Nearly Optimal Strategies for Risk-sensitive Portfolio Optimization on Infinite Horizon

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A personal problem: Solution and Excuse

How to call...

- Freddy sensei (先生),
- Delbaen sensei,
- Delbaen-san (さん),

A personal problem: Solution and Excuse

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- Delbaen sensei,
- Delbaen-san (さん),

- Rita-san lectured me that
  "when I call a person by first name, that person should call me by first name, too. Otherwise..."

Problem

- Long-term growth-rate maximization of expected power-utility ($\theta \in (0, \infty)$: risk-averse parameter),

$$\begin{aligned}\Gamma(\theta) &:= \sup_{\pi} \overline{\lim_{T\to\infty}} \frac{1}{(-\theta)T} \log \mathbb{E} \left(X_T^\pi\right)^{-\theta} \\ &= \sup_{\pi} \overline{\lim_{T\to\infty}} \frac{1}{(-\theta)T} \log \mathbb{E} \exp\left(-\theta \log X_T^\pi\right).\end{aligned}$$

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- Helpful and convenient when we treat, e.g.,

$$\inf_{\pi} \underline{\lim_{T\to\infty}} \frac{1}{T} \log \mathbb{P}\left(X_T^\pi \le X_0 \mathrm{e}^{kT}\right)$$

and

$$X_t^\pi \ge f\left(\sup_{s\in[0,t)} X_s^\pi\right) \quad {}^\forall t \ge 0.$$

Problem (cont.)

- For Markovian model, the natural candidate strategy $\hat{\pi} := (\hat{\pi}_t)_{t\geq 0}$ is always constructed from the stabilizing solution to the associated Ergodic HJB equation.

However..

Verification Result

1) If the agent is "not so risk-averse", we can verify that

$$\Gamma(\theta) = \lim_{T\to\infty} \frac{1}{(-\theta)T} \log \mathbb{E}\left(X_T^{\hat{\pi}}\right)^{-\theta} \in (-\infty, +\infty).$$

2) If the agent is "too risk-averse", the verification can fail and it can occur that

$$\overline{\lim_{T\to\infty}} \frac{1}{(-\theta)T} \log \mathbb{E}\left(X_T^{\hat{\pi}}\right)^{-\theta} = -\infty.$$

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3) We show that the value $\Gamma(\theta)$ is always finite. We construct $\exists \left(\tilde{\pi}^{(p)}\right)_{p\in\mathbb{N}}$: nearly optimal strategy, s.t.

$$\Gamma(\theta) = \lim_{p\to\infty} \lim_{T\to\infty} \frac{1}{(-\theta)T} \log \mathbb{E}\left(X_T^{\tilde{\pi}^{(p)}}\right)^{-\theta}.$$

Remarks:

- 1) and 2) have been observed by Fleming and Sheu ('99), Kuroda and Nagai ('02), and Nagai ('02), using their Markovian models.
- 3) has been proved by Fleming and Sheu ('99) and Hata and S ('10), using specific 1-dim. OU-based models.

Contributions:

- Show 3) in a generalized setting with a "symmetric" model.
- Revise and simplify the original proof by Fleming and Sheu ('99).

Limitation:

- Proof is still indirect (relying on the convergence of the bottom of spectrum).

Market Model

- Market model:

$$\begin{aligned} dS^0_t =& S^0_t r(Y_t)dt, \quad S^0_0 = 1, \\ dS_t =& \mathrm{diag}(S_t)\left[\{r(Y_t)\mathbf{1} + a(Y_t)\}dt + \Sigma(Y_t)dW_t\right], \quad S_0 \in \mathbb{R}^n_{++}, \\ dY_t =& b(Y_t)dt + \Lambda(Y_t)dW_t, \quad Y_0 \in \mathbb{R}^m, \end{aligned}$$

(W: d-dim. BM, $\mathbf{1} := (1, \ldots, 1)^\top \in \mathbb{R}^n$).

- Wealth process of a self-financing investor:

$$\frac{dX_t}{X_t} = \sum_{i=1}^d \pi^i_t \frac{dS^i_t}{S^i_t} + \left(1 - \sum_{i=1}^d \pi^i_t\right)\frac{dS^0_t}{S^0_t}, \quad X_0 \in \mathbb{R}_{++},$$

($\pi := (\pi_t)_{t \geq 0}$, $\pi_t := (\pi^1_t, \ldots, \pi^d_t)^\top$: dynamic investment strategy).

Long-term Portfolio Optimization

$$\Gamma(\theta) := \sup_{\pi} \varlimsup_{T\to\infty} \frac{1}{(-\theta)T} \log \mathbb{E}(X_T^\pi)^{-\theta}$$

- $\theta(>0)$: risk-averse parameter.
- Bielecki and Pliska, Fleming and Sheu, Kuroda and Nagai, Nagai, Nagai and Peng, and so on..
- Risk-sensitive portfolio optimization:

$$\Gamma(\theta) = \sup_{\pi} \varlimsup_{T\to\infty} \frac{1}{(-\theta)T} \log \mathbb{E} \exp\left(-\theta \log X_T^\pi\right).$$

"Risk-sensitized" problem of

$$\sup_{\pi} \varlimsup_{T\to\infty} \frac{1}{T} \mathbb{E} \log X_T^\pi.$$

Recall

$$(X_t)^{-\theta} = (X_0)^{-\theta} M_t(-\theta\Sigma^\top\pi) \exp\left\{-\theta \int_0^t \ell(Y_s, \pi_s; \theta) ds\right\},$$

$$M_t(f) := \exp\left(\int_0^t f_u^\top dW_u - \frac{1}{2}\int_0^t |f_u|^2 du\right),$$

$$\ell(y, p; \theta) := r(y) + p^\top a(y) - \frac{1+\theta}{2} p^\top (\Sigma\Sigma^\top)(y) p.$$

We then see

$$\Gamma(\theta) = \sup_\pi \overline{\lim_{T\to\infty}} \frac{1}{(-\theta)T} \log \mathbb{E}^{(\theta\pi)} \exp\left\{-\theta \int_0^T \ell(Y_u, \pi_u; \theta) du\right\}$$

with

$$dY_t = \left\{b(Y_t) - \theta(\Lambda\Sigma^\top)(Y_t)\pi_t\right\} dt + \Lambda(Y_t) dW_t^{(\theta\pi)}.$$

For

$$\Gamma^{(T)}(\theta) = \sup_{\pi} \frac{1}{(-\theta)} \log \mathbb{E}^{(\theta\pi)} \exp\left\{-\theta \int_0^T \ell(Y_u, \pi_u; \theta) du\right\}$$

and

$$\bar{V}_t^{(T)} := \operatorname{esssup}_{\pi} \frac{1}{(-\theta)} \log \mathbb{E}^{(\theta\pi)} \left[\exp\left\{-\theta \int_t^T \ell(Y_u, \pi_u; \theta) du\right\} \middle| \mathcal{F}_t\right],$$

the associated HJB is

$$\begin{aligned} -\partial_t V =& \frac{1}{2}\left\{\operatorname{tr}\left(\Lambda\Lambda^\top \partial_{yy} V\right) - \theta\left|\Lambda^\top \partial_y V\right|^2\right\} \\ &+ \sup_{\pi\in\mathbb{R}^n} \left[\{b(y) - \theta\Lambda\Sigma^\top\pi\}^\top \partial_y V + \ell(y, \pi; \theta)\right] \\ V(T, y) =& 0, \end{aligned}$$

which is rewriten as

$$
\begin{aligned}
-\partial_t V =& \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda^\top \partial_{yy} V\right) - \frac{1}{2}\partial_y V^\top \Lambda N^{-1} \Lambda^\top \partial_y V \\
& + k(y)^\top \partial_y V + \left(\frac{1}{2(1+\theta)} a^\top (\Sigma\Sigma^\top)^{-1} a + r \right)(y), \\
V(T,y) =& 0,
\end{aligned}
$$

where

$$
\begin{aligned}
N^{-1} :=& N(\theta)^{-1} = \theta \left\{ I - \frac{\theta}{1+\theta} \Sigma^\top (\Sigma\Sigma^\top)^{-1} \Sigma \right\}, \\
k :=& k(\theta) = b - \frac{\theta}{1+\theta} \Lambda \Sigma^\top (\Sigma\Sigma^\top)^{-1} a.
\end{aligned}
$$

Next, regarding

$$V^{(T)}(t,y) \sim \chi \cdot (T-t) + \xi(y) \quad \text{as } T-t \to \infty,$$

where $\chi \in \mathbb{R}$ and $\xi \in C^2(\mathbb{R}^m)$, describe the EHJB:

$$\begin{aligned}\chi =& \frac{1}{2}\left\{\mathrm{tr}\left(\Lambda\Lambda^\top \nabla\nabla\xi\right) - \theta\left|\Lambda^\top\nabla\xi\right|^2\right\} \\ &+ \sup_{\pi\in\mathbb{R}^n}\left[\{b(y) - \theta\Lambda\Sigma^\top\pi\}^\top\nabla\xi + \ell(y,\pi;\theta)\right],\end{aligned}$$

or

$$\begin{aligned}\chi =& \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda^\top\nabla\nabla\xi\right) - \frac{1}{2}\nabla\xi^\top\Lambda N^{-1}\Lambda^\top\nabla\xi + k(y)^\top\nabla\xi \\ &+ \left(\frac{1}{2(1+\theta)}a^\top(\Sigma\Sigma^\top)^{-1}a + r\right)(y).\end{aligned}$$

Results on HJBs (Nagai, '03)

- Assume that a, b, r, Σ, Λ are globally Lipschitz, and that $\epsilon_1 I_n \le \Sigma\Sigma^\top \le \frac{1}{\epsilon_1} I_n$, $\epsilon_2 I_m \le \Lambda\Lambda^\top \le \frac{1}{\epsilon_2} I_m$ with ${}^{\exists}\epsilon_1, \epsilon_2 > 0$.

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Then, for each $T, \theta > 0$, the following are valid.

- $^\exists V^{(T)}$: classical sol. of HJB.
- $\Gamma^{(T)}(\theta) = V^{(T)}(0, Y_0)$.
- Optimal strategy $\hat{\pi}^{(T)} := (\hat{\pi}_t^{(T)})_{0 \leq t \leq T}$ is given by

$$\hat{\pi}_t^{(T)} := \frac{1}{1+\theta} (\Sigma\Sigma^\top)^{-1} \left\{ a - \theta \Sigma \Lambda^\top \partial_y V^{(T)} \right\} (t, Y_t).$$

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Moreover, under a certain additional condition,

- $V^{(T)}(0, \cdot) - V^{(T)}(0, 0) \to {}^\exists \hat{\xi}$, $\frac{1}{T} V^{(T)}(0, \cdot) \to {}^\exists \hat{\chi}$, and $(\hat{\chi}, \hat{\xi}) \in \mathbb{R} \times C^2(\mathbb{R}^m)$ solves EHJB.
- The diffusion $d\hat{Y}_t = \left\{ k - \Lambda N^{-1} \Lambda \nabla^\top \hat{\xi} \right\} (\hat{Y}_t) dt + \Lambda(\hat{Y}_t) dW_t$ is ergodic ("system" is stabilized).

Immediately, it follows that

$$\hat{\chi} = \lim_{T\to\infty} \frac{1}{T}\Gamma^{(T)}(\theta) \geq \Gamma(\theta).$$

We expect that

$$\hat{\chi} = \lim_{T\to\infty} \frac{1}{T}\Gamma^{(T)}(\theta) = \Gamma(\theta),$$

i.e.,

$$\overline{\lim_{T\to\infty}} \sup_{\pi} \frac{1}{(-\theta)T} \log \mathbb{E}\left(X_T^{\pi}\right)^{-\theta} = \sup_{\pi} \overline{\lim_{T\to\infty}} \frac{1}{(-\theta)T} \log \mathbb{E}\left(X_T^{\pi}\right)^{-\theta}.$$

??

- $(\hat{\chi}, \hat{\xi}) \in \mathbb{R} \times C^2(\mathbb{R}^m)$: stabilizing sol. of EHJB.
- $\hat{\pi} := (\hat{\pi}_t)_{t \geq 0}$: candidate strategy, where
$$\hat{\pi}_t := \frac{1}{1+\theta}(\Sigma\Sigma^\top)^{-1}\left\{a - \theta\Sigma\Lambda^\top\nabla\hat{\xi}\right\}(Y_t).$$

We see that

$$-\frac{1}{\theta T}\log \mathbb{E}\left(X_T^{\hat{\pi}}\right)^{-\theta} = \hat{\chi} + \frac{\hat{\xi}(x)}{T} - \frac{1}{\theta T}\log \hat{\mathbb{E}} e^{\theta\hat{\xi}(Y_T)}$$

under a measure-change. $\hat{\mathbb{P}}$-dynamics of Y is

$$dY_t = \left\{k - \Lambda N^{-1}\Lambda\nabla^\top\hat{\xi}\right\}(Y_t)dt + \Lambda(Y_t)d\hat{W}_t,$$

which is ergodic.

Verification Step

- $(\hat{\chi}, \hat{\xi}) \in \mathbb{R} \times C^2(\mathbb{R}^m)$: stabilizing sol. of EHJB.
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- exponential-quadratic integrability of the inv. meas. of $\hat{Y}$.

Verification Theorem (1)

- (Nagai '03): if

$$\left\{ \frac{1}{\theta^2} a^\top (\Sigma\Sigma^\top)^{-1} a - (\nabla \hat{\xi}^{(\theta)})^\top \Lambda \Sigma^\top (\Sigma\Sigma^\top)^{-1} \Sigma \Lambda^\top \nabla \hat{\xi}^{(\theta)} \right\} (y) \to \infty$$

as $|y| \to \infty$, then, the verification is established.

- (Kuroda and Nagai, '02) Assume

(†) $a(y) = a_0 + A_1 y, \quad b(y) = b_0 + B_1 y, \quad r, \Lambda, \Sigma$: constant.

Then,

$$\hat{\xi}^{(\theta)}(y) = \frac{1}{2} y^\top \hat{Q}^{(\theta)} y + y^\top \hat{q}^{(\theta)},$$

where $\hat{Q}^{(\theta)}$: stabilizing sol. of algebraic Riccati eq. and $\hat{q}^{(\theta)}$: sol. of an algebraic linear eq., and a sharper and more explicit result has been obtained.

- (Fleming and Sheu, '99) In addition to (†), assume that

$$d = 1, \quad n = 1, \quad A_1 = B_1, \quad \text{and} \quad \Sigma = \Lambda.$$

Then, the following assertions hold.

- If $0 < \theta < 3$, then,

$$\Gamma(\theta) = \lim_{T\to\infty} \frac{1}{(-\theta)T} \log \mathbb{E}\left(X_T^{\hat{\pi}}\right)^{-\theta} = \hat{\chi}.$$

- If $\theta \geq 3$, then,

$$\overline{\lim_{T\to\infty}} \frac{1}{(-\theta)T} \log \mathbb{E}\left(X_T^{\hat{\pi}}\right)^{-\theta} = -\infty.$$

Remark & Idea

- $[0, T]$: negligible contribution, $[T, \infty)$: dominant part. So, e.g.,

$$\tilde{\pi}_t^{(p)} := \hat{\pi}_t^{(p)} 1_{[0,p]}(t), \quad (t \geq 0, \; p \in \mathbb{N})$$

does not approximate the value $\Gamma(\theta)$.

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$$\tilde{\pi}_t^{(p)} := \hat{\pi}_t^{(p)} 1_{[0,p]}(t), \quad (t \geq 0,\ p \in \mathbb{N})$$

does not approximate the value $\Gamma(\theta)$.

- Take a feedback-form strategy $\pi := (\pi(Y_t))_{t\geq 0}$. Regard

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \log \mathbb{E}\left(X_T^{\pi}\right)^{-\theta} = \overline{\lim_{T\to\infty}} \frac{1}{T} \log \mathbb{E}^{(\theta\pi)} \mathrm{e}^{-\int_0^T U(Y_t)dt}$$

with $U(y) := \theta\ell(y,\pi(y);\theta)$ as the bottom of the spectrum of a Schrödinger operator: $H := -L + U$, if $(Y, \mathbb{P}^{(\theta\pi)})$ is a symmetric diffusion.

Model with Symmetric Diffusion

$$
\begin{aligned}
dS_t^0 &= S_t^0 r(Y_t)dt, \quad S_0^0 = 1, \\
dS_t &= \mathrm{diag}(S_t)\left[\{r(Y_t)\mathbf{1} + a(Y_t)\}dt + \Sigma(Y_t)dW_t\right], \quad S_0 \in \mathbb{R}^n_{++}, \\
dY_t &= b(Y_t)dt + \Lambda(Y_t)dW_t, \quad Y_0 \in \mathbb{R}^m,
\end{aligned}
$$

- a, b, r, Σ, Λ: Lipschitz, $\epsilon_1 I_n \leq \Sigma\Sigma^\top \leq \frac{1}{\epsilon_1} I_n$, $\epsilon_2 I_m \leq \Lambda\Lambda^\top \leq \frac{1}{\epsilon_2} I_m$ with $^\exists\epsilon_1, \epsilon_2 > 0$.
- Moreover, assume one of the following:

(I) $m = 1$, $k(y)^\top y \leq -\kappa_1 |y|^2$, $^\exists\kappa_1 > 0$, where

$$
k := b - \frac{\theta}{1+\theta}\Lambda\Sigma^\top(\Sigma\Sigma^\top)^{-1}a.
$$

(II) $m = n$, $a = b$, $\Sigma = \Lambda$: const. matrix,

$$a(y) = \Lambda\Lambda^\top \nabla \tilde{A}(y), \quad a(y)^\top y \le -\kappa_1 |y|^2 \quad {}^\exists \kappa_1 > 0,$$

■ Example:

$$\frac{dS_t^i}{S_t^i} - \frac{dS_t^0}{S_t^0} = dY_t^i, \quad i \in \{1, \cdots, n\},$$

where $Y := (Y^1, \ldots, Y^n)^\top$ is given by

$$dY_t = \Lambda\Lambda^\top (\tilde{a}_0 + \tilde{A}_1 Y_t)dt + \Lambda dW_t, \quad Y_0 \in \mathbb{R}^n,$$

$\tilde{A}_1 \in \mathbb{S}_n$.

Let $\phi \in C_b^\infty(\mathbb{R})$: smooth, bounded support, $0 \le \phi \le 1$, and $\phi|_{[-1,1]} \equiv 1$. Define $\phi^{(p)} : \mathbb{R}^m \to [0,1]$ by

$$\phi^{(p)}(y) := \phi\left(\frac{|y|}{p}\right), \quad p \in \mathbb{N}.$$

Let $\theta \in (0, +\infty)$. Define $\check{\pi}^{(p)} := (\check{\pi}_t^{(p)})_{t\ge 0}$ ($p \in \mathbb{N}$) by $\check{\pi}_t^{(p)} := \tilde{\pi}^{(p)}(Y_t)$, where

$$\tilde{\pi}^{(p)}(y) := \tilde{\pi}(y)\phi^{(p)}(y)$$

($\hat{\pi}_t := \tilde{\pi}(Y_t)$: candidate of optimal). Then,

$$\Gamma(\theta) = \lim_{p\to\infty} \overline{\lim_{T\to\infty}} \frac{1}{(-\theta)T} \log \mathbb{E} \exp\left(-\theta \log X_T^{\check{\pi}^{(p)}}\right) = \hat{\chi}.$$

Result (II)

Let $\theta \in (0, +\infty)$. Define $\breve{\pi}^{(p)} := (\breve{\pi}_t^{(p)})_{t \geq 0}$ $(p \in \mathbb{N})$ by $\breve{\pi}_t^{(p)} := \tilde{\pi}^{(p)}(Y_t)$, where

$$\tilde{\pi}^{(p)}(y) := \frac{1}{1+\theta} \nabla \left(\tilde{A} - \theta \hat{\xi} \phi^{(p)} \right)(y).$$

Then,

$$\Gamma(\theta) = \lim_{p \to \infty} \overline{\lim_{T \to \infty}} \frac{1}{(-\theta)T} \log \mathbb{E} \exp\left(-\theta \log X_T^{\breve{\pi}^{(p)}}\right) = \hat{\chi}.$$

$$
\begin{aligned}
-\theta\hat{\chi} &= \lim_{T\to\infty}\frac{1}{T}\left\{-\theta\hat{\chi}T - \hat{\xi}(y) + \log\hat{\mathbb{E}}e^{(\theta\hat{\xi}-\hat{B})(Y_T)}\right\} \\
&= \lim_{T\to\infty}\frac{1}{T}\log\mathbb{E}\left(X_T^{\tilde{\pi}}\right)^{-\theta} e^{-\hat{B}(Y_T)} \\
&= \lim_{T\to\infty}\frac{1}{T}\log\mathbb{E}^{(\theta\hat{\pi})}\exp\left\{-\theta\int_0^T \hat{\ell}(Y_t)dt - \hat{B}(Y_T)\right\} \\
&= \lim_{T\to\infty}\frac{1}{T}\log\mathbb{E}\exp\left\{-\int_0^T \hat{U}(y+\Lambda W_t)dt\right\} =: \hat{\lambda}_1,
\end{aligned}
$$

where

$$
\nabla\hat{B}(y) := (\Lambda\Lambda^\top)^{-1}\left\{b - \theta\Lambda\Sigma^\top\tilde{\pi}\right\}(y), \quad \hat{\ell}(y) := \ell\left(y, \tilde{\pi}(y); \theta\right),
$$

$$
\hat{U}(y) := \theta\hat{\ell}(y) + e^{-\hat{B}(y)}(\mathcal{L}e^{\hat{B}})(y), \quad \mathcal{L}f := \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda^\top\nabla\nabla f\right).
$$

Lemma (2)

For $p \in \mathbb{N}$, define

$$\nabla \check{B}^{(p)}(y) := (\Lambda\Lambda^\top)^{-1}\left\{b - \theta\Lambda\Sigma^\top\tilde{\pi}^{(p)}\right\}(y),$$

$$\check{\ell}^{(p)}(y) := \ell\big(y, \tilde{\pi}^{(p)}(y); \theta\big), \quad \check{U}^{(p)} := \theta\check{\ell}^{(p)} + \mathrm{e}^{-\check{B}^{(p)}}\mathcal{L}\mathrm{e}^{\check{B}^{(p)}},$$

Then,

$$\varliminf_{T\to\infty} \frac{1}{T}\log\mathbb{E}\left(X_T^{\check{\pi}^{(p)}}\right)^{-\theta}$$

$$= \varliminf_{T\to\infty} \frac{1}{T}\log\mathbb{E}\exp\left\{-\int_0^T \check{U}^{(p)}(y+\Lambda W_t)dt + \hat{B}^{(p)}(y+\Lambda W_T)\right\}$$

$$\leq \varliminf_{T\to\infty} \frac{1}{T}\log\mathbb{E}\exp\left\{-\int_0^T \check{U}^{(p)}(y+\Lambda W_t)dt\right\} =: \check{\lambda}_1^{(p)}.$$

Schrödinger semigroup

$$Hf := -\frac{1}{2}\mathrm{tr}(\Lambda\Lambda^\top \nabla\nabla f) + Uf, \quad f \in C^2(\mathbb{R}^m),$$

and use same H to denote its self-adjoint extension on $L^2(\mathbb{R}^m)$.
Feynman-Kac semigroup on $L^2(\mathbb{R}^m)$:

$$(\mathrm{e}^{-tH} f)(x) := \mathbb{E}\left[\exp\left\{-\int_0^t U(x+\Lambda W_s)ds\right\} f(x+\Lambda W_t)\right].$$

Fact: The bottom of the spectrum:

$$\inf \sigma(H) = -\lim_{T\to\infty} \frac{1}{T} \log \|\mathrm{e}^{-TH}\|_{2,2}$$

admits the representation (Kac, Simon, Donsker-Varadhan..)

$$\inf \sigma(H) = -\lim_{T\to\infty} \frac{1}{T} \log \mathbb{E}\left[\exp\left\{-\int_0^T U(x+\Lambda W_s)ds\right\}\right].$$

We see, for

$$\hat{H} f := -\frac{1}{2}\mathrm{tr}(\Lambda\Lambda^\top \nabla\nabla f) + \hat{U} f \quad \text{and}$$
$$\check{H}^{(p)} f := -\frac{1}{2}\mathrm{tr}(\Lambda\Lambda^\top \nabla\nabla f) + \check{U}^{(p)} f,$$

that $\lim_{p\to\infty} \|\mathrm{e}^{-T\hat{H}} - \mathrm{e}^{-T\check{H}^{(p)}}\|_{2,2} = 0$, and that

$$-\lim_{p\to\infty} \check{\lambda}_1^{(p)} = \lim_{p\to\infty} \inf \sigma(\check{H}^{(p)}) = \inf \sigma(\hat{H}) = -\hat{\lambda}_1 = \theta\hat{\chi}.$$

From Lemma (1)-(2),

$$\hat{\chi} \le \lim_{p\to\infty} \overline{\lim_{T\to\infty}} \frac{1}{(-\theta)T} \log \mathbb{E}\left(X_T^{\check{\pi}^{(p)}}\right)^{-\theta} \le \Gamma(\theta).$$

Since we generally have $\Gamma(\theta) \le \hat{\chi}$, the proof is complete.

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