Statistics of random linear combinations of Laplace eigenfunctions

05-12-2017

Yaiza Canzani

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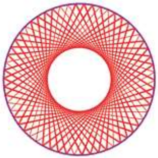

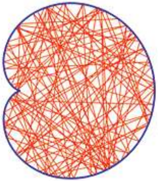

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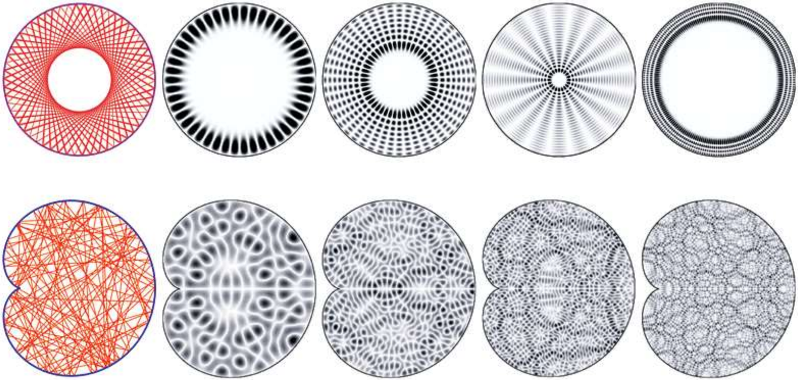

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Random waves: $\Psi_\lambda = \frac{1}{(\#\{\lambda_j=\lambda\})^{1/2}} \sum_{\lambda_j=\lambda} a_j \, \phi_{\lambda_j} \qquad a_j \sim N(0,1)$ iid

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Lemma

Let $x_0 \in \mathbb{S}^n$ or $\mathbb{T}^n$. Then,

$$\lim_{\lambda\to\infty} Cov_{\Psi_\lambda}(x_0 + \tfrac{u}{\lambda}, x_0 + \tfrac{v}{\lambda}) = Cov_{\Psi_\infty}(u,v),$$

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$\Psi_\infty : \mathbb{R}^n \to \mathbb{R}$ is a Gaussian field with $\mathrm{Cov}_{\Psi_\infty}(u,v) = \frac{1}{(2\pi)^n} \int_{\mathbb{S}^{n-1}} e^{i\langle u-v,w\rangle} d\sigma_{S^{n-1}}(w)$

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Let $x_0 \in M$. If measure$\{$geodesic loops closing at $x_0\} = 0$, then

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Random wave conjecture:

$$\phi_\lambda\left(x_0 + \tfrac{u}{\lambda}\right) \quad \text{has same statistics as} \quad \Psi_\infty(u).$$

Answers: Integral Statistics

Theorem (C-Hanin'16)

If measure{geodesic loops closing at x}$= 0$ for a.e $x \in M$, then

$$\lim_{\lambda} \mathbb{E}\left[\frac{\#\{\text{critical points of } \Psi_\lambda\}}{\lambda^n}\right] = A_n$$

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If measure{geodesics joining x, y} $= 0$ for a.e. $x, y \in M$, then

$$Var\left[\frac{\#\{\text{critical points of } \Psi_\lambda\}}{\lambda^n}\right] = O\left(\lambda^{-\frac{n-1}{2}}\right)$$

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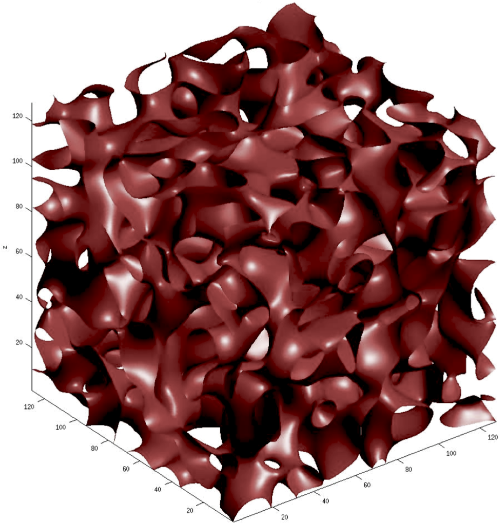
Z_{Ψ_∞} for $n = 3$



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Theorem (Sarnak-Wigman '13 combined with C-Hanin '15**)**

Let (M,g) *be s.t. measure{ geodesic loops at* x *}* $= 0$ *for a.e.* $x \in M$*. Then,*
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Theorem (C-Sarnak '15)

$$supp(\mu_\infty) = \mathcal{H}(n-1).$$

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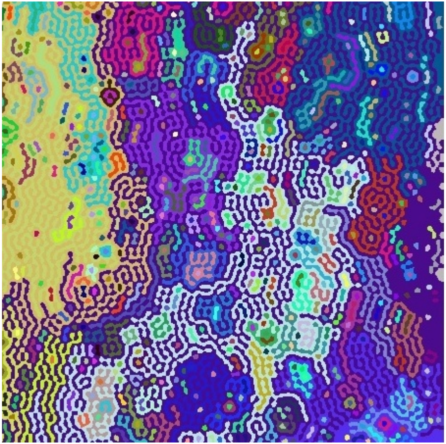
Theorem (C-Sarnak '16 case $n = 2$ done by Sarnak-Wigman '13)

$$\mathit{supp}(\nu_\infty) = \mathcal{T}.$$

Nodal domains for $n = 2$



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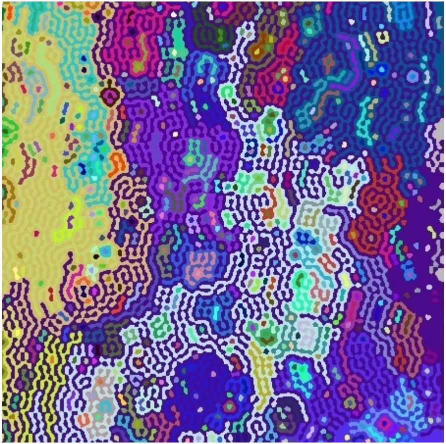

Connectivity	∞-measure
0	0.9447
1	0.0282
2	0.0089
3	0.0044
4	0.0026
5	0.0017
6	0.0012
7	0.0009
8	0.0007
9	0.0006
10	0.0005

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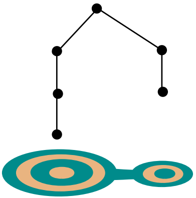

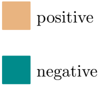

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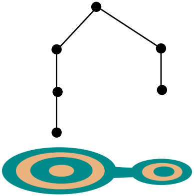
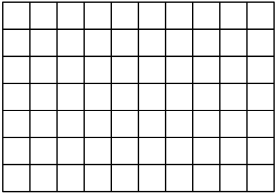
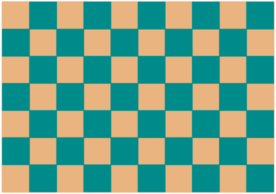




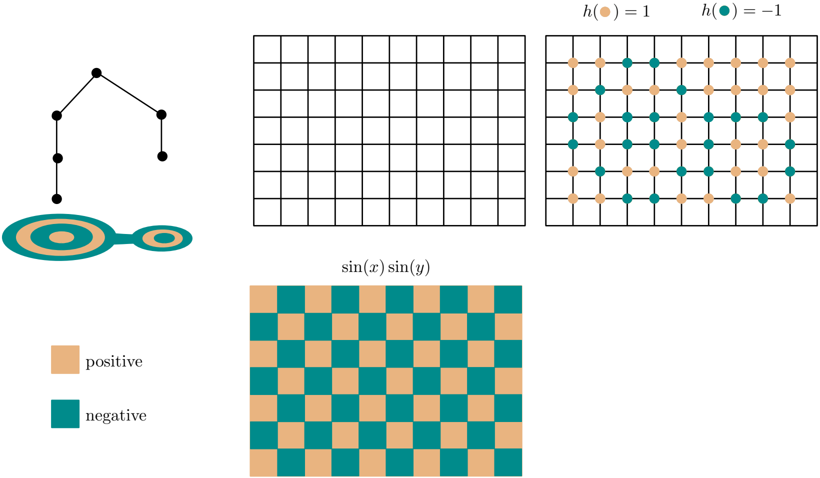
$\sin(x)\sin(y)$



positive

negative

$supp(\nu_\infty) = \mathcal{T}$



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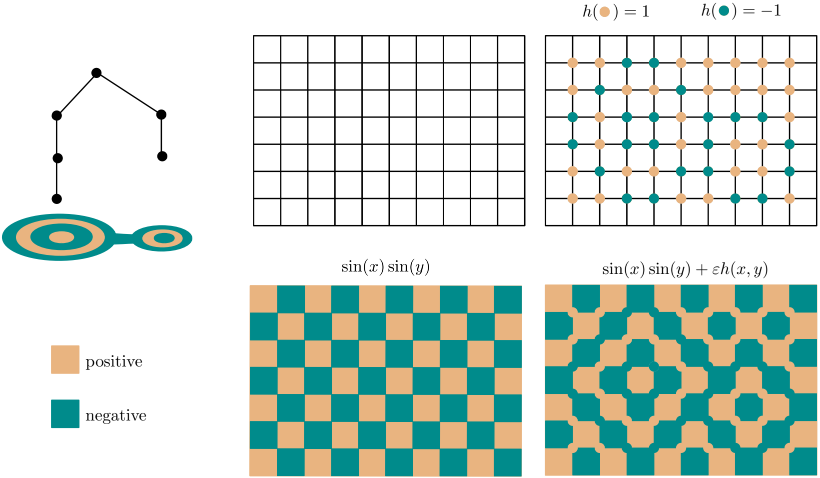
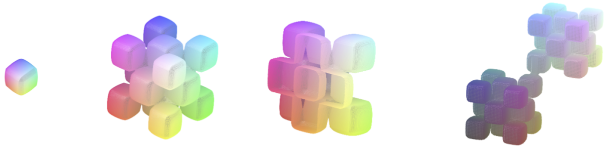
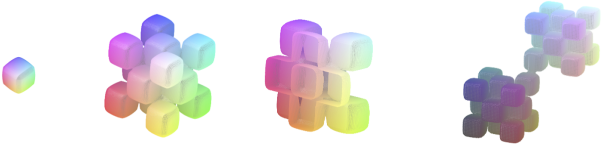
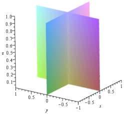
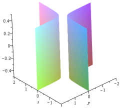




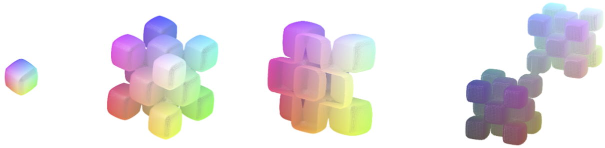


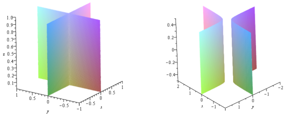


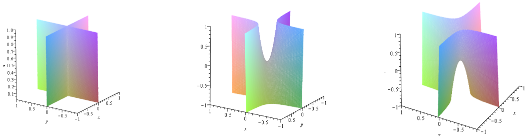
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
z
1
0.5
0
−0.5
−1
y
1
0.5
0
−0.5
−1
x

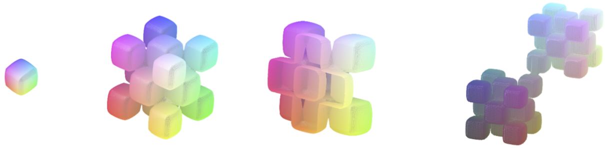


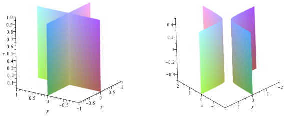
0.4
0.2
0
−0.2
−0.4
2
1
0
−1
x
−2
−1
0
1
y

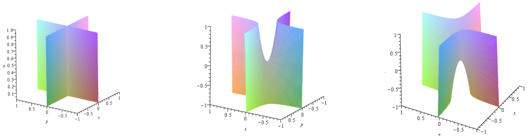


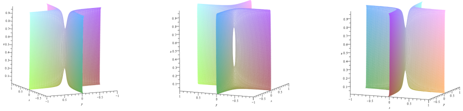


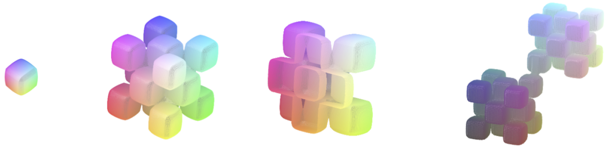


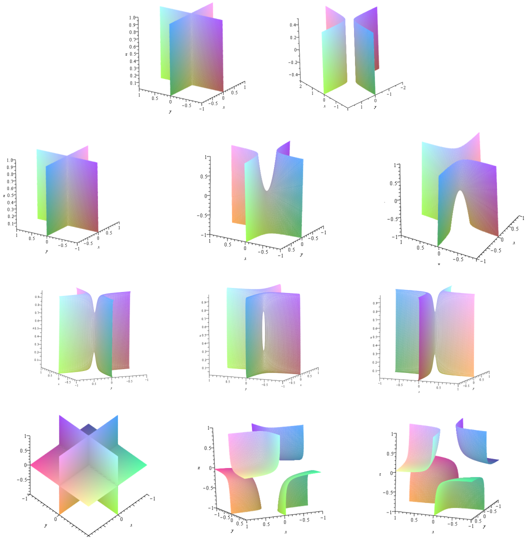












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Thank you!