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PEACOCKS AND ASSOCIATED MARTINGALES, WITH EXPLICIT CONSTRUCTIONS

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WORK and BOOK with:

F. HIRSCH, C. PROFETA, B. ROYNETTE
[HPRY]

1. MOTIVATION, TERMINOLOGY, A GUIDING EXAMPLE

(1.a) Options are risky; it seems natural that their prices increase with maturity, i.e:

$$\begin{cases} T \longrightarrow E((X_T - K)^+) \\ T \longrightarrow E((K - X_T)^+) \end{cases} \uparrow$$

(1.b) The peacock terminology

Let (X_t) be an integrable process,

ie: $E(|X_T|) < \infty$, $\forall T$

We call it a peacock if:

$\forall \varphi$ convex, $T \longrightarrow E[\varphi(X_T)]$

is increasing.

Explanation: In French, PCOC

(1.c) Guiding Example

Carr - Ewald - Xiao (2008) noticed that

$$T \longrightarrow \frac{1}{T}\int_0^T ds \exp\left(\lambda B_s - \frac{\lambda^2 s}{2}\right)$$

is a peacock, which motivated many developments.

(1.d) Kellerer's theorem (1972)

$(X_T)_{T\geq 0}$ is a peacock iff it has the same one-dimensional marginals as a martingale $(M_T)_{T\geq 0}$, which may be chosen Markovian.

(1.e) A clear program: To identify

(Many peacocks) $\longleftrightarrow$ (Associated martingales)

2. On Kellerer's thm: the regular case

Thm K (revisited)

Let $(X_T)_{T \geq 0}$ be a peacock such that: $(T, K) \longrightarrow C(T,K) \equiv E\left((X_T - K)^+\right)$ is C^∞ on $\mathbb{R}_+ \times \mathbb{R}$.

Then, an associated martingale is the solution of:

$$Z_t = Z_0 + \int_0^t \sigma(s, Z_s)\, dB_s$$

where:

$$\boxed{\sigma(s,x) = \left(\frac{2 \frac{\partial}{\partial s} C(s,x)}{\frac{\partial^2}{\partial x^2} C(s,x)} \right)^{1/2}}$$

$\left(\frac{\partial^2}{\partial x^2} C(s,x) \equiv p(s,x), \text{ the density of } X_s \right)$

$\boxed{\text{Dupire's formula}}$

Z_0 is independent from B, and distributed as $p(0,x)\,dx$

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2 approximation steps allow to obtain the desired result in the general case —

$\boxed{\textbf{Thm K'}: \text{ In all generality, a peacock } (X_t) \text{ admits an associated martingale } (M_t).}$

3. Two families of peacocks

(F1) $\boxed{\frac{1}{t}\int_0^t ds\, M_s}$

(F2) $\boxed{\int_0^t ds\,(M_s - M_0)}$; (F'2) $\boxed{tX, \text{ with } E(X)=0}$

4. The sheet's method

(4.a) An associated martingale for the guiding example.

- for fixed t, $\frac{1}{t}\int_0^t ds \exp\left(\lambda B_s - \frac{\lambda^2 s}{2}\right)$

$$\overset{(law)}{=} \int_0^1 du \exp\left(\lambda W_{u,t} - \frac{\lambda^2 ut}{2}\right)$$

(Wiener sheet)

(4.b)

- for every λ fixed,

$$\int \sigma(du) \exp\left(\lambda B_u - \frac{\lambda^2 u}{2}\right)$$

$$\overset{(law)}{=} \int \sigma(du) \exp\left(W_{(\lambda^2, u)} - \frac{\lambda^2 u}{2}\right)$$

(4.c) An extension to Lévy processes

Dalang - Walsh associate to a Lévy process $(L_u)_{u\geq 0}$ a Lévy sheet $(\mathcal{L}_{u,t})_{u,t\geq 0}$

such that :

for fixed t, $L_{\cdot t} \overset{(law)}{=} \mathcal{L}_{\cdot, t}$

Thus :
$$\int \sigma(du) \frac{\exp(L_{ut})}{E[\exp(L_{ut})]}$$

$$\overset{(law)}{=} \int \sigma(du) \frac{\exp(\mathcal{L}_{u,t})}{E[\exp(\mathcal{L}_{u,t})]} \quad (\leftarrow \text{Martingale})$$

5. The time-reversal method

To a space-time harmonic function :
$$h : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R} \quad \ni :$$

$h(0,0) = 0$, and $\dfrac{\partial h}{\partial s} + \dfrac{1}{2}\Delta h = 0$,

we apply time reversal :

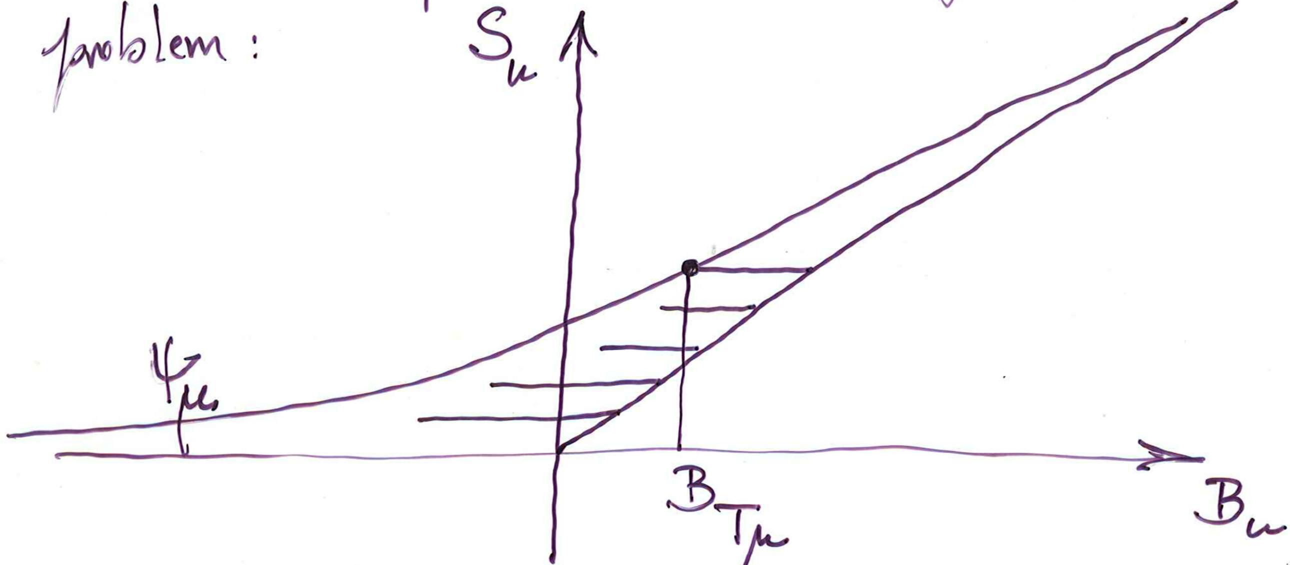
for fixed t, $\displaystyle\int_0^t ds\, h(s, B_s) \overset{(law)}{=} \underbrace{\int_0^t ds\, h(t-s, B_t - B_s)}_{\text{martingale}}$,

which writes (Itô) :
$$\int_0^t dB_u \cdot \int_0^u \nabla h(u-s, B_u - B_s).$$

6. Skorokhod embedding methods

Some SE may be used to show / embed ${}_tX \sim \mu_t$ $(t \geq 0)$ as a martingale in Brownian motion.

Recall the particular solution of Skorokhod problem:



$$T_\mu = \inf\{u : S_u \geq \psi_\mu(B_u)\}$$

with $$\psi_\mu(x) = \frac{1}{\mu[x,\infty)} \int_{[x,\infty)} t \, d\mu(t)$$

If $\psi_{\mu_t}(x)$ is (pointwise in x) increasing in t, then so are the T_{μ_t} and $\left(B_{T_{\mu_t}}\right)$ is a martingale.

A necessary and sufficient condition is:

$$\boxed{\mathbb{R}_+ \ni x \longrightarrow \left(x / \psi_\mu(x) \right) \quad \text{is increasing}}$$

Many examples —

7. Bougerol's identity

- This is : $\boxed{\sinh(B_t) \overset{(law)}{=} \int_0^t \exp(B_s)\, d\gamma_s}$

for fixed t.

Thus, $(\sinh(B_t), t \geq 0)$ is a peacock.

Again, to prove the identity, we use time-reversal

for fixed t, $\sinh(B_t) \overset{(law)}{=} \exp(B_t) \int_0^t \exp(-B_s)\, d\gamma_s$

But, now, the two processes have the same law, that of the solution of :

$$X_t = \int_0^t \sqrt{1+X_s^2}\, d\beta_s + \frac{1}{2}\int_0^t X_s\, ds$$

- More generally, when is the process $(\varphi(B_t))$ a peacock ?; also, the solution of :

$$X_t = \int_0^t \sigma(X_s)\, d\beta_s + \int_0^t b(X_s)\, ds \;??$$

Answer (sufficient conditions)

$(*)$ σ, b Lipschitz ; σ even ; b odd $\exists$:

$$\mathrm{sgn}(b(x)) = \mathrm{sgn}(x)$$

$(**)$ φ increasing and odd, $\exists$:

$$\forall t, \quad E[|\varphi(B_t)|] < \infty$$

8. Questions, Remarks

(8.a) Let $Q_{|\mathcal{F}_t} = D_t \cdot W_{|\mathcal{F}_t}$

(Wiener measure on canonical space)

Question: Under which condition on (D_t) is (X_t) a peacock under Q?

Exercise 6.1 (p. 242) of [HPRY] gives restrictive sufficient condition —

$$E_Q[\psi(X_t)] = \underbrace{E_W[\psi(X_t) E_W(D_t | X_t)]}_{\text{is increasing in } t \;??}$$

(8.b) Let (M_t) be defined as:

$$M_t = \int_0^t \sigma(s)\, dB_s$$

($\uparrow$ predictable)

Gyongy and Krylov show (M_t) has the same 1-dim marginals as

$$X_t = \int_0^t \tilde{\sigma}(s, X_s)\, dB_s$$

where:

$$\tilde{\sigma}(s,x) = \left(E[\sigma^2(s) | M_s = x] \right)^{1/2}$$

(8.c) Non-uniqueness of associated martingale

There exist a sequence of continuous martingales with the same 1-dim. marginals as BM

See Albin / Stat. Prob. Letters (2008)

$$E_W[\psi(X_t)D_t]$$

$$= \psi(0) + E_W\Big[\int_0^t \psi(X_s)\,dD_s + \int_0^t D_s\,\tfrac{1}{2}\psi''(X_s)\,ds + \int_0^t d\langle D, X\rangle_s\,\psi'(X_s)\Big]$$

$$\geq \psi(0) + E\Big[\int_0^t \underbrace{d\langle D, X\rangle_s}_{(=\,\delta(s,X_s)\,ds)}\,\psi'(X_s)\Big]$$