Russo-Seymour-Welsh estimates for the Kostlan ensemble of random polynomials

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Random geometries / Random topologies

Laplace Eigenfunctions: Chladni figures

Interest in the nodal lines of Laplace eigenfunctions has a very long history and goes back to Hooke (XVII century) and Chladni (XVIII century) who observed nodal lines on a vibrating plate.

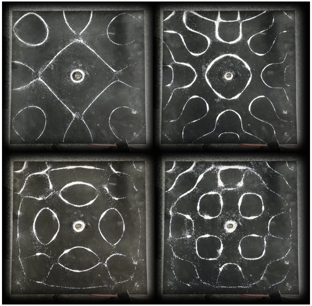


Figure: Chladni figures (MIT Physiscs Demos)

Berry's conjecture

In 1977 M. Berry conjectured that high energy eigenfunctions in the chaotic case have statistically the same behaviour as random plane waves. (Figures from Bogomolny-Schmit paper)

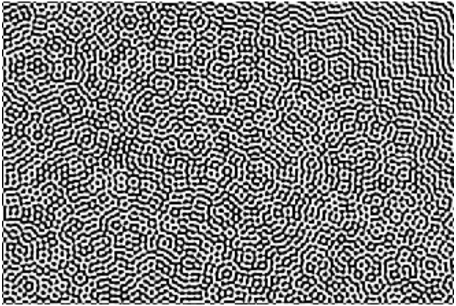


Figure: Nodal domains of an eigenfunction of a stadium

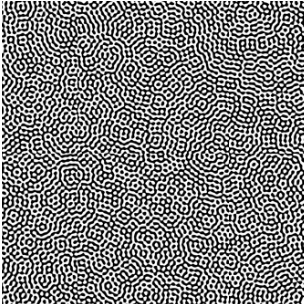


Figure: Nodal domains of a random plane wave

Random Ensembles

$(\mathcal{M}, g)$ Riemannian manifold, $\Delta\phi_n + \lambda_n\phi_n = 0$

Monochromatic weave: random combination of eigenfunctions with close eigenvalues

$$f_{1,T} = \sum_{T-\sqrt{T}<\lambda_i<T} c_n\phi_n, \quad c_i \text{ i.i.d. normal}$$

Band-limited: random combination of eigenfunctions for a wide range of frequencies

$$f_{\alpha,T} = \sum_{\alpha T<\lambda_i<T} c_n\phi_n, \quad c_i \text{ i.i.d. normal}$$

Spherical ensembles

For sphere eigenfunctions are spherical harmonics. There are $2n+1$ spherical harmonics of degree n, eigenvalue $n(n+1)$.

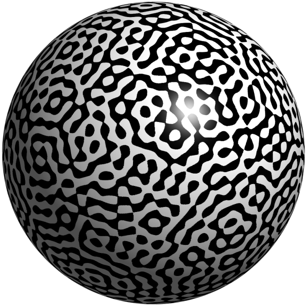


Figure: Random spherical harmonic of degree 80, $\alpha = 1$

Figure: Real Fubini-Study ensemble of degree 80, $\alpha = 0$

Scaling limits of spherical ensembles

It is possible to pass to the (universal) local scaling limit as degree (energy) goes to infinity.

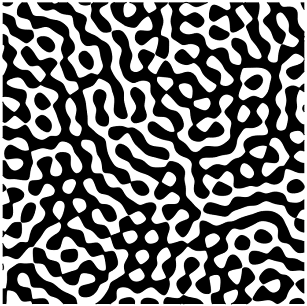


Figure: The random plane wave is the scaling limit of the random spherical harmonic

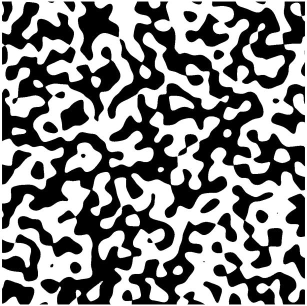


Figure: The band-limited function is the scaling limit of the real Fubini-Study

Scaling Limit: Random Plane Wave

Two ways to (informally) think of the random plane wave

- A "random" or "typical" solution of Helmholtz equation $\Delta f + k^2 f = 0$
- A random superposition of all possible plane waves with the same frequency k

Formal definition:

- A Gaussian field $f(z)$ with covariance kernel

$$K(z, w) = \mathbb{E}[f(w)f(w)] = J_0(k|z-w|)$$

- Random series

$$f(z) = f(re^{i\theta}) = \mathrm{Re} \sum a_n J_{|n|}(r) e^{in\theta}$$

Nodal Lines of Gaussian Spherical Harmonic

Theorem (Bérard, 1985)

For Gaussian spherical harmonic g_n of degree n

$$\mathbb{E}L(g_n) = \pi\sqrt{2\lambda_n} = \sqrt{2}\pi n + O(1)$$

Theorem (Nazarov and Sodin, 2007)

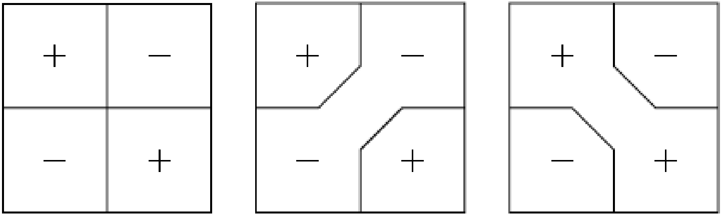
Let g_n be Gaussian spherical harmonic of degree n. Then there is a positive constant a such that

$$\mathbb{P}\left\{\left|\frac{N(g_n)}{n^2} - aVol(S^2)\right| > \epsilon\right\} \leq C(\epsilon)e^{-c(\epsilon)n}$$

where $C(\epsilon)$ and $c(\epsilon)$ are positive constant depending on ϵ only.

Bogomolny-Schmit Percolation Model

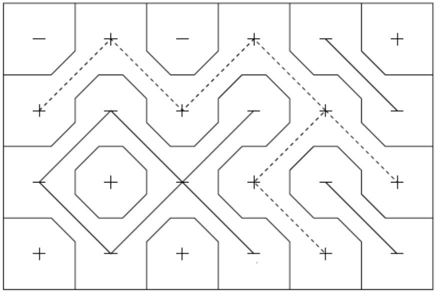
They proposed that the nodal lines form a perturbed square lattice



Picture from Bogomolny-Schmit paper.

Bogomolny-Schmit Percolation Model

Using this analogy we can think of the nodal domains as percolation clusters on the square lattice. This leads to the conjecture that $a = (3\sqrt{3} - 5)/\pi \approx 0.0624$



Picture from Bogomolny-Schmit paper.

Is It Really True?

- Numerical results (Nastasescu (2011), Konrad (2012), B.-Kereta (2013)) show that the number of nodal domains per unit area is 0.0589 instead of 0.0624 predicted by Bogomolny-Schmit.
- Number of clusters per vertex is a non-universal quantity in percolation, it is lattice dependent. Global properties should be universal i.e. lattice independent.
- Numerical evidence that many global 'universal' observables (crossing probabilities, decay rate for the area of nodal domains, one-arm exponent) match percolation predictions.

Off-critical models

Off-critical percolation is a model for excursion and level sets

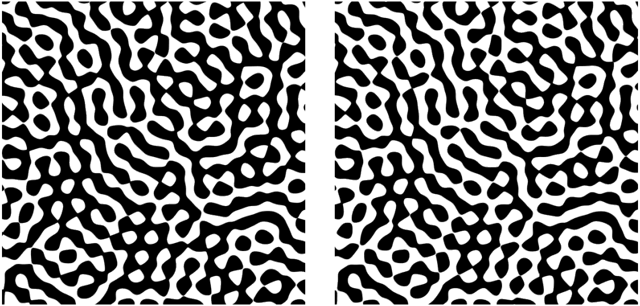


Figure: Excursion sets for levels 0 (nodal domains) and level 0.1

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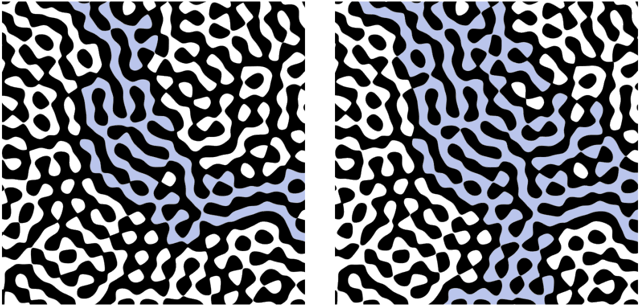

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Kostlan ensemble

Kostlan or complex Fubini-Study ensemble of homogeneous polynomials $\mathbb{R}^3$ (or S^2)

$$f(x) = \sum_{|J|=n} a_J \sqrt{\binom{n}{J}} x^J$$

Covariance kernel $\cos^n(d(x,y))$

Natural measure on space of homogeneous polynomials

Nodal domains is a natural model for real projective curve

Figure: Kostlan ensemble $n = 300$

Bargmann-Fock Random Function

Theorem (Beffara-Gayet 2016)

Russo-Seymour-Welsh estimate for Bargmann-Fock random function.

Bargmann-Fock Gaussian function is the scaling limit of Kostlan ensemble

$$f(x) = \sum a_{i,j} \frac{1}{\sqrt{i!j!}} x_1^i x_2^j e^{-|x|^2/2}$$

Covariance kernel

$$K(x,y) = e^{-|x-y|^2/2}$$

Important:
positive, symmetric, fast decaying

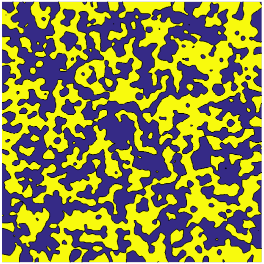
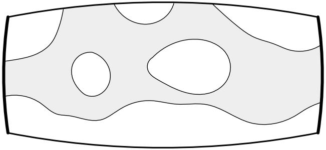


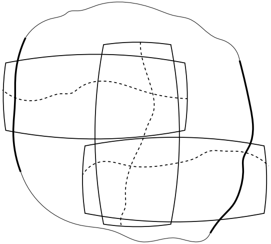
Figure: Nodal domains of Bargmann-Fock function

Russo-Seymour-Welsh for Kostlan Ensemble

Theorem (B.-Muirhead-Wigman)

There is RSW estimate for nodal domains and nodal lines of Kostlan ensemble which is uniform in polynomial degree and 'conformal type' of a domain.
Also true for general symmetric fields with sufficiently fast decay of correlation.





General result

Theorem (Abstract RSW)

Let f_n be a collection of Gaussian fields with covariance kernels κ_n. We assume that there are $s_n \to 0$ such that

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Fields $f_n \in C^3$ and $\kappa_n \in C^6$. Hessian of κ_n at 0 is positive definite

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Let $\Phi : \mathbb{R}^2 \to S^2$ exponential map (at some point), then $K_n(x, y) = \kappa_n(\Phi(s_n x), \Phi(s_n y)) \to K_\infty(x, y)$ locally uniformly together with the first four derivatives.

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Negative part of κ_n is $o(s_n^{12+\epsilon})$. Exponent 12 is not optimal, could be replaced by 8 (assuming convergence of six derivatives). See also Beffara-Gayet.

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$|\kappa_n(x,y)| \lesssim (d(x,y)/s_n)^{-18-\epsilon})$ *for* $d(x,y) \gtrsim s_n$. *Exponent* 18 *could be replaced by* 12. *See also Rivera and Vanneuville (2017).*

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Then the nodal lines satisfy RSW on down to the scale s_n and nodal domains satisfy RSW on all scales.

Ingredients

- For each field there is the minimal scale s_n. All estimates should be uniform

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For Kostlan ensemble $s_n = \sqrt{n}$. On this scale it converges to Bargmann-Fock

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For $\epsilon > 0$ and polygons of controlled shape that are Cs_n separated for large C

$$|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \leq \epsilon$$

where A and B are crossing events

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Problems: no scaling, curvature

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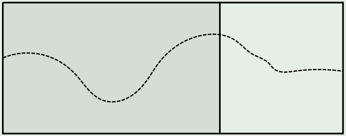

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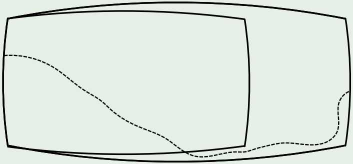

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Can add a small positive constant to the kernel. This allows to ignore sufficiently small negative correlations

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