Topologies of the zero sets of random real projective hypersurfaces and monochromatic random waves

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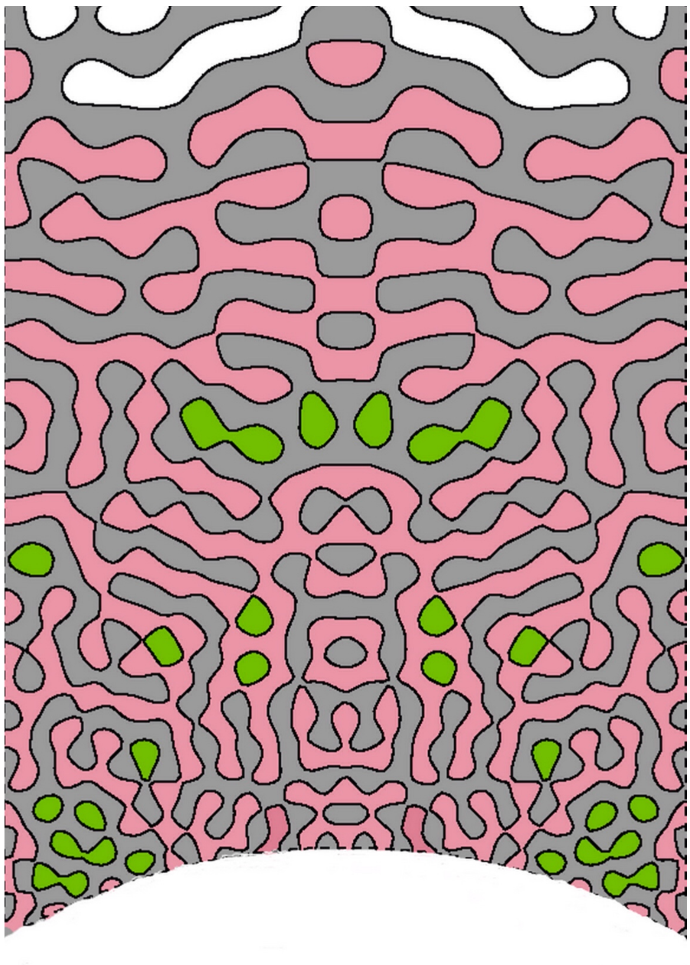
Joint work with I. Wigman and Y. Canzani

Nodal portrait



Setting

- **Monochromatic random waves** model the eigenfunctions of a quantization of a classically chaotic hamiltonian (M. Berry).

- **Random Fubini-Study ensembles** are a model for random real algebraic geometry.

  Single variable:

$$f(x) = \sum_{j=0}^{t} a_j x^j \qquad a_j \in \mathbb{R}$$

$$Z(f) = \{x : \; f(x) = 0\}$$

  Topology of $Z(f)$ is $|Z(f)|$.

  $W_{1,t}$ = vector space of such polynomials f.

What is random? (single variable)

We stick to centered Gaussian ensembles on a (finite) dimensional vector space W. This is **equivalent** to giving an inner product $\langle \ , \ \rangle$ on W.

'Naive' ensemble:

$$\langle f, g \rangle = \sum_{j=0}^{t} a_j b_j \qquad \text{on } W_{1,t}.$$

- equivalent to choosing the $a_j's$ as i.i.d. standard Gaussians.
- **not natural** since it singles out ± 1 as to where most the zeros locate themselves.

What is random? (single variable)

Real Fubini-Study ensemble: $f(x,y) = \sum_{j=0}^{t} a_j x^j y^{t-j}$,

with

$$\langle f, g\rangle = \int_{\mathbb{R}^2} f(x)g(x)e^{-\frac{|x|^2}{2}}dx = * \int_{\mathbb{P}^1(\mathbb{R})} f(\theta)g(\theta)d\theta.$$

- In this ensemble $\{x^j y^{t-j} : j = 0, \ldots, t\}$ are not orthogonal, rather $\sin(\theta k)$ and $\cos(\theta k)$ are.

Complex Fubini-Study ensemble on $W_{1,t}$:

$$\langle f, g\rangle = \int_{\mathbb{P}^1(\mathbb{C})} \tilde{f}(z)\overline{\tilde{g}(z)}d\sigma(z).$$

- $\tilde{f}, \tilde{g}$ are complex extensions of f, g.
- In this ensemble $\{x^j y^{t-j} : j = 0, \ldots, t\}$ are orthogonal.

Kac-Rice formulas (single variable)

Kac-Rice formulas give asymptotically the number of zeros of $f \in W_{1,t}$

- Naive ensemble: $\frac{2}{\pi}\log(t)$
- Real Fubini-Study : $t/\sqrt{3}$
- Complex Fubini-Study : $\sqrt{t}$
- Monochromatic (harmonic): t

$$\mathrm{Cov}_{f_t}(x,y) = \mathbb{E}(f_t(x), f_t(y)) =: K_t(x,y).$$

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}|\{x : |f(x)| < \varepsilon\}| = \sum_{a\in Z(f)}\frac{1}{|f'(a)|}.$$

$$\mathbb{E}(|Z(f)|) = \mathbb{E}\left(\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\{|f|<\varepsilon\}}|f'(y)|dy\right).$$

- This can be computed in terms of $K_t(x,y)$.
- Reduces problem to the asymptotics of $K_t(x,y)$ as $t\to\infty$.

What is random? (several variables)

$W_{n,t}$: space of $f(x_0, x_1, \ldots, x_n)$ homogeneous of degree t.

- same definitions of the naive, real F-S, complex F-S, monochromatic.
- real F-S ($\alpha = 0$):

$$\langle f, g \rangle = \int_{P^n(\mathbb{R})} f(x)g(x)d\sigma(x).$$

- monochromatic random waves ($\alpha = 1$): same $\langle \, , \, \rangle$ but restricted to the subspace $H_{n,t}$ of $W_{n,t}$ consisting of harmonic polynomials.

Denote these two ensembles by $\mathcal{E}_{n,\alpha}$ with $\alpha = 0, 1$.

Zero set: $Z(f) = \{x \in \mathbb{P}^n(\mathbb{R}) : \ f(x) = 0\}$

- For a random f, $Z(f)$ is smooth.
- Let $C(f)$ be the connected components of $Z(f)$. These are compact, $(n-1)$-dimensional manifolds.
- Let $\tilde{H}(n-1)$ be the countable collection of compact, $(n-1)$-dimensional manifolds **mod diffeos**.

$$Z(f) = \bigcup_{c \in C(f)} c, \qquad c \in \tilde{H}(n-1).$$

$$\mathbb{P}^n(\mathbb{R}) \setminus Z(f) = \bigcup_{\omega \in \Omega(f)} \omega.$$

the ω's are the nodal domains of f.

What can we say about the topologies of a random f as $t \to \infty$?

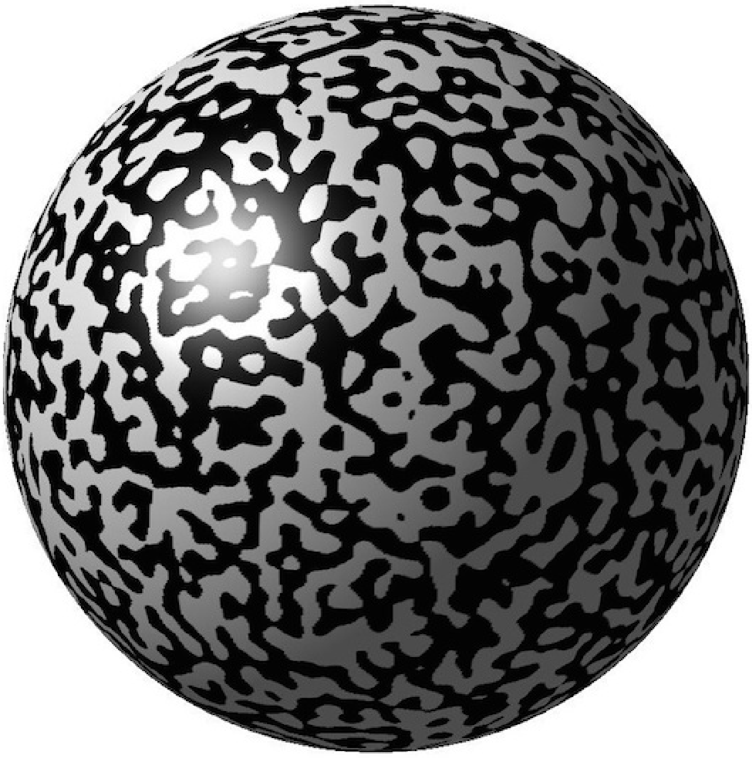
Nesting of nodal domains

- Nesting tree $X(f)$ (Hilbert for ovals).
- The vertices of $X(f)$ are the nodal domains $\omega \in \Omega(f)$. Two vertices ω and ω' are joined if they have a common boundary $c \in \mathcal{C}(f)$.
- $X(f)$ is a tree (Jordan-Brouwer).

$$|\Omega(f)| = |\mathcal{C}(f)| - 1.$$

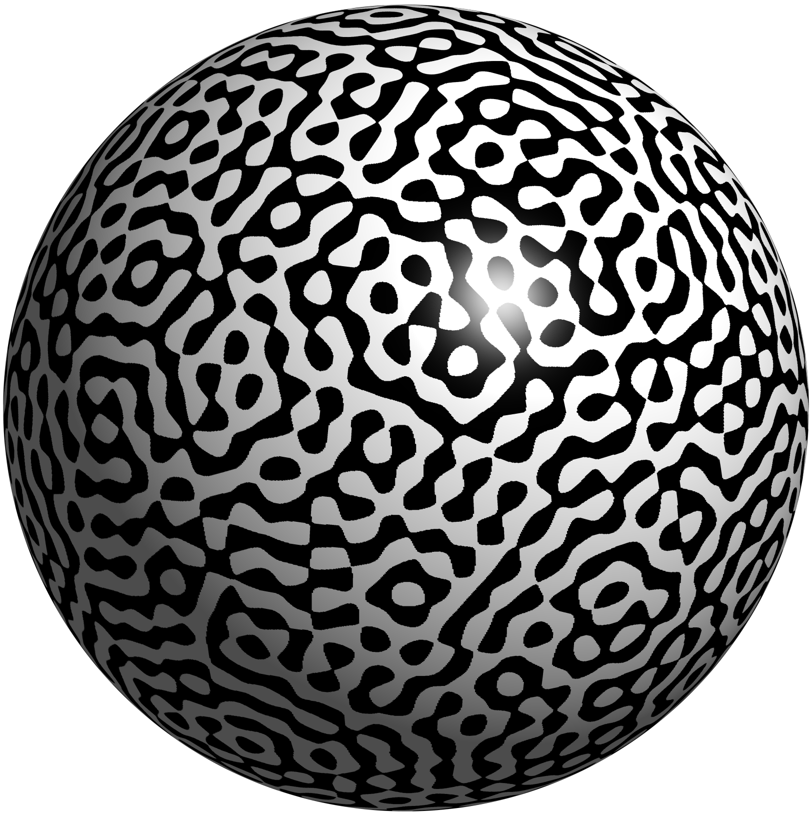
- $X(f)$ carries all the combinatorial information about the connectivities $m(\omega)$ for $\omega \in \Omega(f)$.

Nodal portrait: **Fubini-Study ensemble ($\alpha = 0$)**



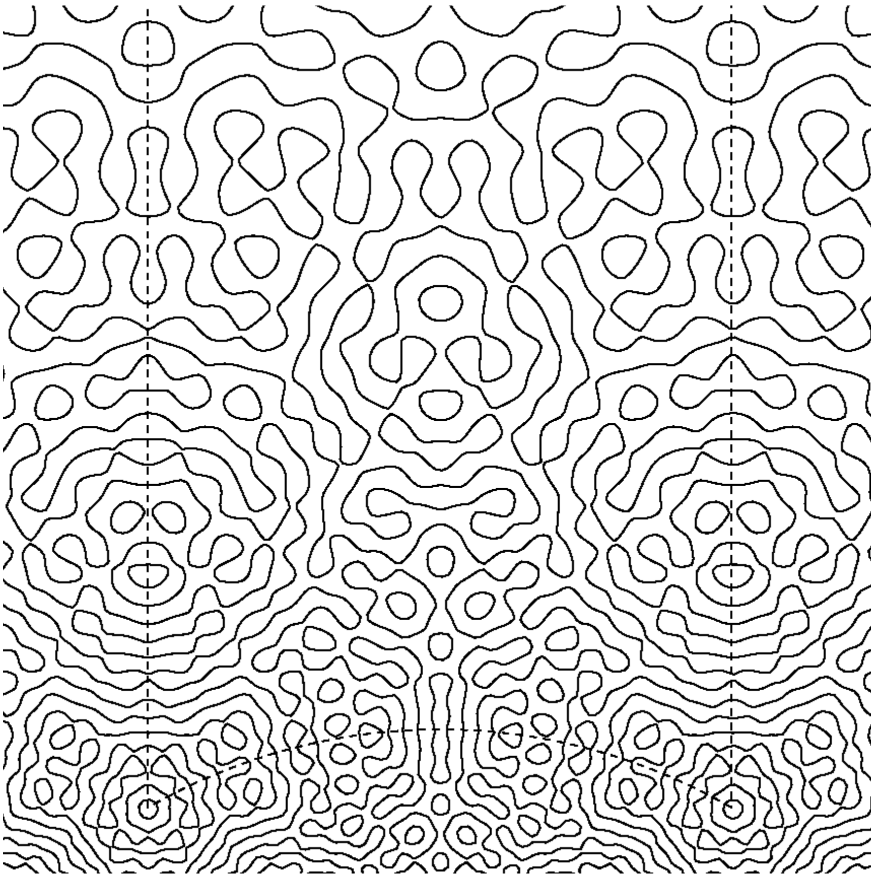
Sum of random spherical harmonics of degree ≤ 80 (A. Barnett).

Nodal portrait: Random spherical harmonic ($\alpha = 1$)

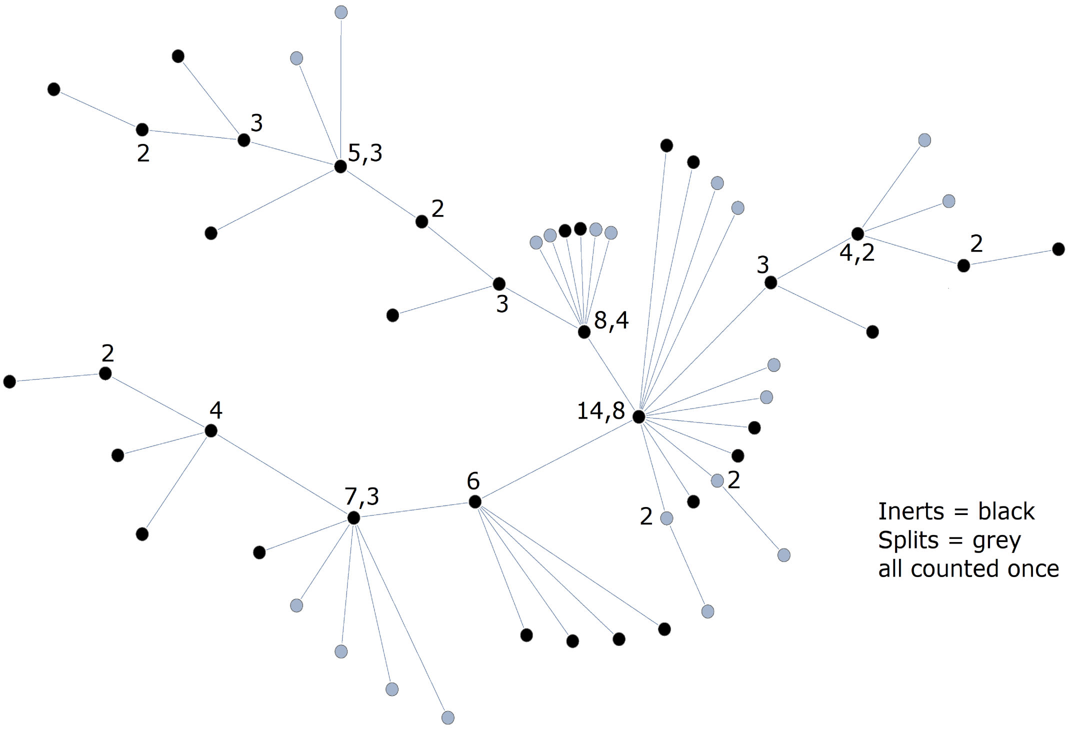


random spherical harmonic of degree $= 80$. (A. Barnett)

Zero set



Nesting tree



Inerts = black
Splits = grey
all counted once

Local and global quantities

For a Gaussian ensemble the Kac-Rice formula allows for the explicit computation of the expected values of **local** quantities.

- $|Z(f)|$ the induced $(n-1)$ dimensional volume of $Z(f)$.
- The Euler number $\chi(Z(f))$.
- The number of critical points of f.

The question of **global** topology of $Z(f)$ is much more difficult.

Nazarov and Sodin [NS] have introduced some powerful "soft" techniques to study the problem of the number of connected components of $Z(f)$ for random f.

Their methods show that most of the components $c \in C(f)$ are small occuring at a scale of $1/t$ and thus semi-localising this count.

Nazarov-Sodin

Theorem (Nazarov-Sodin 2013,2016)

There are positive constants $\beta_{n,\alpha}$ *such that*

$$|C(f)| \sim \beta_{n,\alpha} t^n \qquad as \;\; t \to \infty$$

for the random f *in* $\mathcal{E}_{n,\alpha}(t)$, *for* $\alpha = 0, 1$.

- Their 'soft' proof offers no effective lower bounds for these N-S constants $\beta_{n,\alpha}$.
- Their barrier method (2008) can be made effective but the resulting bounds are extremely small.
  - $\beta_{2,0} \geq 10^{-320}$ Nastasescu,
  - $\beta_{2,0} \geq 10^{-70}$ deCourcy-Ireland,
  - $\beta_{n,0} \geq e^{-e^{257n^3/2}}$ Gayet-Welschinger
- For a random f the set $Z(f)$ has many components and we can ask about their topologies.

Topologies and Nestings

For $f \in \mathcal{E}_{n,\alpha}(t)$ set

$$(A) \quad \mu_{\mathcal{C}(f)} := \frac{1}{|\mathcal{C}(f)|} \sum_{c \in \mathcal{C}(f)} \delta_{t(c)}$$

where $t(c)$ is the topological type of c in $\tilde{H}(n-1)$ and $\delta_{t(c)}$ is the point mass at $t(c)$.

$\mu_{\mathcal{C}(f)}$ is a probability measure on $\tilde{H}(n-1)$.

$$(B) \quad \mu_{\mathcal{X}(f)} := \frac{1}{|\mathcal{C}(f)|} \sum_{c \in \mathcal{C}(f)} \delta_{e(c)}$$

where $e(c)$ is the smallest of the two rooted trees that one gets from $X(f)$ after removing the edge $c \in \mathcal{C}(f)$.

$\mu_{\mathcal{X}(f)}$ is a probability measure on $\mathcal{T}$ (the space of finite rooted trees).

Topologies and Nestings: main result

Theorem[Wigman-S 2015, Canzani-S 2017]

(i) There are **probability** measures $\mu_{C,n,\alpha}$ on $\tilde{H}(n-1)$ and $\mu_{X,n,\alpha}$ on $\mathcal{T}$ such that for random $f \in \mathcal{E}_{n,\alpha}(t)$

$$\mu_{C(f)} \to \mu_{C,n,\alpha}, \qquad\qquad \mu_{X(f)} \to \mu_{X,n,\alpha}$$

as $t \to \infty$, and the convergence is tight.

(ii) $\quad \text{supp}(\mu_{C,n,\alpha}) = H(n-1) \quad$ and $\quad \text{supp}(\mu_{X,n,\alpha}) = \mathcal{T}$.

Obs. $H(n-1)$ is the subset of diffeomorphism types in $\tilde{H}(n-1)$ that can be embedded into $\mathbb{R}^n$.

Obs. These give universal laws for the distributions of the topologies of the components of random real hypersurfaces ($\alpha = 0$) and monochromatic waves ($\alpha = 1$), as well as for nesting ends.

Betti numbers and connectivities

The theorem implies universal laws for the distribution of the Betti numbers of the components as well as for the connectivities of the domains.

For $f \in \mathcal{E}_{n,\alpha}(t)$ set

$$(A) \quad \nu_{Betti(f)} := \frac{1}{|\mathcal{C}(f)|} \sum_{c \in \mathcal{C}(f)} \delta_{B(c)}$$

where $B(c) = (b_1(c), \ldots, b_{n-2}(c))$ is the collection of Betti numbers.

$$(B) \quad \nu_{con(f)} := \frac{1}{|\Omega(f)|} \sum_{\omega \in \Omega(f)} \delta_{m(\omega)}$$

where $m(\omega)$ is the number of boundary components of ω.
The universal limits are

$$\nu_{Betti,n,\alpha} \text{ on } (\mathbb{Z}_{\geq 0})^{n-2}, \qquad \nu_{con,n,\alpha} \text{ on } \mathbb{N}.$$

Remarks

- The existence of the universal measures follows the 'soft' methods of N-S. However, the tightness of the convergence (with the consequence that all universal measures are probability measures) and the determination of their supports (especially when $\alpha = 1$) is a challenge.

- Gayet and Welschinger (2013) used the barrier method, in the context of the Kostlan distribution and its generalizations, to show that every topological type $c \in H(n-1)$ occurs with positive probability.

- Lelario-Lunderberg (2013) used the barrier method to give lower bounds for the number of connected components for random Fubini-Study ($\alpha = 0$).

How do the universal measures look like?

Barnett/Jin (2013, 2017) carried out Monte-Carlo simulations $n = 2, 3$.

- When $n = 2$ we have $H(1)$ is a point.
- The connectivity measures $\nu_{con(f)}$ satisfy

$$\mathbb{E}\left(\nu_{con(f)}\right) = \sum_{m=1}^{\infty} m \cdot \nu_{con(f)}(m) = \sum_{\omega \in \Omega(f)} \frac{m(\omega)}{|\Omega(f)|} = 2 + o(1).$$

m	1	2	3	4	5	6	7	8
$\nu_{con,2,0}$	0.973	0.027	0.009	0.003	0.002	0.002	0.001	0.001

m	1	2	3	4	5	6	7	8
$\nu_{con,2,1}$	0.906	0.055	0.010	0.006	0.003	0.002	0.001	0.0009

Observations

- It appears that
$$\mathbb{E}(\nu_{con,\alpha,2}) < 2$$
corresponding to the persistence of many domains of large connectivity.

- The N-S constants $\beta_{2,\alpha}$ are of order 10^{-2} and for $\alpha = 2$ the random plane curve is 4% Harnack (that is, it has 4% of the maximum number of ovals that such a curve can have). M. Natasescu(2012).

- When $n = 3$ we have $H(2)$ is the set of compact orientable surfaces; determined by their genus $g \in \mathbb{Z}_{\geq 0}$. So $\mu_{\mathcal{C},3,\alpha}$ is a probability measure on $Z_{\geq 0}$.

$\mu_{\mathcal{C}(f)}$

A Kac-Rice computation (Podkoytov 2001) gives

$$\mathbb{E}\left(|\chi(Z(f)|\right) \sim \begin{cases} \frac{t^3}{3^{3/2}}, & \alpha = 0 \\ \frac{t^3}{5^{3/2}}, & \alpha = 1. \end{cases}$$

Thus,

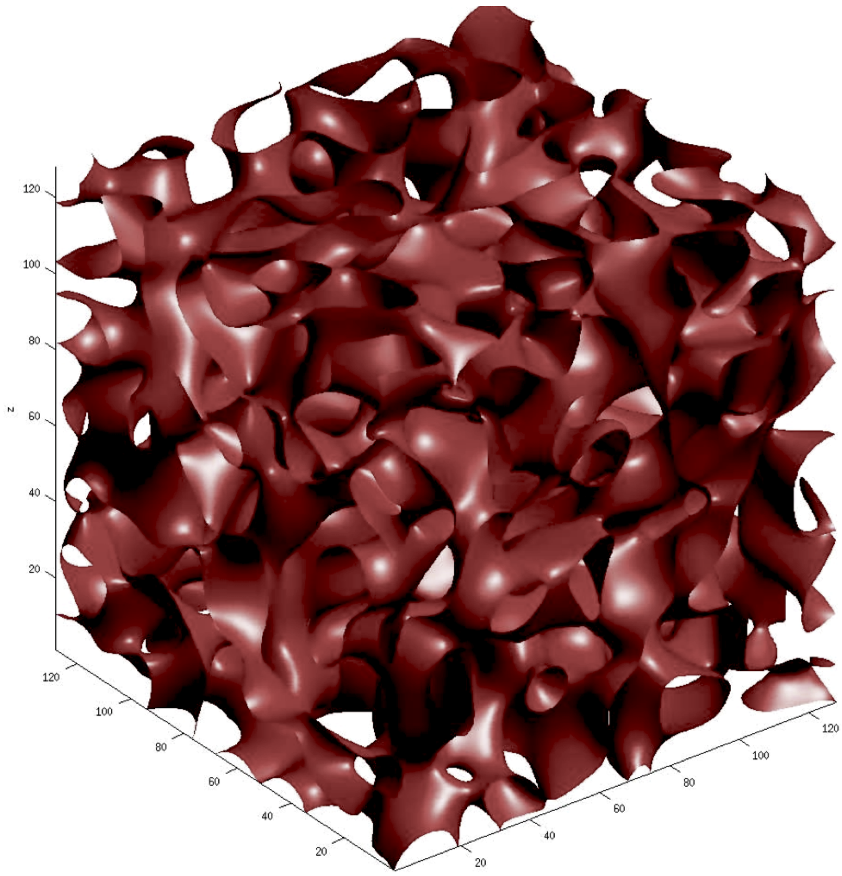
$$\mathbb{E}\left(\mu_{\mathcal{C}(f)}\right) = \sum_{g=0}^{\infty} g \cdot \mu_{C(f)}(g) \sim \begin{cases} 2 + \frac{1}{3^{3/2}\beta_{3,0}} = A_0, & \alpha = 0 \\ 2 + \frac{1}{5^{3/2}\beta_{3,1}} = A_1, & \alpha = 1. \end{cases}$$

In particular,

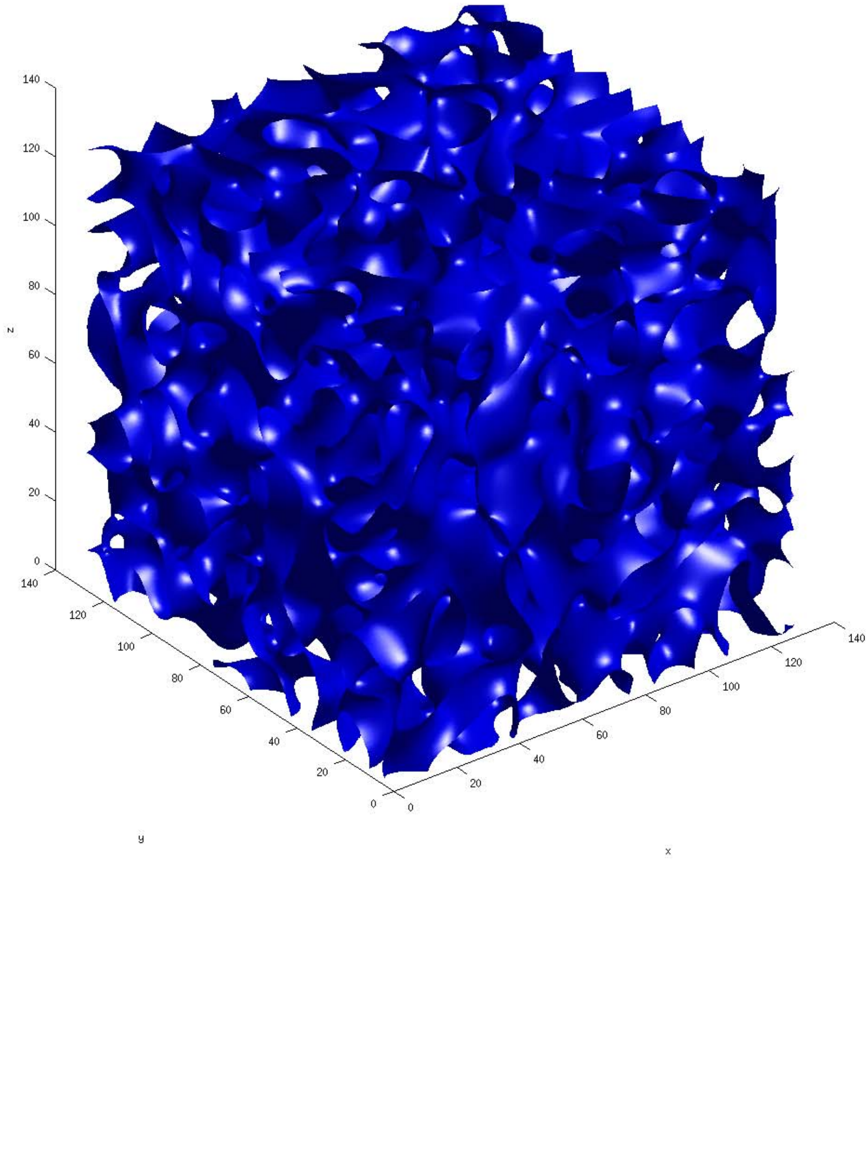
$$\mathbb{E}\left(\mu_{C,3,\alpha}\right) \leq A_\alpha.$$

What Barnett-Jin find for $\mu_{\mathcal{C}(f)}$ is dramatic.

Zero set



Zero set



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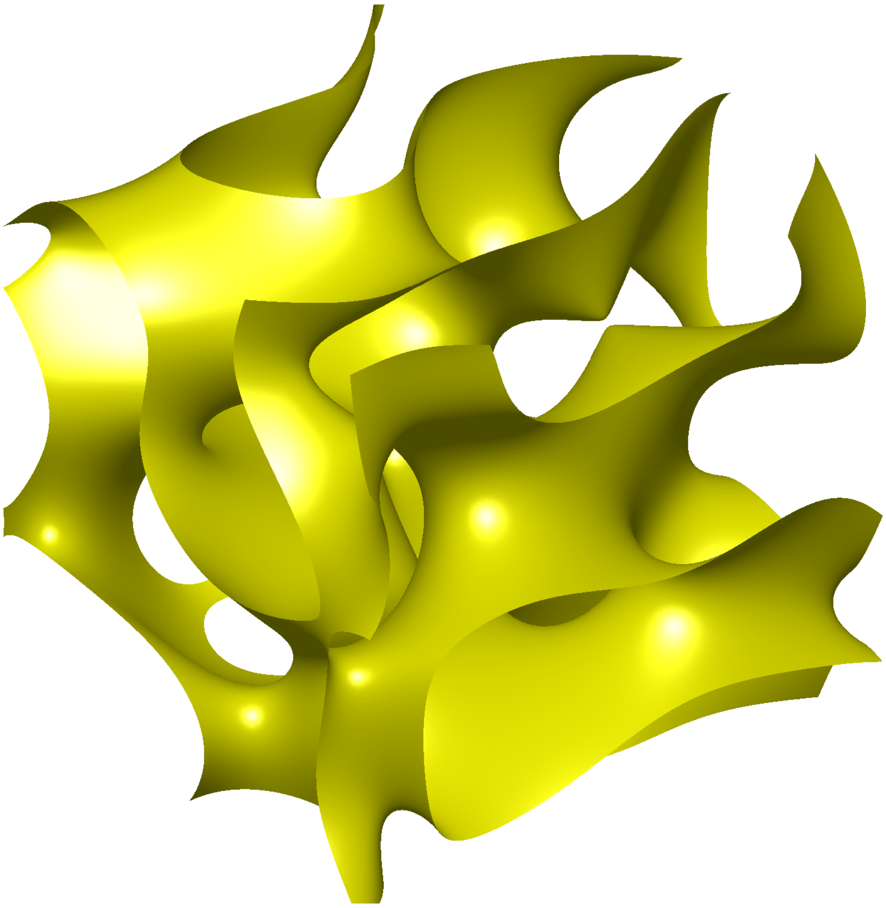
Zero set



Observations

- Apparently we are in a super critical regime with a unique giant percolating component $\pi(f) \in \mathcal{C}(f)$.
- The N-S constants $\beta_{3,0}, \beta_{3,1}$ are very small ($\approx 10^{-7}$) and the feasibility of observing $\mu_{\mathcal{C},3,\alpha}, \mu_{\mathcal{X},3,\alpha}$ is problematic.
- A_0, A_1 are very large so there is a dramatic loss of mean in going from the finite measures to their limits.
- In the main equidistribution theorems each topological component is counted with equal weight. So there is no contradiction as $\pi(f)$ is treated as equal to others.
- Clearly, to complete the basic understanding of $Z(f)$, the topology of $\pi(f)$ needs to be examined.

Speculations/Questions

- As an element of the discrete $H(n-1)$, $\pi(f) \to \infty$ as $t \to \infty$ for random f.

- Betti($\pi(f)$):

$$\lim_{t\to\infty} \frac{B(\pi(f))}{t^n} = \begin{cases} 0 \in (\mathbb{Z}_{\geq 0})^{n-2} & n-1 \text{ odd} \\ (0,\dots,0,\delta_{\frac{n-1}{2}},0,\dots,0) & n-1 \text{ even} \end{cases}$$

with $\delta_{\frac{n-1}{2},\alpha} > 0$.

That is, for $n-1$ even the homology of the percolating component is $\delta\%$ of the homology of that of a complex hypersurface $f = 0$.

To explain the source of the super critical percolation we need to go into some of the analysis.

Brief comments about proofs

Covariance:

$$K_{n,\alpha}(t; x, y) = \mathbb{E}_{f \in \mathcal{E}_{n,\alpha}(t)}(f(x)f(y)).$$

As $t \to \infty$ one shows using well known asymptotics of special functions and micro-local analysis in the more general setting of 'band limited functions' on a manifold, Canzani-Hanin (2015)

$$\frac{K_{n,\alpha}(t; x, y)}{\dim \mathcal{E}_{n,\alpha}(t)} = \begin{cases} B_{n,\alpha}(t\, d(x, y)) + O(1/t), & td(x, y) \leq 1, \\ O(1/t), & td(x, y) \geq 1, \end{cases}$$

where

$$B_{n,\alpha}(\omega) = B_{n,\alpha}(|\omega|) = \frac{1}{|\Omega_\alpha|} \int_{\Omega_\alpha} e^{i\langle \omega, \xi \rangle} d\xi$$

with $\Omega_\alpha = \{\omega : \ \alpha \leq |\omega| \leq 1\}$.

Brief comments about proofs

- Following N-S we show that our quantities can be studied semi locally, i.e. in neighborhoods of size $1/t$.
- After scaling one arrives at a Gaussian translation invariant isotropic field on $\mathbb{R}^n$ (with slow decay of spatial correlations).
- The existence of the limiting measures, as well as the convergence in measure, follows from soft ergodic theory of the action of $\mathbb{R}^n$.
- The properties of the universal μ's, that of being probability measures (i.e. no escape of topology for them) and that they charge every admissible atom positively, is much harder earned.

Brief comments about proofs

- To control the escape of topology, that is the tightness of the convergence, we show that **most** components of the scaled Gaussian are geometrically controlled (specifically their curvatures) and eventually apply a form of Cheeger finiteness.

- To show that the support is full in the case $\alpha = 1$ requires one to prescribe topological configurations locally for "1-harmonic'" entire functions
$$\Delta\psi + \psi = 0 \qquad \text{on } \mathbb{R}^n.$$
For this we prove versions of Runge type approximation/interpolation theorems for such ψ's.

- The nesting prescription is the most challenging and is achieved in $n = 3$ by deformation
$$f = f_0 + \varepsilon f_1$$
$f_0 = \sin(x)\sin(y)\sin(z)$ and f_1 a suitable 1-harmonic function.

Percolating component

To end we explain the source of the dominant percolating $\pi(f)$. For $\alpha = 1$ and $n = 3$ the scaling limit mean zero Gaussian field on $\mathbb{R}^3$ has

$$\text{Cov}(x, y) = K(x, y) = * \frac{\sin(|x - y|)}{|x - y|} \qquad x, y \in \mathbb{R}^3$$

for this field or any similar Gaussian field define the critical level h_* by:

- For $h > h_*$ the set $\{x : f(x) \geq h\}$ has no infinite component with probability 1.

- For $h < h_*$ the set $\{x : f(x) \geq h\}$ has an infinite component with probability 1.

h_* is a function of the field.

Conjecture

Conjecture: If $n \geq 3$, then $h_* > 0$.

- In particular, the zero levels $h = 0$ are supercritical. Note that for $n = 2$ it is known that $h_* = 0$ (Alexander '96).
- Evidence towards this conjecture is provided by the recent proof (Rodriguez, Drewitz, Prevost) of the 1987 conjecture of Brimont-Lebowitz-Maes, that for the discrete analogue on $\mathbb{Z}^3$ of the Gaussian free field ($K(x, y) = \frac{1}{|x-y|}$) one has $h_* > 0$.

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