{t_1})ds - \int_t^{T-\kappa} \partial_x u(s, B_s - B_{t_1})dB_s - (K_{T-\kappa} - K_t),$$

□

Sketch of Proof of Theorem.

where

$$K_t = \frac{1}{2}\int_{t_1}^{t} \partial_{xx}^2 u(\cdot) d\langle B\rangle_s - \int_{t_1}^{t} G(\partial_{xx}^2 u(\cdot)) ds$$

$$|u(t,x) - u(s,y)| \leq L_1(\sqrt{|t-s|} + |x-y|).$$

$\tilde{u}$ is the solution of PDE:

$$\partial_t \tilde{u} + G(\partial_{xx}^2 \tilde{u}) + h(\tilde{u}, \partial_x \tilde{u}) = 0,$$
$$\tilde{u}(T,x) = \varphi(x + x_0).$$

□

Sketch of Proof of Theorem.

$$u(t, x + x_0) \leq u(t, x) + L_\varphi |x_0| \exp(L_h(T - t)),$$

Since x_0 is arbitrary, we get $|u(t, x) - u(t, y)| \leq \hat{L}|x - y|$, where $\hat{L} = L_\varphi \exp(L_h T)$. From this we can get $|\partial_x u(t, x)| \leq \hat{L}$ for each $t \in [0, T]$, $x \in \mathbb{R}$. On the other hand, for each fixed $\bar{t} < \hat{t} < T$ and $x \in \mathbb{R}$, applying Itô's formula to $u(s, x + B_s - B_{\bar{t}})$ on $[\bar{t}, \hat{t}]$, we get

$$u(\bar{t}, x) = \hat{\mathbb{E}}[u(\hat{t}, x + B_{\hat{t}} - B_{\bar{t}}) + \int_{\bar{t}}^{\hat{t}} h(u, \partial_x u)(s, x + B_s - B_{\bar{t}}) ds].$$

□

Sketch of Proof of Theorem.

From this we deduce that

$$|u(\bar{t},x)-u(\hat{t},x)| \leq \hat{\mathbb{E}}[\hat{L}|B_{\hat{t}}-B_{\bar{t}}|+\tilde{L}|\hat{t}-\bar{t}|] \leq (\hat{L}\bar{\sigma}+\tilde{L}\sqrt{T})\sqrt{|\hat{t}-\bar{t}|},$$

where $\tilde{L}=\sup_{(x,y)\in\mathbb{R}^2}|h(x,y)|$. Thus we get (**??**) by taking $L_1=\max\{\hat{L},\hat{L}\bar{\sigma}+\tilde{L}\sqrt{T}\}$. Letting $\kappa\downarrow 0$ in Itô's equation, it is easy to verify that

$$\hat{\mathbb{E}}[|Y_{T-\kappa}-\xi|^2+\int_{T-\kappa}^{T}|Z_t|^2dt+(K_{T-\kappa}-K_T)^2]\to 0,$$

where $Y_t=u(t,B_t-B_{t_1})$ and $Z_t=\partial_x u(t,B_t-B_{t_1})$. Thus $(Y_t,Z_t,K_t)_{t\in[t_1,T]}$ is a solution of equation (GBSDE) with terminal value $\xi=\varphi(B_T-B_{t_1})$. Furthermore, it is easy to check that $Y\in S_G^{\alpha}(t_1,T)$, $Z\in H_G^{\alpha}(t_1,T)$ and $K_T\in L_G^{\alpha}(\Omega_T)$ for any $\alpha>1$. □

Sketch of Proof of Theorem.

Part 2) $\xi = \psi(B_{t_1}, B_T - B_{t_1})$:

$$\begin{aligned} u(t,x,B_t - B_{t_1}) =& u(T,x,B_T - B_{t_1}) + \int_t^T h(u, \partial_y u)(s,x,B_s - B_{t_1}) ds \\ & - \int_t^T \partial_y u(\cdot) dB_s - (K_T^x - K_t^x), \end{aligned}$$

$$K_t^x = \frac{1}{2}\int_{t_1}^t \partial_{yy}^2 u(\cdot) d\langle B\rangle_s - \int_{t_1}^t G(\partial_{yy}^2 u(\cdot)) ds.$$

$$Y_t = Y_T + \int_t^T h(Y_s, Z_s) ds - \int_t^T Z_s dB_s - (K_T - K_t),$$

□

Sketch of Proof of Theorem.

where

$$Y_t := u(t, B_{t_1}, B_t - B_{t_1}), \quad Z_t := \partial_y u(\cdot),$$

$$K_t := \frac{1}{2}\int_{t_1}^{t} \partial_{yy}^2 u(\cdot) d\langle B\rangle_s - \int_{t_1}^{t} G(\partial_{yy}^2 u(\cdot)) ds.$$

Need to prove $(Y, Z, K) \in \mathfrak{S}_G^{\alpha}(0, T)$. By partition of unity theorem, $\exists$ $h_i^n \in C_0^{\infty}(\mathbb{R})$ s.t.

$$\lambda(\mathrm{supp}(h_i^n)) < 1/n, \quad 0 \leq h_i^n \leq 1,$$

$$I_{[-n,n]}(x) \leq \sum_{i=1}^{k_n} h_i^n \leq 1.$$

□

Sketch of Proof of Theorem.

We have

$$Y_t^n = Y_T^n + \int_t^T \sum_{i=1}^n h(y_s^{n,i}, z_s^{n,i}) h_i^n(B_{t_1}) ds - \int_t^T Z_s^n dB_s - (K_T^n - K_t^n),$$

where

$$y_t^{n,i} = u(t, x_i^n, B_t - B_{t_1}), \quad z_t^{n,i} = \partial_y u(t, x_i^n, B_t - B_{t_1}),$$

$$Y_t^n = \sum_{i=1}^n y_t^{n,i} h_i^n(B_{t_1}), \quad Z_t^n = \sum_{i=1}^n z_t^{n,i} h_i^n(B_{t_1}),$$

$$K_t^n = \sum_{i=1}^n K_t^{x_i^n} h_i^n(B_{t_1}).$$

□

Sketch of Proof of Theorem.

Thus

$$|Y_t - Y_t^n| \leq \sum_{i=1}^{k_n} h_i^n(B_{t_1})|u(t, x_i^n, B_t - B_{t_1}) - u(t, B_{t_1}, B_t - B_{t_1})|$$

$$+ |Y_t| I_{[|B_{t_1}|>n]} \leq \frac{L_2}{n} + \frac{||u||_\infty}{n}|B_{t_1}|.$$

Thus

$$\hat{\mathbb{E}}[\sup_{t\in[t_1,T]} |Y_t - Y_t^n|^\alpha] \leq \hat{\mathbb{E}}[(\frac{L_2}{n} + \frac{||u||_\infty}{n}|B_{t_1}|)^\alpha] \to 0.$$

By the estimates

$$\hat{\mathbb{E}}[(\int_{t_1}^T |Z_s - Z_s^n|^2 ds)^{\alpha/2}] \leq C_\alpha\{\hat{\mathbb{E}}[\sup_{t\in[t_1,T]} |Y_t - Y_t^n|^\alpha]$$

$$+(\hat{\mathbb{E}}[\sup_{t\in[t_1,T]} |Y_t - Y_t^n|^\alpha])^{1/2}\} \to 0.$$

Thus $Z \in M_G^\alpha(0, T)$, $K_t \in L_G^\alpha(\Omega_t)$.

Sketch of Proof of Theorem.

[Sketch of Proof of Theorem] prove K is G-martingale. Following [Li-P.], we take

$$h_i^n(x) = I_{[-n+\frac{i}{n}, -n+\frac{i+1}{n})}(x), \quad i = 0, \dots, \quad 2n^2 - 1,$$
$$h_{2n^2}^n = 1 - \sum_{i=0}^{2n^2-1} h_i^n$$

$$\tilde{Y}_t^n = \sum_{i=0}^{2n^2} u(t, -n + \frac{i}{n}, B_t - B_{t_1}) h_i^n(B_{t_1}), \ \tilde{Z}_t^n = \sum_{i=0}^{2n^2} \partial_y u(\cdot) h_i^n(B_{t_1})$$

solves

$$\tilde{Y}_t^n = \tilde{Y}_T^n + \int_t^T h(\tilde{Y}_s^n, \tilde{Z}_s^n) ds - \int_t^T \tilde{Z}_s^n dB_s - (\tilde{K}_T^n - \tilde{K}_t^n),$$

□

Sketch of Proof of Theorem.

We have $\hat{\mathbb{E}}[(\int_{t_1}^{T}|Z_s-\tilde{Z}_s^n|^2ds)^{\alpha/2}]\to 0$. Thus $\hat{\mathbb{E}}[|K_t-\tilde{K}_t^n|^\alpha]\to 0$ and $\hat{\mathbb{E}}_t[K_s]=K_t$. For $Y_{t_1}=u(t_1,B_{t_1},0)$, we can use the same method as Part 1 on $[0,t_1]$.

Step 2) $f(t,\omega,y,z)=\sum_{i=1}^{N}f^ih^i(y,z)$ with $f^i\in M_G^0(0,T)$ and $h^i\in C_0^\infty(\mathbb{R}^2)$. □

Sketch of Proof of Theorem.

Step 3) $f(t,\omega,y,z)=\sum_{i=1}^N f^i h^i(y,z)$ with $f^i\in M_G^\beta(0,T)$ bounded and $h^i\in C_0^\infty(\mathbb{R}^2)$, $h^i\geq 0$ and $\sum_{i=1}^N h^i\leq 1$:
Choose

$$f_n^i\in M_G^0(0,T) \text{ s.t. } |f_n^i|\leq \|f^i\|_\infty,\ \sum_{i=1}^N \|f_n^i-f^i\|_{M_G^\beta}<1/n.$$

Set $f_n:=\sum_{i=1}^N f_n^i h^i(y,z)$.
Let (Y^n,Z^n,K^n) be the solution of (GBSDE) with generator f_n.

$$\begin{aligned}\hat{f}_s^{m,n}&:=|f_m(s,Y_s^n,Z_s^n)-f_n(s,Y_s^n,Z_s^n)|\\&\leq\sum_{i=1}^N|f_n^i-f^i|+\sum_{i=1}^N|f_m^i-f^i|=:\hat{f}_n+\hat{f}_m,\end{aligned}$$

□

Sketch of Proof of Theorem.

We have, for any $1 < \alpha < \beta$,

$$\hat{\mathbb{E}}_t[(\int_0^T \hat{f}_s^{m,n} ds)^\alpha] \leq \hat{\mathbb{E}}_t[(\int_0^T (|\hat{f}_n(s)| + |\hat{f}_m(s)|) ds)^\alpha].$$

By Theorem 2.10, $\forall \alpha \in (1, \beta)$

$$\hat{\mathbb{E}}\left[\sup_t \hat{\mathbb{E}}_t[|\int_0^T \hat{f}_s^{m,n} ds|^\alpha]\right] \to 0, \ \ m, n \to \infty$$

By Proposition 3.9 $\{Y^n\}$ is Cauchy under $\|\cdot\|_{S_G^\alpha}$. By Proposition 3.7, 3.8, $\{Z^n\}$ is a also Cauchy under $\|\cdot\|_{H_G^\alpha}$ thus $\{\int_0^T f_n(s, Y_s^n, Z_s^n) ds\}$ under $\|\cdot\|_{L_G^\alpha}$ thus $\{K_T^n\}$ is also Cauchy under $\|\cdot\|_{L_G^\alpha}$. □

Sketch of Proof of Theorem.

Step 4). f is bounded, Lipschitz. $|f(t,\omega,y,z)| \le CI_{B(R)}(y,z)$ for some $C,R>0$. Here $B(R)=\{(y,z)|y^2+z^2\le R^2\}$.
For any n, by the partition of unity theorem, there exists $\{h_n^i\}_{i=1}^{N_n}$ such that $h_n^i \in C_0^\infty(\mathbb{R}^2)$, the diameter of support $\lambda(\text{supp}(h_n^i))< 1/n$, $0\le h_n^i\le 1$, $I_{B(R)}\le \sum_{i=1}^N h_n^i \le 1$. Then $f(t,\omega,y,z)=\sum_{i=1}^N f(t,\omega,y,z)h_n^i$. Choose y_n^i, z_n^i such that $h_n^i(y_n^i,z_n^i)>0$. Set

$$f_n(t,\omega,y,z)=\sum_{i=1}^N f(t,\omega,y_n^i,z_n^i)h_n^i(y,z)$$

□

Sketch of Proof of Theorem.

Then

$$|f(t,\omega,y,z)-f_n(t,\omega,y,z)| \leq \sum_{i=1}^{N} |f(t,\omega,y,z)-f(t,\omega,y_n^i,z_n^i)|h_n^i \leq L/n$$

and

$$|f_n(t,\omega,y,z)-f_n(t,\omega,y',z')| \leq L(|y-y'|+|z-z'|+2/n).$$

Noting that $|f_m(s,Y_s^n,Z_s^n)-f_n(s,Y_s^n,Z_s^n)| \leq (L/n+L/m)$, □

Sketch of Proof of Theorem.

we have

$$\hat{\mathbb{E}}_t[|\int_0^T (|f_m(s, Y_s^n, Z_s^n) - f_n(s, Y_s^n, Z_s^n)| + \frac{2L}{m})ds|^\alpha] \leq T^\alpha(\frac{L}{n} + \frac{3L}{m})^\alpha.$$

So by the estimates $\{Y^n\}$ cauchy under $\|\cdot\|_{S_G^\alpha}$. $\{Z^n\}$ is cauchy under $\|\cdot\|_{H_G^\alpha}$. is also cauchy $\{\int_0^T f_n(s, Y_s^n, Z_s^n)ds\}$ under $\|\cdot\|_{L_G^\alpha}$. □

Sketch of Proof of Theorem.

Step 5). f is bounded, Lipschitz.
For any $n \in \mathbb{N}$, choose $h^n \in C_0^\infty(\mathbb{R}^2)$ such that $I_{B(n)} \leq h^n \leq I_{B(n+1)}$ and $\{h^n\}$ are uniformly Lipschitz w.r.t. n. Set $f_n = fh^n$, which are uniformly Lipschitz. Noting that for $m > n$

$$\begin{aligned} &|f_m(s, Y_s^n, Z_s^n) - f_n(s, Y_s^n, Z_s^n)| \\ &\leq |f(s, Y_s^n, Z_s^n)| I_{[|Y_s^n|^2 + |Z_s^n|^2 > n^2]} \\ &\leq \|f\|_\infty \frac{|Y_s^n| + |Z_s^n|}{n}, \end{aligned}$$

□

Sketch of Proof of Theorem.

we have

$$\hat{\mathbb{E}}_t[(\int_0^T |f_m(s, Y_s^n, Z_s^n) - f_n(s, Y_s^n, Z_s^n)|ds)^\alpha]$$
$$\leq \frac{\|f\|_\infty^\alpha}{n^\alpha}\hat{\mathbb{E}}_t[(\int_0^T |Y_s^n| + |Z_s^n|ds)^\alpha]$$
$$\leq \frac{\|f\|_\infty^\alpha}{n^\alpha}C(\alpha, T)\hat{\mathbb{E}}_t[\int_0^T |Y_s^n|^\alpha ds + (\int_0^T |Z_s^n|^2 ds)^{\alpha/2}],$$

where $C(\alpha, T) := 2^{\alpha-1}(T^{\alpha-1} + T^{\alpha/2}])$. □

Sketch of Proof of Theorem.

So by Theorem 2.10 and Proposition 3.4 we get $|| \int_0^T \hat{f}_s^{m,n} ds||_{\alpha,\mathcal{E}} \longrightarrow 0$ as $m, n \longrightarrow \infty$ for any $\alpha \in (1, \beta)$. By Proposition 3.5, we conclude that $\{Y^n\}$ is cauchy under $\|\cdot\|_{S_G^\alpha}$. $\{Z^n\}$ cauchy sequence under $\|\cdot\|_{H_G^\alpha}$. $\{\int_0^T f_n(s, Y_s^n, Z_s^n)ds\}$ is cauchy under $\|\cdot\|_{L_G^\alpha}$:

$$
\begin{aligned}
&|f_n(s, Y^n, Z^n) - f_m(s, Y^m, Z^m)| \\
&\leq |f_m(s, Y^n, Z^n) - f_m(s, Y^m, Z^m)| + |f_n(s, Y^n, Z^n) - f_m(s, Y^n, Z^n)| \\
&\leq L(|\hat{Y}_s| + |\hat{Z}_s|) + |f(s, Y_s^n, Z_s^n)| 1_{[|Y_s^n| + |Z_s^n| > n]},
\end{aligned}
$$

which implies the desired result. □

Sketch of Proof of Theorem.

Step 6). For the general f.

Set $f_n = [f \vee (-n)] \wedge n$, which are uniformly Lipschitz. Choose $0 < \delta < \frac{\beta-\alpha}{\alpha} \wedge 1$. Then $\alpha < \alpha' = (1+\delta)\alpha < \beta$. Since for $m > n$

$$|f_n(s, Y^n, Z^n) - f_m(s, Y^n, Z^n)| \leq |f(s, Y_s^n, Z_s^n)| I_{[|f(s, Y_s^n, Y_s^n)|>n]} \leq \frac{1}{n^\delta}|f(s, Y_s^n$$

we have

$$\hat{\mathbb{E}}_t[(\int_0^T |f_n(s, Y^n, Z^n) - f_m(s, Y^n, Z^n)|ds)^\alpha]$$
$$\leq \frac{1}{n^{\alpha\delta}}\hat{\mathbb{E}}_t[(\int_0^T |f(s, Y_s^n, Z_s^n)|^{1+\delta} ds)^\alpha],$$
$$\leq \frac{C(\alpha, T, L, \delta)}{n^{\alpha\delta}}\hat{\mathbb{E}}_t[\int_0^T |f(s, 0, 0)|^{\alpha'} ds + \int_0^T |Y_s^n|^{\alpha'} ds + (\int_0^T |Z_s^n|^2 ds)^{\frac{\alpha'}{2}}],$$

where $C(\alpha, T, L, \delta) := 3^{\alpha'-1}(T^{\alpha-1} + L^{\alpha'} T^{\frac{\alpha(1-\delta)}{2}} + T^{\alpha-1}L^{\alpha'})$. □

Sketch of Proof of Theorem.

So by Song's estimate and a priori estimate, we get $||\int_0^T \hat{f}_s^{m,n} ds||_{\alpha,\mathcal{E}} \longrightarrow 0$ as $m, n \longrightarrow \infty$ for any $\alpha \in (1, \beta)$. We know that $\{Y^n\}$ is a cauchy sequence under the norm $\|\cdot\|_{S_G^\alpha}$. And consequently $\{Z^n\}$ is a cauchy sequence under the norm $\|\cdot\|_{H_G^\alpha}$. Now we prove $\{\int_0^T f_n(s, Y_s^n, Z_s^n) ds\}$ is a cauchy sequence under the norm $\|\cdot\|_{L_G^\alpha}$. In fact,

$$
\begin{aligned}
&|f_n(s, Y^n, Z^n) - f_m(s, Y^m, Z^m)| \\
&\leq |f_m(s, Y^n, Z^n) - f_m(s, Y^m, Z^m)| + |f_n(s, Y^n, Z^n) - f_m(s, Y^n, Z^n)| \\
&\leq L(|\hat{Y}_s| + |\hat{Z}_s|) + \frac{3^\delta}{n^\delta}(|f_s^0|^{1+\delta} + |Y_s^n|^{1+\delta} + |Z_s^n|^{1+\delta}),
\end{aligned}
$$

which implies the desired result. □

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