Modeling Insider Trading

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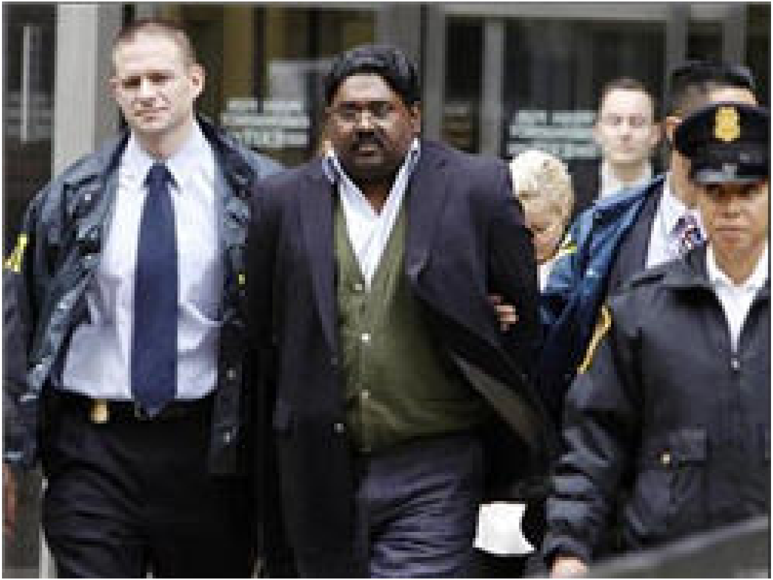
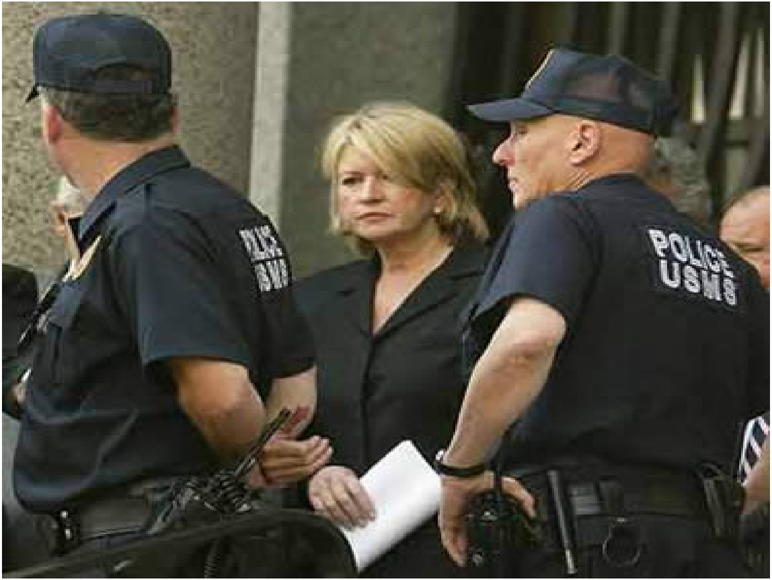
A Mathematical Model of Insider Trading

- To model insider trading, one first needs to model "normal" trading
- We take the minimalist approach of No Arbitrage, pioneered by Harrison, Pliska and Kreps
- Perfected by Delbaen and Schachermayer (Many others in-between: Stricker, Kabanov, etc.)
- In these models, there is a standard underlying space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ represents the collection of observable events that the market can see
- The idea of insider trading presupposes some participants have material information not shared with the rest of the market, and act on it
- Recent examples **that are known to us** include the Galleon Group, Martha Stewart, LIBOR

Raj Rajaratnam



Martha Stewart



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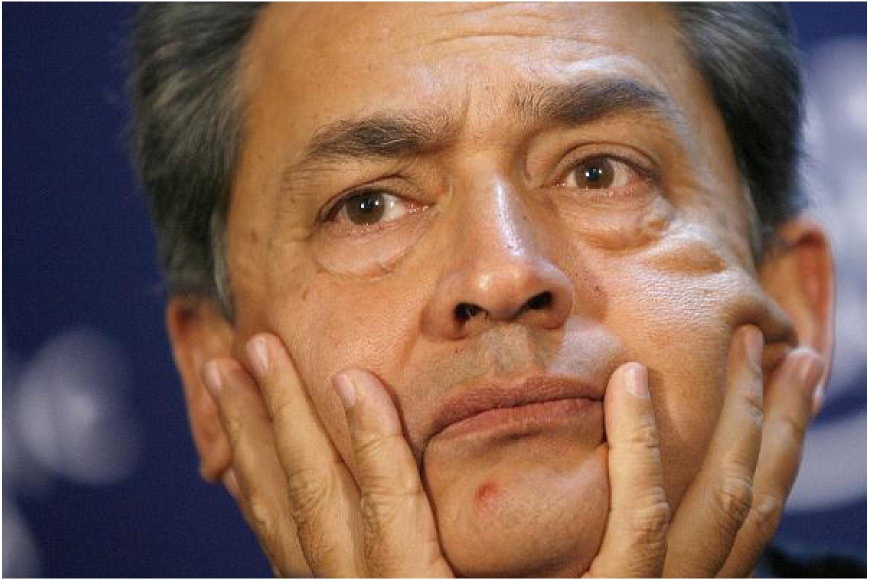
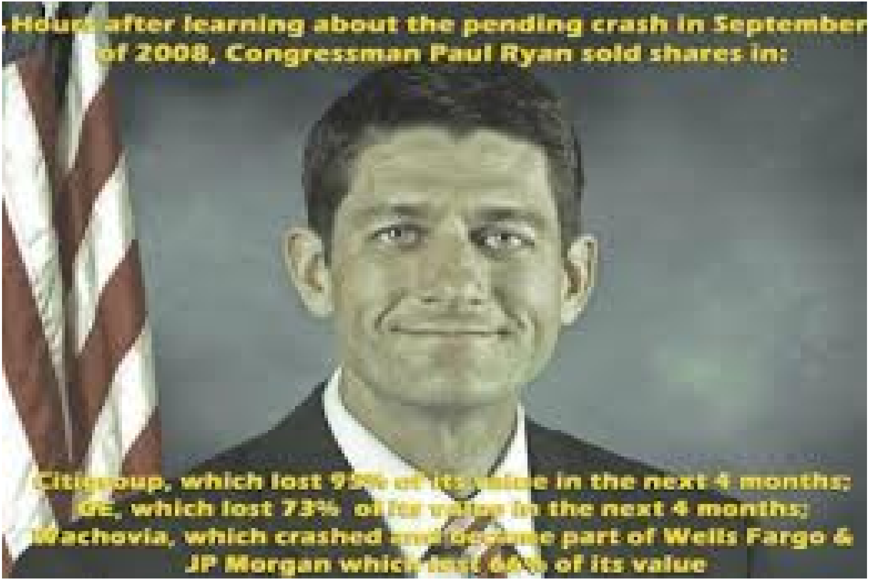


Paul Ryan; September 18, 2008



Initial Expansions

- To model the inclusion of extra information, we use the theory of the **expansion of filtrations**
- This was originally proposed by K. Itô in 1976:

$$\int_0^t B_1 H_s dB_s = B_1 \int_0^t H_s dB_s$$

- Itô added $\sigma(B_1)$ to the σ algebras of the filtration to get $\mathbb{G}$, where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(B_1)$ for $t \geq 0$
- Under $\mathbb{G}$ the Brownian motion B is no longer a Brownian motion, but it remains a semimartingale
- This became known as an **initial expansion**, where one adds the inside information at time 0
- An item of interest is: When does such an expansion preserve the semimartingale property: If X is an $\mathbb{F}$ semimartingale, does it remain a semimartingale in $\mathbb{G}$? If it does, what is its $\mathbb{G}$ decomposition?

Progressive Expansions

- A second kind of expansion, introduced by Martin Barlow in his PhD thesis, is a **progressive expansion**
- A progressive expansion takes a positive random variable, and adds it dynamically to the filtration in order to make it into a stopping time
- For example let $L \geq 0$ be a random variable, and $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ with $\mathbb{G}_t = \mathcal{F}_t \vee \sigma(L \wedge t)$
- An **honest time** is a random variable that is the right end of an optional set (in $\mathbb{R}_+ \times \Omega$)
- If L is honest, then $\mathbb{F}$ semimartingales remain $\mathbb{G}$ semimartingales, although the decompositions change

Filtration Expansions as Models of Insider Trading: Prior Work

- Much prior work exploring this idea: S. Ankirchner, K. Back, F. Baudoin, F. Biagini, D. Coculescu, R.J. Elliott, D. Kreher, H. Föllmer, A. Grorud, P. Imkeller, M. Jeanblanc, Y. Kchia, A. Kyle, M. Larsson, A. Nikeghbali, B. Øksendal, M. Pontier, F. Weisz, M. Yor, and J. Zwierz
- How does insider knowledge affect things? Two basic outcomes:
  - (a) The new $\mathbb{G}$ decomposition can lead to a different risk neutral measure
  - (b) The new $\mathbb{G}$ decomposition can lead to the non existence of a risk neutral measure, and hence the existence of arbitrage
- Most of the prior research revolves around (b), but actually (a) is arguably more interesting

Scalable Arbitrage through Insider Trading

- **Scalable arbitrage might not be scalable:** A scalable arbitrage allows unbounded profits. Because acting on insider knowledge is illegal, it cannot be obvious it is going on, leading to constraints on otherwise scalable arbitrage opportunities
- The approach of Kyle and Back (1980's and 1990s) takes this into account, by having a feedback loop, with equilibrium considerations, so that the distribution of the stock price is not overly perturbed, and therefore not noticed by regulators.
- This approach may seem quaint today, when wire tapping is allowed, and trading history is (in theory at least) discoverable
- It is possible to hide insider trading via complicated means: for example, buying a sector index, and shorting all the stocks in the index except the one where insider information indicates a near future rise

Scalable Arbitrage through Insider Trading

- It is easy to construct insider trading examples leading to scalable arbitrage
- **P. Imkeller** has worked out in detail the example where L is the last time a price process, following a recurrent diffusion, crosses 0 before a fixed time T
- If one knows the time L, and $a > 0$, one can buy at L and sell at time T; if $a < 0$, one sells short naked at time L, and covers at time T
- Imkeller shows mathematically that no risk neutral measure can exist under $\mathbb{G}$; **J. Zwierz** extends his result to a more general situation

Background

- We begin with $(\Omega, \mathcal{F}, P)$ and $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$
- Let $S \geq 0$ be the price process of a risky asset, and let $r = 0$ (r is the spot interest rate)
- Under mild assumptions, there exists an equivalent probability measure Q under which S is a local martingale
- Such a Q is called a *risk neutral measure*; if it is unique, the market is *complete*
- The measure Q can be used to price financial derivatives (contingent claims)

The Four Questions

1. Does the risk neutral measure change under an expansion of filtrations?
2. If the risk neutral measure does indeed change, exactly how does it change?
3. When does the risk neutral measure not exist under a filtration expansion, thereby introducing arbitrage opportunities?
4. If the risk neutral measure does not exist as in (3), how might we exploit these arbitrage opportunities?

Expansion of Filtrations Dynamically via Stochastic Processes: A New Approach

Preliminaries

- Review of the Jeulin-Yor theory
- τ is an **honest time** if it is the end of an optional set, such as a last exit time, or the second to last exit time
- Let $\mathbb{G} \subset \mathbb{H}$ be two filtrations and let τ be an $\mathbb{H}$ stopping time. $\mathbb{G}$ and $\mathbb{H}$ **coincide after τ** if for every $\mathbb{H}$ optional process X the process $1_{[\tau,\infty)}$ is $\mathbb{G}$ adapted
- Define:

$$\begin{array}{rcl} Z_t & = & P(\tau > t|\mathcal{F}_t), \text{ the optional projection of } 1_{t\geq\tau} \text{ onto } \mathbb{F} \\ \mu & \text{is} & \text{the martingale part of the Doob-Meyer decomposition of } Z \\ J & \text{is} & \text{the dual predictable projection of } \Delta M_\tau 1_{t\geq\tau} \text{ onto } \mathbb{F} \end{array}$$

Preliminaries, Continued

- Related to results of **Yan Zeng**, and **Xin Guo**
- Theorem (**Y. Kchia, M. Larsson** and Protter, 2011): Let M be an $\mathbb{F}$ local martingale. Let $\mathbb{H}$ coincide with $\mathbb{G}$ after τ. Suppose there exists an $\mathbb{H}$ predictable finite variation process A such that $M - A$ is an $\mathbb{H}$ local martingale. Then M is a G semimartingale and

$$M_{t\wedge\tau} - \int_0^{t\wedge\tau} \frac{d\langle M,\mu\rangle_s + dJ_s}{Z_{s-}} - \int_{t\wedge\tau}^{t} dA_s \tag{1}$$

is the local martingale part of its $\mathbb{G}$ decomposition up to $t \wedge \tau$

A Recent Result of Y. Kchia and M. Larsson, 2011

- Kchia and Larsson treated a progressive expansion of a filtration with positive random variables $(\tau_i)_{i \in I}$, where I is a subset of $\{1, 2, \ldots, n\}$, with the τ_i *not necessarily ordered*
- This decomposition is expressed as a sum of decompositions of the form (1), where each separate decomposition takes place on a stochastic interval of the form $[\sigma_I, \rho_I)$, where $\sigma_I = \max_{i \in I} \tau_i$ and $\rho_I = \min_{j \notin I} \tau_j$
- Their results extend results of **Jeanblanc and Le Cam** (2009), and also **El Karoui, Jeanblanc, and Jiao** (2009 and 2010). Kchia and Larsson include jump sizes

A New Procedure for Dynamic Enlargement

- Due to the previous results of Kchia and Larsson, we know how to expand a filtration with a marked point process with unordered arrivals
- We start with a base filtration $\mathbb{F}$ and we want to expand it to a larger filtration $\mathbb{H}$, tracking what happens to $\mathbb{F}$ semimartingales in the larger $\mathbb{H}$ filtration
- We also want to use a càdlàg process X as our vehicle for expanding $\mathbb{F}$ to $\mathbb{H}$
- **Step 1** is to approximate X with a sequence $(X^n)_{n\geq 1}$ of càdlàg processes that are marked point processes with possibly unordered jumps, and then expand with X^n to get a larger filtration $\mathbb{G}^n$

- **Step 2:** We choose the approximations X^n in such a way that we know that if M is an $\mathbb{F}$ semimartingale, then it is also an $\mathbb{G}^n$ semimartingale, and we can calculate N^n and A^n of its $\mathbb{G}^n$ Doob-Meyer decomposition:

$$M_t^n = N_t^n + A_t^n \tag{2}$$

- We need some sort of a control on N^n and A^n in (2) as n increases to ∞, to get a convergence of of the components of M^n, which is the $\mathbb{F}$ semimartingale M after the expansion
- We take an old lemma of **Martin Barlow** and myself, and feed it steroids
- We combine this with the (somewhat obscure) theory of the convergence of filtrations, developed by **F. Antonelli, A. Kohatsu-Higa, F. Coquet**, and others

- **Step 3:** We say that a semimartingale Y is an $\mathbb{L}$ **nicely integrable** semimartingale if $Y = N + A$ is its canonical decomposition in $\mathbb{L}$ and there exists a constant K such that

$$E\left(\int_0^T |dA_s|\right) \leq K, \quad \text{and} \quad E\left(\sup_{o\leq s\leq T} |N_s|\right) \leq K. \tag{3}$$

- For a given semimartingale X we are using for our expansion, we approximate X with X^n, where

$$X_t^n = \sum_{i=0}^{n+1} (X_{t_n^i} - X_{t_n^{i-1}}) 1_{\{t \geq t_n^i\}} \tag{4}$$

- In his famous paper of 1987, "Grossissement initial, hypothèse (H') et théorème de Girsanov," Jacod gave conditions for an initial expansion by a random variable, involving the existence of conditional densities, that assure Hypothesis (H') holds (i.e., semimartingales stay semimartingales in the expanded filtration)
- Using Equation (4) we expand the filtration initially at each time t_n^{i-1}, with $(X_{t_n^i} - X_{t_n^{i-1}})$, for each n. To do this we will assume there exists a sequence $(\pi_n)_{n\geq 1} = (\{t_n^i\})_{n\geq 1}$ of subdivisions of $[0, T]$ whose mesh tends to zero and is such that $(X_{t_0^n}, X_{t_1^n} - X_{t_0^n}, \ldots, X_T - X_{t_n^n})$ satisfies Jacod's criterion for each n. We call this a **dynamic Jacod criterion**.

- Let $(N^n)_{n\geq 1}$ be a sequence of càdlàg processes converging in probability under the Skorohod topology to a càdlàg process N and let $\mathbb{N}^n$ and $\mathbb{N}$ be their natural filtrations. Define the filtrations $\mathbb{G}^{0,n} = \mathbb{F} \vee \mathbb{N}^n$ and $\mathbb{G}^n$ by $\mathcal{G}^n_t = \bigcap_{u>t} \mathcal{G}^{0,n}_u$. Let also $\mathbb{G}^0$ (resp. $\mathbb{G}$) be the smallest (resp. the smallest right-continuous) filtration containing $\mathbb{F}$ and to which N is adapted.

- **A consequence of a theorem of Mémin:**

Theorem: Let $(\mathbb{G}^n)_{n\geq 1}$ be a sequence of right-continuous filtrations and let $\mathbb{G}$ be a filtration such that $\mathcal{G}_t^n \xrightarrow{w} \mathcal{G}_t$ for all t. Let X be a stochastic process such that for each n, X is a $\mathbb{G}^n$ semimartingale with canonical decomposition $X = M^n + A^n$ such that there exists $K > 0$, $E(\int_0^T |dA_s^n|) \leq K$ and $E(\sup_{0\leq s\leq T} |M_s^n|) \leq K$ for all n. Then

(i) If X is $\mathbb{G}$ adapted, then X is a $\mathbb{G}$ special semimartingale.

(ii) Assume moreover that $\mathbb{G}$ is right-continuous and let $X = M + A$ be the canonical decomposition of X. Then M is a $\mathbb{G}$ martingale and $\sup_{0\leq s\leq T} |M_s|$ and $\int_0^T |dA_s|$ are integrable.

(iii) Furthermore, assume that X is $\mathbb{G}$ quasi-left continuous and $\mathbb{G}^n \xrightarrow{w} \mathbb{G}$. Then (M^n, A^n) converges in probability under the Skorohod J_1 topology to (M, A).

- **Theorem:** Let X be an $\mathbb{F}$ semimartingale such that for each n, X is a $\mathbb{G}^n$ semimartingale with canonical decomposition $X = M^n + A^n$. Assume $E(\int_0^T |dA_s^n|) \leq K$ and $E(\sup_{0\leq s\leq T} |M_s^n|) \leq K$ for some K and all n. Assume either N is quasi-left continuous, or that N^n is a discretization of N along some refining subdivision $(\pi_n)_{n\geq 1}$ such that each fixed time of discontinuity of N belongs to $\cup_n \pi_n$. Then
  (i) X is a $\mathbb{G}^0$ special semimartingale.
  (ii) Moreover, if $\mathbb{F}$ is the natural filtration of some càdlàg process then X is a $\mathbb{G}$ special semimartingale with canonical decomposition $X = M + A$ such that M is a $\mathbb{G}$ martingale and $\sup_{0\leq s\leq T} |M_s|$ and $\int_0^T |dA_s|$ are integrable.
  (iii) Furthermore, assume that X is $\mathbb{G}$ quasi-left continuous and $\mathbb{G}^n \xrightarrow{w} \mathbb{G}$. Then (M^n, A^n) converges in probability under the Skorohod J_1 topology to (M, A).

A Generalized Jacod's Criterion

- **Generalized Jacod's criterion:** There exists a sequence $(\pi_n)_{n\geq 1} = (\{t_i^n\})_{n\geq 1}$ of subdivisions of $[0, T]$ whose mesh tends to zero and such that for each n, $(X_{t_0^n}, X_{t_1^n} - X_{t_0^n}, \ldots, X_T - X_{t_n^n})$ satisfies Jacod's criterion, i.e. there exists a σ-finite measure η_n on $\mathcal{B}(\mathbb{R}^{n+2})$ such that $P\big((X_{t_0^n}, X_{t_1^n} - X_{t_0^n}, \ldots, X_T - X_{t_n^n}) \in \cdot \mid \mathcal{F}_t\big)(\omega) \ll \eta_n(\cdot)$ a.s

A Key Result

- We let $\mathbb{G}^0$ (resp. $\mathbb{G}$) be the smallest (resp. the smallest right-continuous) filtration containing $\mathbb{F}$ and relative to which X is adapted.
- **Theorem:** Assume X and $\mathbb{F}$ satisfy the Generalized Jacod's Criterion, and that either X is quasi-left continuous, or the sequence of subdivisions $(\pi_n)_{n\geq 1}$ is refining and all fixed times of discontinuity of X belong to $\cup_n \pi_n$.

  Let M be a continuous $\mathbb{F}$ martingale such that $E(\sup_{s\leq T}|M_s|) \leq K$ and $E(\int_0^T |dA_s^{(n)}|) \leq K$ for some K and all n. Then

  (i) M is a $\mathbb{G}^0$ special semimartingale, and

  (ii) Moreover, if $\mathbb{F}$ is the natural filtration of some càdlàg process Z, then M is a $\mathbb{G}$ special semimartingale with canonical decomposition $M = N + A$ such that N is a $\mathbb{G}$ martingale and $\sup_{0\leq s\leq T}|N_s|$ and $\int_0^T |dA_s|$ are integrable.

A First Application

- Start with a Brownian filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, $\mathcal{F}_t = \sigma(B_s, s \leq t)$ and consider the stochastic differential equation
$$dX_t = \sigma(X_t)dB_t + b(X_t)dt$$
Assume the existence of a unique strong solution $(X_t)_{0 \leq t \leq T}$
- Indeed, assume the transition density $\pi(t, x, y)$ exists and is twice continuously differentiable in x and continuous in t and y.
- This is guaranteed for example if b and σ are infinitely differentiable with bounded derivatives and if the H örmander condition holds for any x, and we assume that this holds. In this case, π is even infinitely differentiable

- Define the time reversed process $Z_t = X_{T-t}$, for all $0 \leq t \leq T$. Let $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t < \frac{T}{2}}$ be the smallest right-continuous filtration containing $(\mathcal{F}_t)_{0 \leq t < \frac{T}{2}}$ and to which $(Z_t)_{0 \leq t < \frac{T}{2}}$ is adapted
- We would like to prove that B remains a special semimartingale in $\mathbb{G}$ and give its canonical decomposition
- Such questions have been considered before by Jeulin, Pardoux, Jacod & Protter
- Take $T = 1$. The reversed Brownian motion is $\tilde{B}_t = B_{1-t} - B_1$ and the filtration $\tilde{\mathbb{G}} = (\tilde{\mathcal{G}}_t)_{0 \leq t < \frac{1}{2}}$ is defined by

$$\tilde{\mathcal{G}}_t = \bigcap_{t < u < \frac{1}{2}} \sigma(B_s, \tilde{B}_s, 0 \leq s < u).$$

- **A Standard Theorem:** Both B and $\tilde{B}$ are $\mathbb{G}$ semimartingales

- **Theorem:** Assume there exists a nonnegative function ϕ such that $\int_0^1 \phi(s)ds < \infty$ and for each $0 \leq s < t$,

$$E\left(\left|\frac{1}{\pi}\frac{\partial \pi}{\partial x}(t-s, X_s, X_t)\right|\right) \leq \phi(t-s)$$

Then the process $(B_t)_{0 \leq t < \frac{1}{2}}$ is a $\mathbb{G}$ semimartingale and if b and σ are bounded, and σ is bounded away from zero,

$$B_t - \int_0^t \frac{1}{\pi}\frac{\partial \pi}{\partial x}(1-2s, X_s, X_{1-s})ds$$

is a $\mathbb{G}$ Brownian motion

- We cannot prove something like Hypothesis (H') in our context, but we can give in concrete examples sufficient conditions for an $\mathbb{F}$ semimartingale to remain an $\mathbb{G}$ semimartingale, specify the decomposition, and check to see if it provides scalable arbitrage opportunities, or not.
- An example is the following (obtained with **Jean Jacod**): We expand the filtration $\mathbb{F}$ dynamically, with the process

$$X_t = W_1 + \epsilon V_{1-t} \tag{5}$$

where W is a standard Brownian motion, and V is also a BM, independent of W and of $\mathbb{F}$. We define

$$\mathcal{H}_t = \mathcal{F}_t \vee \sigma(W_1 + \epsilon V_{1-s}; s \leq t) = \mathcal{F}_t \vee \sigma(X_s : s \leq t).$$

- **Theorem:** Let H be predictable with $\int_0^1 H_s^2 ds < \infty$ a.s. Define $M_t = \int_0^t H_s dW_s$, an $\mathbb{F}$ local martingale. If H is a.s. of the order $H_s = \frac{1}{1-s}^{1/2+\alpha}$ with $\alpha < \frac{1}{2}$ then M remains a semimartingale in $\mathbb{H}$, and the finite variation term of its decomposition is

$$A_t^{\mathbb{H}} = -\int_0^t H_s \frac{X_s - W_s}{(1+\epsilon^2)(1-s)} ds. \tag{6}$$

- **Final Remark:** We see this as a beginning, and the number of questions related to the expansion of filtrations and concomitant insider trading application is large.