Random Planar Geometry

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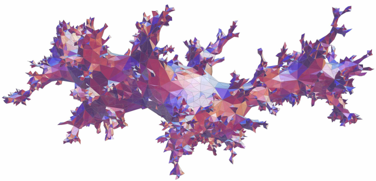
(partly joint with Nicolas Curien)

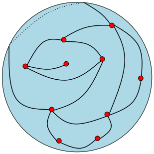
Université Paris-Sud Orsay

Workshop “Random geometries / Random topologies”



To define a canonical random geometry in two dimensions (motivations from physics: 2D quantum gravity)

- Replace the sphere $\mathbb{S}^2$ by a discretization, namely a graph drawn on the sphere (= planar map).
- Choose such a planar map uniformly at random in a suitable class and equip its vertex set with the graph distance.



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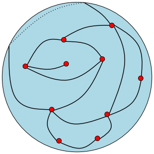
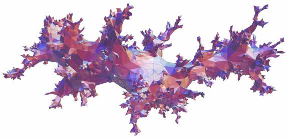


- Replace the sphere $\mathbb{S}^2$ by a discretization, namely a graph drawn on the sphere (= planar map).
- Choose such a planar map uniformly at random in a suitable class and equip its vertex set with the graph distance.
- Let the size of the graph tend to infinity and pass to the limit after rescaling to get a random metric space: the Brownian map.
- This convergence is robust: it still holds if we make local modifications of the graph distance: Universality of the Brownian map.



Goal of the lecture: Present the Brownian map and related models (the Brownian disk, the Brownian plane).

1. The geometry of large random planar maps

Definition

A planar map is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere). Loops and multiple edges allowed.

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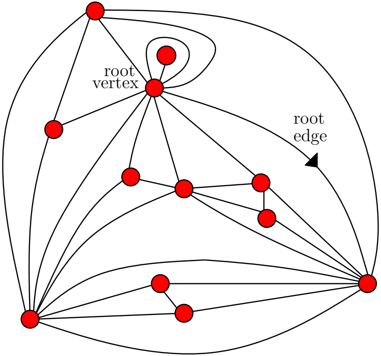


A rooted triangulation with 20 faces

Faces = connected components of the complement of edges

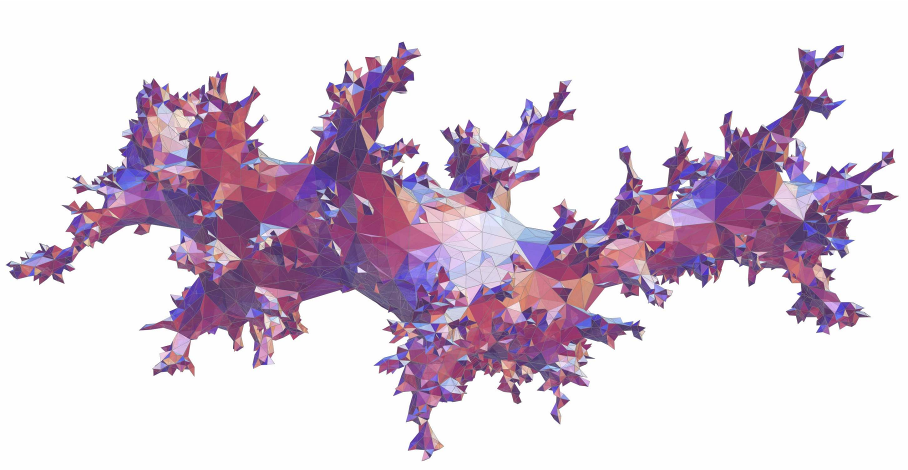
p-angulation:

- each face is incident to p edges

$p = 3$: triangulation
$p = 4$: quadrangulation

Rooted map: distinguished oriented edge

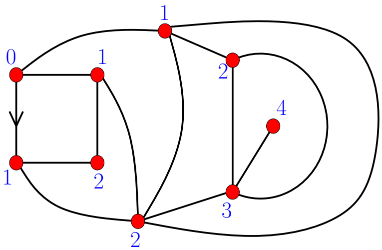
A large triangulation of the sphere
Can we get a continuous model out of this ?



Planar maps as metric spaces

M planar map

- $V(M)$ = set of vertices of *M*
- $d_{\rm gr}$ graph distance on $V(M)$
- $(V(M), d_{\rm gr})$ is a (finite) metric space



In blue : distances from root

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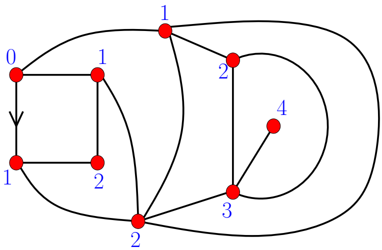


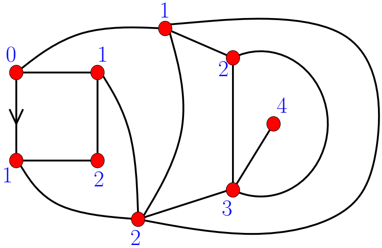
In blue : distances from root

$\mathbb{M}_n^p = \{$rooted $p-$angulations with n faces$\}$

$\mathbb{M}_n^p$ is a finite set (*finite number of possible "shapes"*)

Choose M_n uniformly at random in $\mathbb{M}_n^p$.

Planar maps as metric spaces



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View $(V(M_n), d_{\rm gr})$ as a random variable with values in

$$\mathbb{K} = \{\text{compact metric spaces, modulo isometries}\}$$

which is equipped with the Gromov-Hausdorff distance. (A sequence (E_n) of compact metric spaces converges if one can embed all E_n's isometrically in the same big space E so that they converge for the Hausdorff metric on compact subsets of E.)

The Brownian map

$\mathbb{M}_n^p = \{$rooted $p-$ angulations with n faces$\}$

M_n uniform over $\mathbb{M}_n^p$, $V(M_n)$ vertex set of M_n, d_{gr} graph distance

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Theorem (LG 2013, Miermont 2013 for p=4)

Suppose that either $p = 3$ (triangulations) or $p \geq 4$ is even. Set

$$c_3 = 6^{1/4} \quad , \quad c_p = \Big(\frac{9}{p(p-2)}\Big)^{1/4} \quad \text{if } p \text{ is even.}$$

Then,

$$(V(M_n), c_p\, n^{-1/4}\, d_{\rm gr}) \xrightarrow[n\to\infty]{\rm (d)} (\mathbf{m}_\infty, D^*)$$

in the Gromov-Hausdorff sense. The limit $(\mathbf{m}_\infty, D^*)$ is a random compact metric space that does not depend on p (universality) and is called the Brownian map (after Marckert-Mokkadem).

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Remarks

- The case $p = 3$ (triangulations) solves a question of Schramm (2006)
- Extensions to other classes of random planar maps: Abraham, Addario-Berry-Albenque, Beltran-LG, Bettinelli-Jacob-Miermont, etc.

Two properties of the Brownian map

Theorem (Hausdorff dimension)

$$\dim(\mathbf{m}_\infty, D^*) = 4 \qquad\qquad a.s.$$

(Already “known” in the physics literature.)

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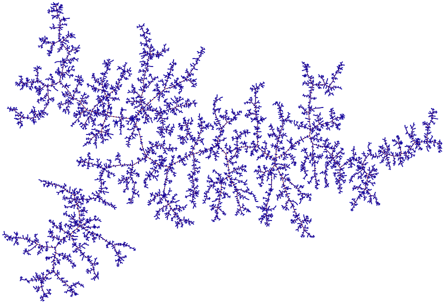
Theorem (topological type)

Almost surely, $(\mathbf{m}_\infty, D^*)$ *is homeomorphic to the* 2*-sphere* $\mathbb{S}^2$*.*

Constructions of the Brownian map

The Brownian map $(\mathbf{m}_\infty, D^*)$ is constructed as a quotient space of the Brownian tree $\mathcal{T}$ (also called the CRT), for an equivalence relation defined in terms of Brownian motion $(Z_a)_{a \in \mathcal{T}}$ indexed by $\mathcal{T}$ (here Z_a is viewed as a Brownian label assigned to the vertex a of the tree $\mathcal{T}$).



A simulation of the CRT

Two points a and b of the CRT are glued if they have the same label $Z_a = Z_b$ and if one can go from a to b around the tree (clockwise or counterclockwise) meeting only points with greater label.

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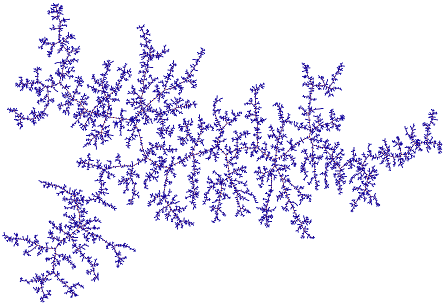

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Recent work of Miller and Sheffield providing a new construction of the Brownian map with conformal invariance properties, related to the so-called Quantum Loewner Evolution $QLE(\frac{8}{3}, 0)$.

Outline

2. Universality of the Brownian map
The Brownian map still appears in the limit if one performs "local modifications" of the distance (universality property)

3. The Brownian disk
("Brownian map with a boundary")

4. The Brownian plane
("Infinite volume Brownian map")

Both the Brownian disk and the Brownian plane are variants of the Brownian map that play an important role in the study of this random metric space.

2. Universality of the Brownian map

(joint work with Nicolas Curien, in revision for Ann. ENS)

Assign i.i.d. random weights (lengths) w_e to the edges of a (random) planar map M.
Define the weight $w(\gamma)$ of a path γ as the sum of the weights of the edges it contains.
The first passage percolation distance d_{FPP} is defined on the vertex set $V(M)$ by

$$d_{\mathrm{FPP}}(v, v') = \inf\{w(\gamma) : \gamma \text{ path from } v \text{ to } v'\}.$$

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Goal: In large scales, d_{FPP} behaves like the graph distance d_{gr} (asymptotically, balls for d_{FPP} are close to balls for d_{gr}).

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Consequence: The scaling limit of the metric space associated with $d_{\rm FPP}$ will again be the Brownian map! (Universality of the limit!)

The Uniform Infinite Planar Triangulation (UIPT)

Let Δ_n be uniformly distributed over $\{$triangulations with n faces$\}$.

For every $r \geq 1$, let $B_r(\Delta_n)$ be the ball of radius r in Δ_n, defined as the union of all faces incident to a vertex at distance strictly less than r from the root vertex ρ (distinguished vertex of Δ_n).

One can prove (Angel-Schramm 2003, Stephenson 2014) that

$$\Delta_n \xrightarrow[n\to\infty]{\text{(d)}} \Delta_\infty$$

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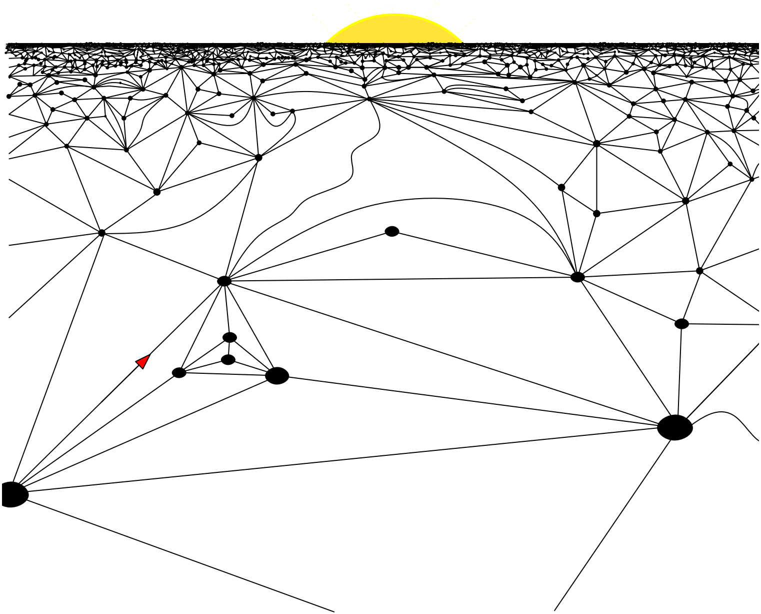
The convergence holds in the sense of local limits: for every r and for every fixed planar map M,

$$\mathbb{P}(B_r(\Delta_n) = M) \xrightarrow[n\to\infty]{} \mathbb{P}(B_r(\Delta_\infty) = M).$$

This is very different from the Gromov-Hausdorff convergence: Here we do no rescaling and thus the limit is a non-compact (infinite) random lattice.



An artistic representation of the UIPT (artist: N. Curien)

First-passage percolation in the UIPT

Assign i.i.d. weights w_e with common distribution ν to the edges of the UIPT Δ_∞ and consider the associated first-passage percolation distance $d_{\rm FPP}$. Assume ν is supported on $[c, C]$, where $0 < c \leq C < \infty$.

For every real $r \geq 0$, let $B_r^{\rm FPP}(\Delta_\infty)$ be the ball of radius r for $d_{\rm FPP}$.

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For every real $r \geq 0$, let $B_r^{\rm FPP}(\Delta_\infty)$ be the ball of radius r for $d_{\rm FPP}$.

Theorem

There exists a constant c_0 with $c \leq c_0 \leq C$, such that, for every $\varepsilon > 0$, we have

$$\lim_{r\to\infty} \mathbb{P}\Big(\sup_{x,y\in B_r(\Delta_\infty)} \big|d_{\rm FPP}(x,y) - c_0\, d_{\rm gr}(x,y)\big| > \varepsilon r\Big) = 0.$$

In particular,

$$B_{(1-\varepsilon)r/c_0}(\Delta_\infty) \subset B_r^{\rm FPP}(\Delta_\infty) \subset B_{(1+\varepsilon)r/c_0}(\Delta_\infty)$$

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The ball of radius r for the FPP distance is asymptotically close to the ball of radius r/c_0 for the graph distance.

Remark. In general one cannot compute the constant c_0, except in special cases.

First-passage percolation in finite triangulations

Δ_n is uniformly distributed over $\{$triangulations with n faces$\}$

$d_{\rm FPP}$ first-passage percolation distance on $V(\Delta_n)$ defined using weights i.i.d. according to ν (same assumption on ν).

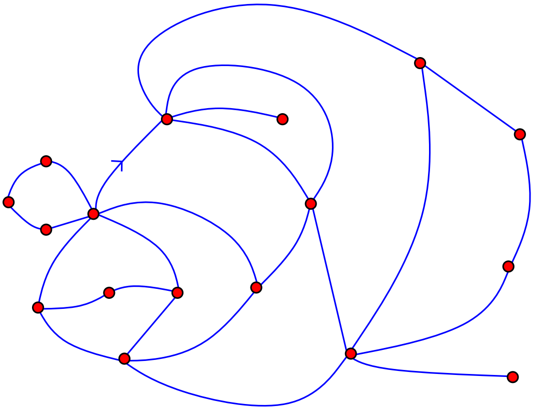
Theorem

$$(V(\Delta_n), 6^{1/4}\, n^{-1/4}\, d_{\rm FPP}) \xrightarrow[n\to\infty]{\rm (d)} (\mathbf{m}_\infty, c_0\, D^*)$$

in the Gromov-Hausdorff sense. Here c_0 is the same constant as in the UIPT case, and $(\mathbf{m}_\infty, D^*)$ is the Brownian map.

Idea of the proof: Use absolute continuity arguments to relate large (finite) triangulations to the UIPT, and then apply the theorem about the UIPT.

3. Brownian disks
Quadrangulations with a boundary



A quadrangulation with a boundary of size 14.

(All faces have degree 4 except for one face called the outer face whose degree is the boundary size.)

For every fixed $p \geq 1$, let $\mathbf{Q}_p$ be a Boltzmann quadrangulation with boundary size $2p$, meaning that

$$\mathbb{P}(\mathbf{Q}_p = Q) = c_p \, 12^{-n} \text{ if } Q \text{ has } n \text{ faces (and boundary size } 2p)$$

(the number of faces is not fixed!)

Convergence to the Brownian disk

$\mathbf{Q}_p$ Boltzmann quadrangulation with boundary size $2p$:

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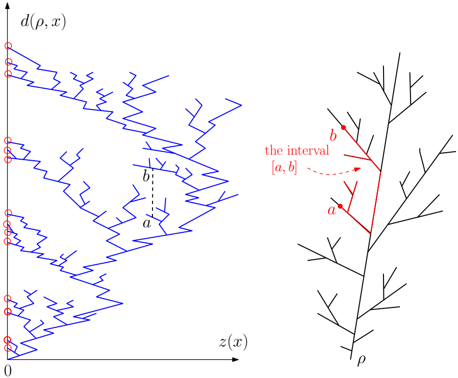
Equip the vertex set $V(\mathbf{Q}_p)$ with the graph distance $d_{\rm gr}$. Then Bettinelli and Miermont proved that

$$\big(V(\mathbf{Q}_p), p^{-1/2} d_{\rm gr}\big) \xrightarrow[p\to\infty]{\rm (d)} \big(\mathbb{D}, D^\partial\big)$$

in the Gromov-Hausdorff sense. The limit $(\mathbb{D}, D^\partial)$ is a random compact metric space homeomorphic to the disk, called the free Brownian disk with perimeter 1. (One can also define the Brownian disk with perimeter r and volume v.)

(See also Gwynne and Miller for the extension to the simple boundary case)

Constructing free Brownian disks



Labels $(z(x))_{x \in \mathcal{T}^\bullet}$. Tree $\mathcal{T}^\bullet$

We start with a pair $(\mathcal{T}^\bullet, (z(x))_{x \in \mathcal{T}^\bullet})$, which is the Brownian tree equipped with Brownian labels, conditioned on the event that labels stay nonnegative.
The cyclic structure on $\mathcal{T}^\bullet$ allows us to define "cyclic intervals" $[a, b]$ for every $a, b \in \mathcal{T}^\bullet$.

We set $\partial\mathcal{T}^\bullet = \{x : z(x) = 0\}$ and $\mathcal{T}^\circ = \mathcal{T}^\bullet \backslash \partial\mathcal{T}^\bullet$.

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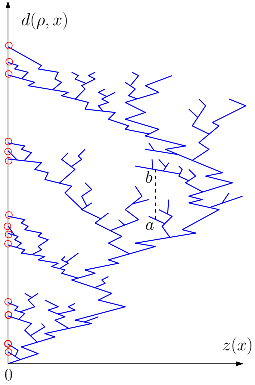

We glue $a, b \in \mathcal{T}^\circ$ if

- they have the same label $z(a) = z(b) > 0$
- $z(c) \geq z(a)$ for every c belonging to the cyclic interval $[a, b]$.

The result of this gluing procedure is a Brownian disk $(\mathbb{D}, D^\partial)$ (equipped with a volume measure $\mathrm{Vol}(\mathrm{d}x)$), with the interpretation of labels:

$z(c) = D^\partial(c, \partial\mathbb{D})$ coincides with the distance from (the equivalence class of) c to $\partial\mathbb{D}$.

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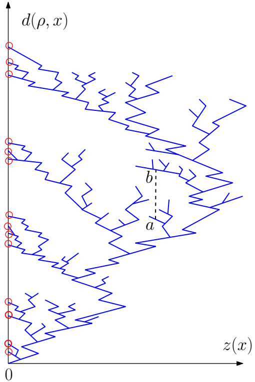

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One can use this to construct the uniform measure on the boundary.

Proposition

The formula $\langle \mu, \varphi \rangle = \lim_{\varepsilon \to 0} \varepsilon^{-2} \int_{\mathbb{D}} \mathrm{Vol}(\mathrm{d}x)\, \varphi(x)\, \mathbf{1}_{\{D^\partial(x,\partial\mathbb{D}) < \varepsilon\}}$ *defines a finite measure on the boundary.*

Brownian disks in the Brownian map

Combining

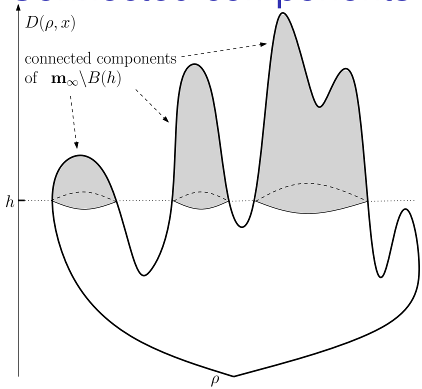
- the construction of the Brownian map from Brownian motion indexed by the Brownian tree
- excursion theory for the latter process (Abraham-LG, JEMS)
- the preceding construction of Brownian disks

we can identify various subsets of the Brownian map as Brownian disks.

Let $(\mathbf{m}_\infty, D^*)$ be the Brownian map. Then $\mathbf{m}_\infty$ has a distinguished point ρ (playing no special role: re-rooting invariance property).

For $h > 0$, let $B(h)$ be the ball of radius h centered at the distinguished point ρ. Then, the connected components of the complement of $B(h)$ are Brownian disks!

Connected components of the complement of a ball



For $h > 0$, let $B(h)$ be the ball of radius h centered at the distinguished point ρ

Let $\mathcal{D}_j$, $j \in J$ be the connected components of $\mathbf{m}_\infty \backslash B(h)$. We can equip each $\mathcal{D}_j$ with its intrinsic metric $D^{(j)}$

Write Vol for the volume measure on $\mathbf{m}_\infty$.

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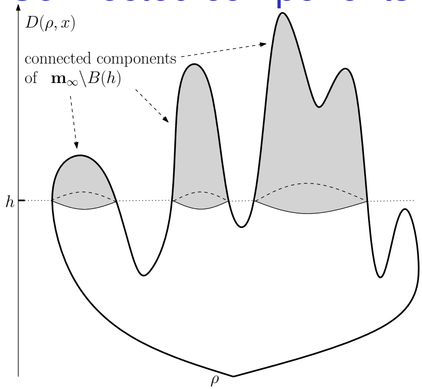

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Theorem

The metric $D^{(j)}$ has a continuous extension to $\bar{\mathcal{D}}_j = \mathcal{D}_j \cup \partial\mathcal{D}_j$, and the limit $|\partial\mathcal{D}_j| := \lim_{\varepsilon\to 0} \varepsilon^{-2}\mathrm{Vol}\{x \in \mathcal{D}_j : D^{(j)}(x, \partial\mathcal{D}_j) < \varepsilon\}$ exists, for every j. Conditionally on $(|\partial\mathcal{D}_j|, \mathrm{Vol}(\mathcal{D}_j))_{j\in J}$, the metric spaces $(\bar{\mathcal{D}}_j, D^{(j)})$ are independent Brownian disks with the prescribed volumes and perimeters.

4. The Brownian plane

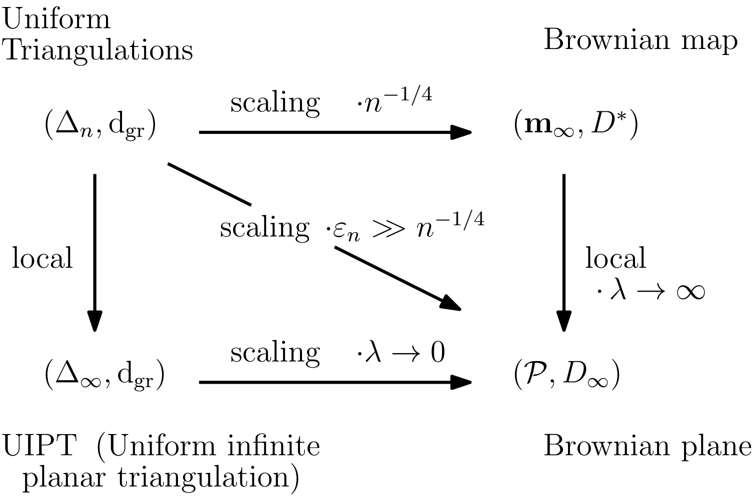
(mostly joint work with N. Curien)

The Brownian plane $(\mathcal{P}, D_\infty)$ is an infinite volume version of the Brownian map (again with a distinguished point ρ),
with scale invariance property: $(\mathcal{P}, \lambda D_\infty) \overset{\text{(d)}}{=} (\mathcal{P}, D_\infty)$

- tangent cone of the Brownian map: $(\mathbf{m}_\infty, \lambda D^*) \xrightarrow[\lambda\to\infty]{\text{(d)}} (\mathcal{P}, D_\infty)$
  (in the sense of Gromov-Hausdorff for pointed metric spaces)
- scaling limit of the Uniform Infinite Planar Triangulation (UIPT)
- scaling limit of finite triangulations, with scaling factor $\varepsilon_n \gg n^{-1/4}$

Same local properties as the Brownian map: One can couple $\mathbf{m}_\infty$ and $\mathcal{P}$ so that, for some (random) $r > 0$, the balls of radius r in $\mathbf{m}_\infty$ and in $\mathcal{P}$ centered at the distinguished point are the same (as metric spaces).

Convergence to the Brownian plane



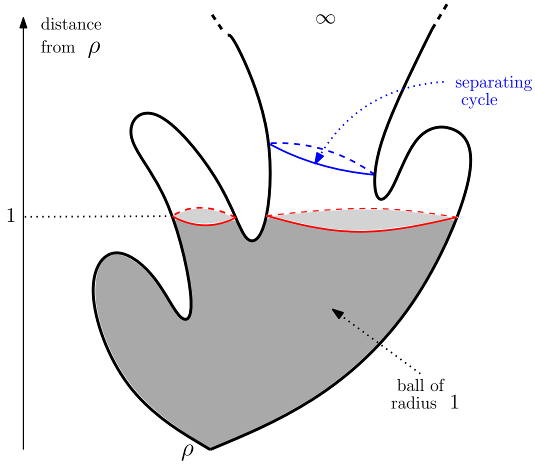
Geometric properties of the Brownian plane

- Scale invariance : $\lambda\mathcal{P} \overset{(\mathrm{d})}{=} \mathcal{P}$
- $\dim \mathcal{P} = 4$, $\mathcal{P}$ homeomorphic to the plane
- Confluence of geodesic rays to infinity ($g : [0,\infty) \longrightarrow \mathcal{P}$ is a geodesic ray if $D_\infty(g(s), g(t)) = |s-t|$ for all s, t)
  Any two geodesic rays merge in finite time
- The construction is based on an infinite Brownian tree $\mathcal{T}_\infty$ equipped with Brownian labels Z^∞. These labels Z^∞ are interpreted as "distances from infinity":

$$Z^\infty_x - Z^\infty_y = \lim_{z\to\infty} (D^\infty(x,z) - D^\infty(y,z))$$

  (similar to a result of Curien-Ménard-Miermont for UIPQ)
- Estimates for lengths of separating cycles and isoperimetric inequalities (work in progress of A. Riera)

Separating cycles



Consider the Brownian plane $\mathcal{P}$.

Let L be the minimal length of a cycle separating the ball of radius 1 centered at the distinguished vertex ρ from infinity.

Proposition (Riera)

$$c_1 \varepsilon^2 \leq P(L \leq \varepsilon) \leq c_2 \varepsilon^2$$

Isoperimetric inequalities in the Brownian plane

Let $\mathcal{O}$ be the class of all simply connected (bounded) open subsets of $\mathcal{P}$ containing the distinguished point ρ.

For $O \in \mathcal{O}$ let $|O|$ be the volume of O and let $|\partial O|$ be the length of the boundary of O.

Proposition (Riera, Lehéricy-LG for the UIPQ)

For $\varepsilon > 0$,

$$\inf_{O\in\mathcal{O}} \frac{|\partial O|}{|O|^{1/4}\,(1+|\log|O||)^{-\frac{1}{2}-\varepsilon}} > 0,$$

and

$$\inf_{O\in\mathcal{O}} \frac{|\partial O|}{|O|^{1/4}\,(1+|\log|O||)^{-\frac{1}{2}+\varepsilon}} = 0.$$

Similar results for the Brownian map (recall the Brownian map and the Brownian plane have the same local properties).

(Work in progress!)