Optimal Insurance Design under Rank-Dependent Expected Utility

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Outline

1. Insurance Design under Rank-Dependent Expected Utility
2. Models: Indemnity Frame and Retention Frame
3. Quantile Formulation
4. A One-dimensional Optimization Problem
5. Discussion and Comparisons
6. Numerical Examples
7. Conclusion

- An *insured* faces a random loss X. He can purchase an insurance contract from an insurance company (i.e. *insurer*). He pays a premium π and obtains an indemnity $Y = I(X)$, where I is so-called *indemnity function* with $0 \leq I(x) \leq x$.
- Example (*deductible insurance contract*): $I(X) = (X - C)_+$ for some constant level $C > 0$.

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Insurance Design under RDEU (Cont'd)

- The insured chooses premium π and indemnity function I to maximizes the rank-dependent expected utility (RDEU) of his final wealth W:

$$V^{rdeu}(W) = (T \circ \mathbb{P})(U(W)),$$

- where $W = W_0 - X + I(X) - \pi$, W_0 is the initial wealth; U is a *utility function*, T is a *probability distortion function*, and $T \circ \mathbb{P}(U(W))$ is the *Choquet integral* of $U(W)$ w.r.t. $T \circ \mathbb{P}$.

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Insurance Design under RDEU (Cont'd)

- It can be written explicitly as

$$V^{rdu}(W) = \int U(x) d\left[-T(1 - F_W(x))\right]$$

- Empirical evidence supports a *concave* utility function and a *reversed S-shaped* probability distortion function: large and small payoffs occurring with small probability are overweighted.

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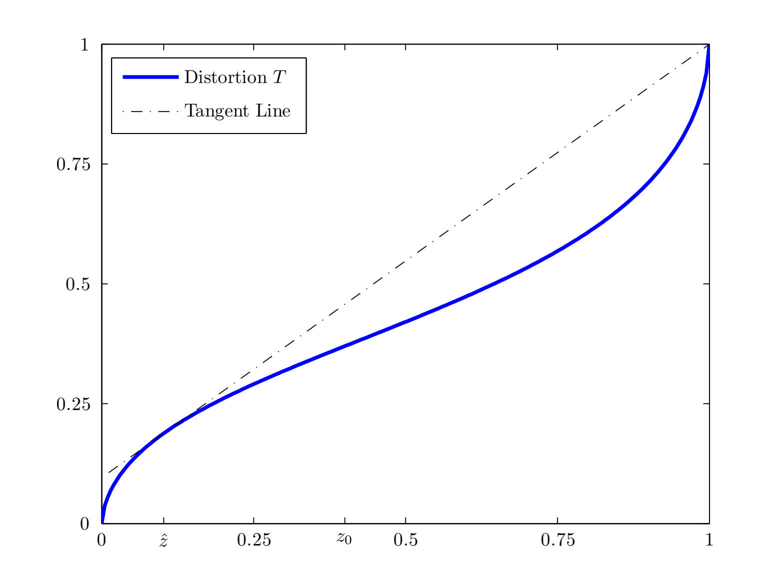
A Reversed S-shaped Distortion Function



Problem 2.1: Optimal Indemnity Design

- To solve the problem, we first fixe a premium π and find the optimal indemnity function I, and then find the optimal premium π^*. As the second step is easy, we focus on the first step.
- Denote by $\mathcal{I}$ the set of all indemnity functions. For a fixed premium π, the insured's optimization problem can be written as

$$\max_{I(\cdot)\in\mathcal{I}} \quad V^{rdu}(W_0 - X + I(X) - \pi)$$
$$\text{subject to} \quad (1+\rho)\mathbb{E}[I(X)] \leq \pi,$$

- where $\rho > 0$ is the insurer's *safety loading*.

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Literature

- Classical works in the EUT framework: Arrow(1963, 1971), Raviv (1979), Gollier(1996), and Gollier-Sch (1996)
- Related works in the RDEU framework: Chateauneuf et al. (2000), Dana and Scarsini(2007), Carlier and Dana (2011), Sung et al. (2011), and Carlier et al. (2008).
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Retention Functions

- We assume the insured's loss is bounded by M.
- In order to apply the quantile formulation technique, we consider the retention, $R(X) := X - I(X)$, i.e., the part of loss retained by the insured, where $R(x) = x - I(x), x \in [0, M]$ is the so-called *retention function*.
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Problem 2.2: Optimal Retention Design

- We then reformulate Optimal Indemnity Design in terms of the retention function:

$$\max_{R(\cdot)\in\mathcal{R}} \quad V^{rdu}(W_0 - R(X) - \pi)$$
$$\text{subject to} \quad \mathbb{E}[R(X)] \geq \Delta,$$

where

$$\Delta := \mathbb{E}[X] - \frac{\pi}{1+\rho}.$$

Assumptions

- **Assumption 2.1** The loss X has no atom, i.e., the CDF of X is continuous. Moreover, its quantile function $F_X^{-1} : (0,1) \to \mathbb{R}_+$ is continuous.
- **Assumption 2.2** [Concave Utility] U is strictly increasing and is continuously differentiable on $(0,\infty)$. Furthermore, $U'(\cdot)$ is strictly decreasing on $(0,\infty)$.
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Assumptions (Cont'd)

- $z_0 \in (0,1)$ such that T' is strictly decreasing on $(0, z_0)$ and strictly increasing on $(z_0, 1)$. Furthermore, $T'(0+) := \lim_{z \downarrow 0} T'(z) > 1$ and $T'(1-) := \lim_{z \uparrow 1} T'(z) = +\infty$.
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Quantile Formulation

- First of all, we make a change-of-variable

$$
\begin{aligned}
V^{rdu}(W_0 - R(X) - \pi) &= \int U(x)d\left[-T(1 - F_{W_0-R(X)-\pi}(x))\right] \\
&= \int_0^1 U(F^{-1}_{W_0-R(X)-\pi}(z))T'(1-z)dz \\
&= \int_0^1 U(W_0 - \pi - F^{-1}_{R(X)}(1-z))T'(1-z)dz \\
&= \int_0^1 U\left(W_0 - \pi - F^{-1}_{R(X)}(z)\right) T'(z)dz
\end{aligned}
$$

- Let us denote $G := F_{R(X)}^{-1}$, the quantile function of $R(X)$. The previous calculations show that one can express the objective functional of the insured as a functional of G, which is concave in G, because the utility function U is concave.
- Intuitively, a reasonable retention functions should be non-decreasing. The following proposition shows that we can restrict ourselves to the retention functions in the form of $R(x) = G(F_X(x))$ where $G(\cdot)$ is a quantile function.

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- **Proposition 2.1** Under Assumption 2.1, for any feasible solution $R(\cdot)$ to Problem 2.2, $\tilde{R}(x) := F^{-1}_{R(X)}(F_X(x))$ is also feasible with respect to Problem 2.2 and $\tilde{R}(X)$ has the same law as $R(X)$.
- For retention functions of the form $R(x) = G(F_X(x))$, the constraint $0 \leq R(x) \leq x, x \in [0, M]$ is equivalent to $0 \leq G(z) \leq F_X^{-1}(z), 0 < z < 1$.
- On the other hand, it is easy to see that $\mathbb{E}[R(X)] \geq \Delta$ is equivalent to $\int_0^1 G(z)dz \geq \Delta$.

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Proof Denote $Z := F_X(X)$. Because X has no atom, Z is a uniform random variable on $[0,1]$. As a result, $\tilde{R}(X) = F^{-1}_{R(X)}(Z)$ has the same law as $R(X)$. Recalling that $R(x) \le x$, $x \in [0, M]$, we immediately have

$$F^{-1}_{R(X)}(z) = \inf\{s : F_{R(X)}(s) \ge z\} = \inf\{s : \Pr(R(X) \le s) \ge z\} \\ \le \inf\{s : \Pr(X \le s) \ge z\} = F_X^{-1}(z)$$

for any $z \in (0,1)$. It follows that $\tilde{R}(x) \le F_X^{-1}(F_X(x)) \le x$, $x \in [0, M]$.

Problem 2.3: Optimal Quantile of Retention

- Hence, we can rewrite Problem 2.2 as the following problem, where the quantile function $G(\cdot)$ becomes the decision variable.

$$
\begin{aligned}
\underset{G(\cdot)\in\mathbb{G}}{\text{Max}}\; V(G(\cdot)) \quad & := \int_0^1 U(W_0 - \pi - G(z))T'(z)dz, \\
\text{Subject to} \quad & 0 \le G(z) \le F_X^{-1}(z), 0 < z < 1, \\
& \int_0^1 G(z)dz \ge \Delta,
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where $\mathbb{G}$ is the set of all *quantile* functions.

Problem 2.4: Auxiliary Problem

- If G^* is the optimal quantile, then $R^*(X) := G^*(F_X(X))$ is the optimal retention.
- We apply the Lagrange dual method to remove the second constraint in Problem 2.3.
- Apply multiplier λ to the second constraint, leading to the following partially constrained problem

$$\begin{aligned} \max_{G(\cdot)\in\mathbb{G}} V_\lambda(G(\cdot)) \quad &:= \int_0^1 \left[U(W_0 - \pi - G(z))T'(z) + \lambda G(z)\right] dz - \lambda\Delta, \\ \text{subject to} \quad & 0 \le G(z) \le F_X^{-1}(z), \quad 0 < z < 1. \end{aligned}$$

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Solving the Auxiliary Problem

- We first solve Problem 2.4 to find the optimal solution for any given multiplier λ.
- Ignoring all the constraints in Problem 2.4 for the present, we can derive the optimal solution to the problem by performing the pointwise optimization

$$\max_{y} \quad \{U(W_0 - \pi - y)T'(z) + \lambda(y - \Delta)\}$$

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Solving the Auxiliary Problem (Cont'd)

- The *pointwise optimizer* can be easily derived as follows:

$$H_\lambda(z) := W_0 - \pi - (U')^{-1}\left(\frac{\lambda}{T'(z)}\right), \quad 0 < z < 1.$$

  Here, we define $(U')^{-1}(y) := 0$ for any $y > U'(0+)$.

- Because of Assumption 2.2, $H_\lambda(\cdot)$ is strictly increasing on $(0, z_0)$ and strictly decreasing on $(z_0, 1)$.

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Solving the Auxiliary Problem (Cont'd)

- If we take the constraint $0 \le G(z) \le F_X^{-1}(z), 0 \le z \le 1$ into account, we then need to consider the pointwise optimization:

$$\max_{y \in [0, F_X^{-1}(z)]} \left\{ U(W_0 - \pi - y) T'(z) + \lambda(y - \Delta) \right\}, \ 0 < z < 1,$$

leading to the pointwise optimizer

$$\widetilde{H}_\lambda(z) := \max(0, \min(H_\lambda(z), F_X^{-1}(z))).$$

Solving the Auxiliary Problem (Cont'd)

- If $\widetilde{H}_\lambda(\cdot)$ were non-decreasing, then it would automatically become the optimal solution to Problem 2.4.
- However, $\widetilde{H}_\lambda(\cdot)$ fails to be globally non-decreasing on $(0, 1)$ because $H_\lambda(\cdot)$ is decreasing on $(z_0, 1)$.
- Following the idea of He and Zhou (2012), we can modify a pointwise optimizer.

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- Because $H_\lambda(\cdot)$ is strictly decreasing and $F_X^{-1}(\cdot)$ is increasing on $(z_0, 1)$, the intersection point of $H_\lambda(\cdot)$ and $F_X^{-1}(\cdot)$ on $(z_0, 1)$, if it exists, is unique. Denote by $z_2(\lambda)$ this intersection point when it exists.
- Otherwise, define $z_2(\lambda) = 1$ if $H_\lambda(z) > F_X^{-1}(z), z_0 < z < 1$; and $z_2(\lambda) = z_0$ if $H_\lambda(z) < F_X^{-1}(z), z_0 < z < 1$.
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Solving the Auxiliary Problem (Cont'd)

- The following proposition is a key step toward the final result:
- **Proposition 3.1** For any feasible solution $G(\cdot)$ of Problem 2.4, there exists $c \in (0, z_2(\lambda)]$ such that

$$G^c(z) := \widetilde{H}_\lambda(z) I_{z \le c} + \widetilde{H}_\lambda(c) I_{z > c}, \quad 0 < z < 1$$

satisfies (i) $V_\lambda(G(\cdot)) \le V_\lambda(G^c(\cdot))$; (ii) the equality holds if and only if $G(z) = G^c(z), 0 < z < 1.$

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Solving the Auxiliary Problem (Cont'd)

- So any feasible solution to Problem 2.4 is dominated by a simple modification of $\widetilde{H}_\lambda$ parameterized by c.
- As a result, Problem 2.4 can be reduced to

$$\operatorname*{Max}_{G(\cdot)\in\mathbb{S}_\lambda} V_\lambda(G(\cdot)) := \int_0^1 [U(W_0-\pi-G(z))T'(z)+\lambda G(z)]\,dz-\lambda\Delta.$$

where

$$\mathbb{S}_\lambda := \{G^c(\cdot) \mid G^c(z) := \widetilde{H}_\lambda(z)I_{z\le c}+\widetilde{H}_\lambda(c)I_{z>c}, 0<z<1, \text{ for some } c\in(0,z_2(\lambda)]\}.$$

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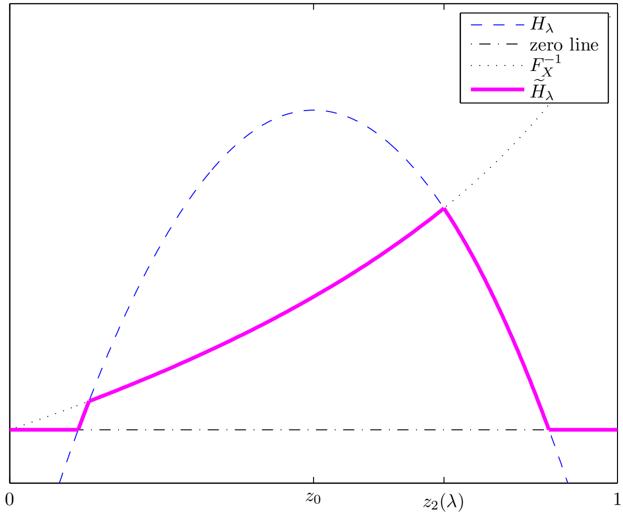
Graphic Demonstration



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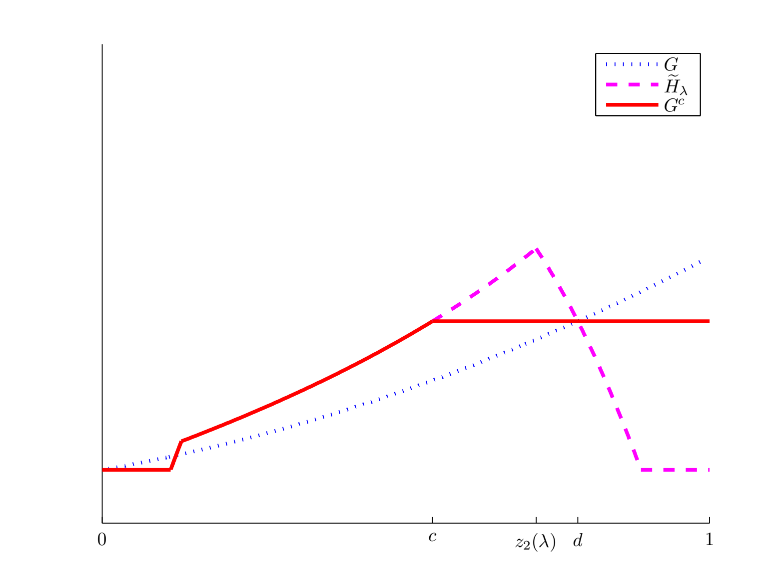

A One-dimensional Optimization Problem

- For any $G^c(\cdot) \in \mathbb{S}_\lambda$, denote $f(c) := V_\lambda(G^c(\cdot))$. Straightforward computation leads to

$$f(c) = \int_{(0,c]} h(z) d\widetilde{H}_\lambda(z) + U(W_0 - \pi - \widetilde{H}_\lambda(0+)) + \lambda \widetilde{H}_\lambda(0+) - \lambda \Delta,$$

- where

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A One-dimensional Optimization Problem (Cont'd)

- Explicit solutions (for three cases) can be found (illustrated graphically in the following).
- **Case 1—Small** λ: $\lambda \leq U'(W_0 - \pi)T'(\hat{z})$.
- **Case 2—Medium** λ: $U'(W_0 - \pi)T'(\hat{z}) < \lambda < U'(W_0 - \pi - F_X^{-1}(\hat{z}))T'(\hat{z})$.
- **Case 3—Large** λ: $\lambda \geq U'(W_0 - \pi - F_X^{-1}(\hat{z}))T'(\hat{z})$.

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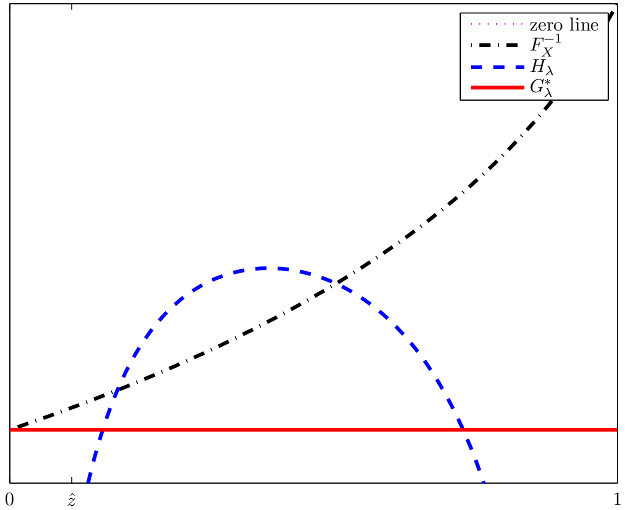
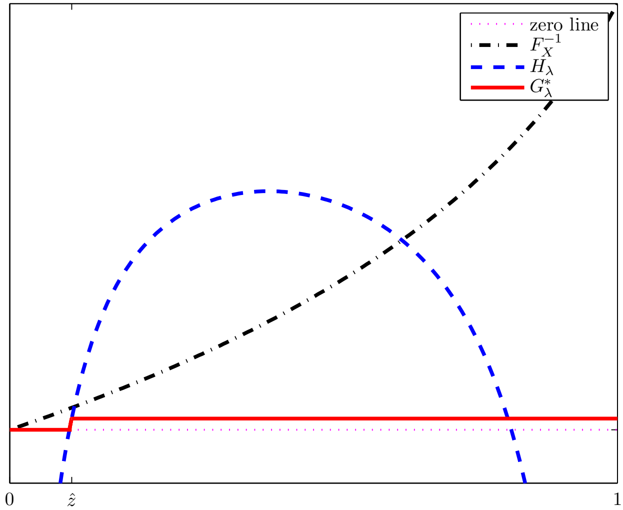
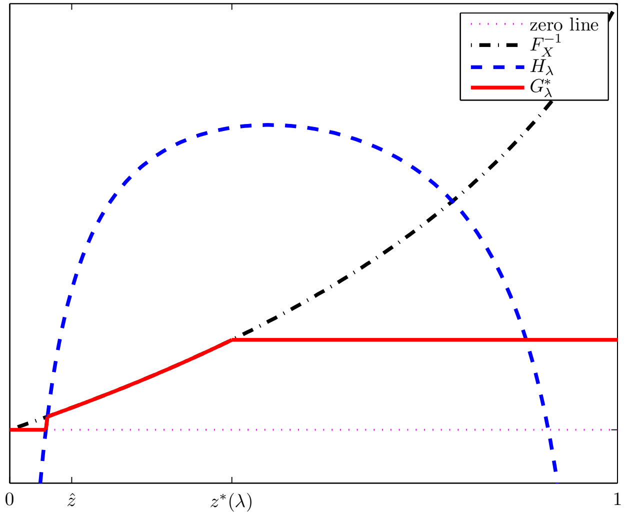
Optimal Solution: Case 1—Small λ (Low π)



Optimal Solution: Case 2—Medium λ (Medium π)



Optimal Solution: Case 3—Large λ (High π)



- **Convex Distortions** The optimal indemnity function is given by $I^*(x) = (x - F_X^{-1}(c^*))_+$ where c^* is such that $\mathbb{E}[I^*(X)] = \frac{\pi}{1+\rho}$.
- **Concave Distortions** The optimal indemnity function is given by

$$I^*(x) = \left[x - \max\left(W_0 - \pi - (U')^{-1}\left(\frac{\lambda}{T'(F_X(x))}\right), 0\right)\right]_+,$$

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Discussion and Comparisons (Cont'd)

Table: Summary of the results when U is concave

	Convex Distortion	Concave Distortion	Reversed S-shaped Distortion
Indemnity	Deductible	Complex contract	Complex contract
Small losses	No insurance	Full insurance.	Full insurance.
Medium Losses	No insur. *or* FIAD	CC(PFI)	CC(PD)
Large Losses	FIAD	No insurance	FIAD

"CC(PD)" stands for "Complex Contract (possibly decreasing)".
" CC(PFI)" stands for "Complex Contract (possibly full insurance)".
"FIAD" stands for "full insurance above a deductible".

A Numerical Example: Parameter Specifications

- X follows a truncated exponential distribution with density $f(x) = \frac{me^{-mx}}{1-e^{-mM}}$, $x \in [0, M]$, where $m = 0.1$ and $M = 10$.
- The utility function is exponential: $U(x) = 1 - e^{-\gamma x}$ with $\gamma = 0.2$.
- Probability distortion function (Tversky and Kahneman(1992))

$$T(z) = \frac{z^a}{(z^a + (1-z)^a)^{\frac{1}{a}}}, \quad 0 < z < 1.$$

We take three values for a: 0.5, 0.65, and 0.8.

- The safety loading $\rho = 0.2$.
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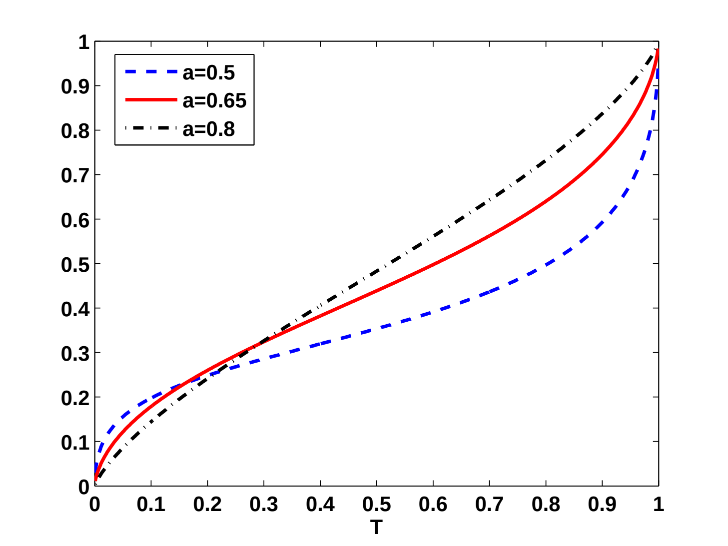
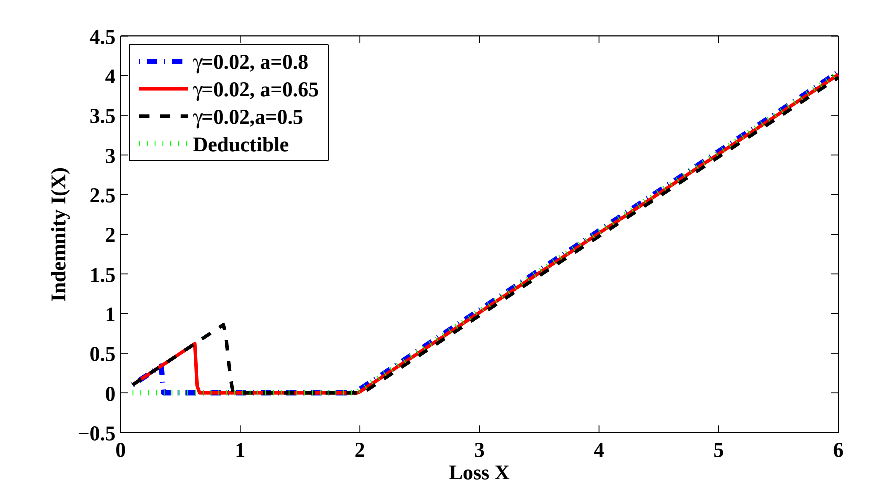
A Numerical Example: Distortion Function



A Numerical Example: Optimal Indemnity



Conclusion

To summarize, the contribution of our paper is threefold.

- First, we formulate and solve an optimal insurance problem in which the insured has RDEU preferences.
- Second, we work out the optimal indemnity explicitly.
- Third, we demonstrate that RDEU is able to explain the demand for insurance for small losses, which is consistent with observed behaviors that EU fails to explain.

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