# Nodal sets of random spherical harmonics

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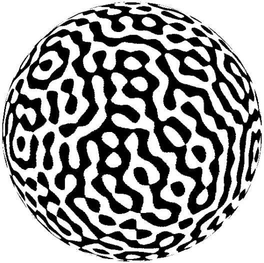

Random geometries / Random topologies

ETH, December 2017

# Random spherical harmonics:

$\mathcal{H}_n$ real Hilbert space of 2D spherical harmonics equipped with the $L^2(\mathbb{S}^2)$-norm, $\dim \mathcal{H}_n = 2n+1$

$(Y_k)$ orthonormal basis in $\mathcal{H}_n$
$(\xi_k)$ Gaussian IIDs, $\mathbb{E}|\xi_k|^2 = \frac{1}{2n+1}$

$f_n = \sum_{k=-n}^{n} \xi_k Y_k$ random spherical harmonic of degree $n$

The distribution of $f_n$

- is independent of the choice of the ONB in $\mathcal{H}_n$
- is invariant w.r.t. rotations of the sphere $\mathbb{S}^2$

$Z(f_n) = f^{-1}\{0\}$ the zero set of $f_n$

$N(f_n)$ the number of connected components of $Z(f_n)$

# Major difficulties:

- "non-locality" of topological observables (contrary to the length);
- slow off-diagonal decay of the covariance $\mathbb{E}[f_n(x)f_n(y)]$

$\mathbb{E}[f_n(x)f_n(y)] = P_n(\cos\Theta(x,y))$,
$P_n$ Legendre polynomial of degree $n$, $P_n(1) = 1$,
$\Theta(x,y)$ angle between $x, y \in \mathbb{S}^2$.

Scaled covariance: $P_n\left(cos\frac{z}{n}\right) \sim J_0(z)$, the 0-th Bessel function

It is more natural to think of $f_n$ as defined on the sphere $n\mathbb{S}^2$ of radius $n$ and of area $\simeq n^2$. In this scale the covariance decays as $\mathrm{dist}^{-1/2}$.

# Bogomolny and Schmit percolation model

In 2001, Bogomolny and Schmit proposed a remarkable percolation-like model for description of the topology of the zero set $Z(f_n)$. Their model completely ignores slow decaying correlations and is very far from being rigorous.

On the other hand, attempts to digest their work stimulated much of the progress recently achieved in this area.

# LLN + Exponential concentration:

THEOREM 1 (F.Nazarov, M.S., Amer. J. Math., 2009)

$$\mathbb{P}\Big[\big|N(f_n) - \nu n^2\big| > \varepsilon n^2\Big] < Ce^{-c(\varepsilon)n}$$

with some $\nu > 0$ and $c(\varepsilon) \gtrsim \varepsilon^{15}$.

The proof gives

$$\nu = \lim_{n\to\infty} \mathbb{E}\Big[\frac{1}{\mathrm{area}(G_n)}\Big],$$

where $G_n$ is a nodal domain of $f_n$ on $n\,\mathbb{S}^2$ that contains a marked point $x$.

Afterwards, we have shown that the Law of Large Numbers with a positive limit (but without the exponential concentration) holds for rather general classes of smooth Gaussian fields on $\mathbb{R}^d$ and of smooth Gaussian ensembles on manifolds (J. Math. Phys., Analysis, Geometry, 2016).

# Far-reaching extensions

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- "derandomization" on the torus: Bourgain, Buckley-Wigman, Ingremeau;
- other topological observables: Gayet-Welschinger, Lerario-Lundberg (upper and lower bounds for mean values), Sarnak-Wigman, Canzani-Sarnak (the Law of Large Numbers, nesting configurations, especially, for so called monochromatic waves);
- fields and ensembles with positive correlations: Malevich (1972, sic!), Beffara-Gayet, Beliaev-Muirhead-Wigman, Rivera-Vanneuville.

# Do large nodal domains exist?

$G_n$ nodal domain of $f_n$ on $n\mathbb{S}^2$ that contains a marked point

The only thing we know about the distribution of area$(G_n)$ is that, for some positive constants $C, c$,

$$\mathbb{P}\big[\text{area}(G_n) < C\big] \geqslant c$$

which yields positivity of the limiting constant $\nu = \lim_{n\to\infty} \mathbb{E}\big[\frac{1}{\text{area}(G_n)}\big]$ in Theorem 1.

QUESTON 1 Is it true that $\lim_{C\to\infty} \limsup_{n\to\infty} \mathbb{P}\big[\text{area}(G_n) \geqslant C\big] = 0$?

We do not know the answer to a much weaker question:

QUESTON 1a Show that for any $\delta > 0$, $\lim_{n\to\infty} \mathbb{P}\big[\text{area}(G_n) \geqslant \delta n^2\big] = 0$.

We also do not know anything about domains of a large diameter that contain a given point.

# Level sets:

Though the sets $\{f_n > \varepsilon\}$ and $\{f_n < \varepsilon\}$ have roughly the same areas, topologically, the former one should look as a collection of many small islands in ocean formed by the latter one.

Given $\varepsilon, \delta$ consider that event $\mathcal{X}_n(\varepsilon, \delta)$ that the level set $\{f_n > \varepsilon\}$ has a connected component of diameter at least $\delta n$.

QUESTION 2 Show that for any $\varepsilon, \delta > 0$, $\lim\limits_{n\to\infty} \mathbb{P}\big[\mathcal{X}_n(\varepsilon, \delta)\big] = 0$.

# A version of the Michael Berry prediction:

Consider high-energy Laplace eigenfunctions on the sphere endowed with a generic smooth Riemannian metric close to the constant one.

QUESTION 3 Do they (or at least some portion of them) behave similarly to random spherical harmonics?

Instead of perturbing the round metric on the sphere $\mathbb{S}^2$, one can add a small random potential to the Laplacian on the round sphere. The question remains just as hard.

# Size of fluctuations of $N(f_n)$:

$N(f_n)$ the number of connected components of the zero set $Z(f_n)$.

QUESTION 4 Estimate the variance of $N(f_n)$.

- Trivial bounds: $1 \lesssim \mathrm{Var}[N(f_n)] \lesssim n^4$.
- The Bogomolny and Schmit prediction says that $\mathrm{Var}[N(f_n)]$ grows as $n^2$, that is, as $\mathbb{E}[N(f_n)]$.
- The exponential concentration

$$\mathbb{P}\Big[\big|N(f_n) - \nu n^2\big| > \varepsilon n^2\Big] < Ce^{-c\varepsilon^{15} n}$$

yields the upper bound: $\mathrm{Var}[N(f_n)] \lesssim n^{4-\frac{2}{15}}$.

# Recent "little advance":

THEOREM 2 (work in progress with Fedya Nazarov)

$$\mathrm{Var}[N(f_n)] \gtrsim n^{\sigma}$$

with some $\sigma > 0$

REMARK This lower bound holds for *any* non-degenerated isotropic smooth Gaussian fields on $n\,\mathbb{S}^2$ with decay of correlations $\gtrsim \mathrm{dist}^{-c}$ with some $c > 0$.

The proof upper bound $\mathrm{Var}[N(f_n)] \lesssim n^{4-\sigma}$ requires some additional restrictions on the ensemble (which are, likely, unnecessary).

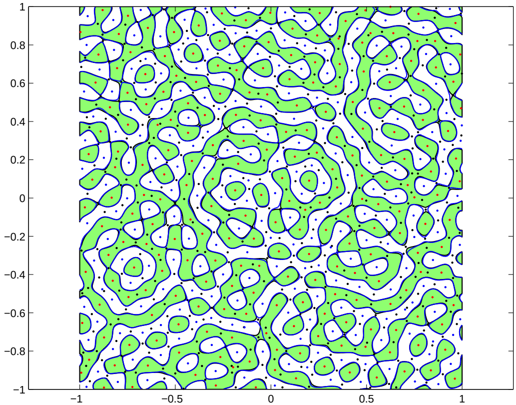
In what follows we will discuss main ideas from the proof of the lower bound.

# Saddle points with small critical values:

Heuristically, most of the fluctuations are caused by saddle points of $f_n$ with small critical values that yield so called "avoided crossings" of the zero set $Z(f_n)$.

I.e., switches in the topology of the zero set of $f_n$ are caused by a point process that has a low intensity but strong long range dependence, as illustrated on the following simulation produced by Dima Beliaev.

Instead of random spherical harmonics Beliaev simulated so called random plane waves (RPWs) but one may safely ignore the difference (the RPW is a scaling limit of the our random spherical harmonics on $n\mathbb{S}^2$ as $n \to \infty$).




Blue lines are zero lines of a RPW $F_0$, blue and red points are maxima and minima of $F_0$, and black points are saddle points of $F_0$. Black lines are zero lines of the sum $F_0 + \frac{1}{10}F_1$, where $F_1$ is another RPW, equidistributed with $F_0$ and independent of $F_0$, green domains are connected components of the set where this sum is positive.

# Step 1: Low level critical points

$f = f_n$ random spherical harmonic of degree $n$ on $n\mathbb{S}^2$, $\mathbb{E}|f|^2 = 1$

$\mathrm{Cr}(\alpha) = \left\{z \in n\mathbb{S}^2 \colon \nabla f(z) = 0, |f(z)| \leqslant \alpha\right\}$, $\alpha = n^{-2+\varepsilon}$

"With high probability" (w.h.p.) means except of an event of probability $O(n^{-c})$ with some $c > 0$.

Non-degeneracy: w.h.p., $\max_{\mathrm{Cr}(\alpha)} |(\nabla^2 f)^{-1}| \lesssim n^{3\varepsilon}$. That is, the Hessian $\nabla^2 f$ does not degenerate on $\mathrm{Cr}(\alpha)$.

LEMMA 1 W.h.p., the set $\mathrm{Cr}(\alpha)$ is relatively large: $|\,\mathrm{Cr}(\alpha)| \gtrsim n^{c\varepsilon}$, and the points in this set are $n^{1-C\varepsilon}$-separated.

# Step 2: Introducing a small perturbation

$f_\alpha = \sqrt{1-\alpha^2} f + \alpha g$, $g$ is an independent copy of $f$

We condition on $f$ and estimate from below the conditional variance $\mathrm{Var}\big[N(f_\alpha)\big| f\big]$

LEMMA 2 W.h.p., topology of $Z(f_\alpha)$ is determined by the collection of signs of $f_\alpha(z)$ at $z \in \mathrm{Cr}(\alpha')$. Here $\alpha' = \alpha n^\varepsilon = n^{-2+2\varepsilon}$.

This lemma allows us "to localize" the problem. Its proof needs a caricature of a quantitative Morse theory.

# Step 3: Reduction to independent percolation

Recall: $\alpha = n^{-2+\varepsilon}$, $\alpha' = n^{-2+2\varepsilon}$, $f_\alpha = \sqrt{1-\alpha^2} f + \alpha g$, $g$ is an independent copy of $f$

We replace $g$ by its independent copy $g_z$ (some linear algebra with estimates, cf. IMRN, 2011).

This step needs a good separation between the points of $\mathrm{Cr}(\alpha')$ (Lemma 1).

Define a collection of independent random functions $\widetilde{f_\alpha} = \sqrt{1-\alpha^2} f + \alpha g_z$, $z \in \mathrm{Cr}(\alpha')$.

LEMMA 3 W.h.p., $\mathrm{sgn}(f_\alpha(z)) = \mathrm{sgn}(\widetilde{f_\alpha}(z))$, $z \in \mathrm{Cr}(\alpha')$.

This reduces the problem to the independent percolation process on a graph with the degree of each vertex either 4 (saddle points of $f$) or 0 (maxima and minima of $f$).

Discard the latter case and assume that the degree of each vertex is 4.

# Step 4: Independent percolation on some graphs

$G = G(V, E)$ a graph embedded in $n\mathbb{S}^2$

The degree of each vertex $v \in V$ is 4.

In each vertex $v$, we independently replace the edges crossing by one of two possible avoided crossing configuration, $p(v), 1 - p(v)$ are the corresponding probabilities.

$\Gamma$ random percolation configuration of loops,

$N(\Gamma)$ the number of loops in $\Gamma$.

LEMMA 4 : For any $p_0 > 0$,

$$\mathrm{Var}[N(\Gamma)] \geqslant c(p_0)|V(p_0)|,$$

where $V(p_0) = \{v \in V \colon p_0 \leqslant p(v) \leqslant 1 - p_0\}$.

The End