Generalized McKean-Vlasov control problems and an application to optimal liquidation

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Outline

Motivation

N-player stochastic differential game

→ N players with **private state processes** evolving as

$$dX_t^i = b(t, X_t^i, \alpha_t^i, \bar{\nu}_t^i)dt + \sigma(t, X_t^i, \alpha_t^i, \bar{\nu}_t^i)dW_t^i, \quad i = 1, ..., N$$

- $W^1, ..., W^N$ independent Wiener processes
- $\alpha^1, ..., \alpha^N$ controls of the N players
- $\bar{\nu}_t^i = \frac{1}{N-1} \sum_{j \neq i} \delta_{X_t^j}$ empirical distrib. states of the other players

→ **Objective of player i**: choose a control $\alpha^i \in \mathbb{A}$ that minimizes

$$\mathbb{E}\left[\int_0^T f(t, X_t^i, \alpha_t^i, \bar{\nu}_t^i)dt + g(X_T^i, \bar{\nu}_T^i)\right]$$

→ Statistically identical players: same functions b, σ, f, g

N-player stochastic differential game

→ N players with **private state processes** evolving as

$$dX_t^i = b(t, X_t^i, \alpha_t^i, \bar{\xi}_t^i)dt + \sigma(t, X_t^i, \alpha_t^i, \bar{\xi}_t^i)dW_t^i, \quad i = 1, ..., N$$

- $W^1, ..., W^N$ independent Wiener processes
- $\alpha^1, ..., \alpha^N$ controls of the N players
- $\bar{\xi}_t^i = \frac{1}{N-1} \sum_{j \neq i} \delta_{(X_t^j, \alpha_t^j)}$ empirical distrib. states & controls

→ **Objective of player i**: choose a control $\alpha^i \in \mathbb{A}$ that minimizes

$$\mathbb{E}\left[\int_0^T f(t, X_t^i, \alpha_t^i, \bar{\xi}_t^i)dt + g(X_T^i, \bar{\nu}_T^i)\right]$$

→ Statistically identical players: same functions b, σ, f, g

N-player stochastic differential game

Problems:

- search for equilibria: very difficult
- even when they exist, difficult to characterize

Idea:

$\hookrightarrow$ for large symmetric games, some averaging/independence are expected when the number of players tends to infinity

$\hookrightarrow$ use theory of propagation of chaos (Sznitman 1991): approximation by asymptotic arguments

$\hookrightarrow$ formulation of the problem for a representative agent

Some literature: Carmona-Delarue (2013), Fischer (2015), Lacker (2015), Lacker (2016)

Asymptotic argument

N-player game – – – – – – – –– > $N \to \infty$

I. Nash equilibrium (competitive) – – –– > Mean Field Game

II. Pareto equilibrium (cooperative - social planner) – – –– > McKean Vlasov

→ **I. Nash equilibrium:** each agent chooses the control as best answer to other agents' actions

→ **II. Pareto equilibrium:** agents choose their controls so as to minimize their average cost

Problem Formulation

Generalized McKean-Vlasov control problem

→ Asymptotic formulation of **cooperative** equilibria:
Generalized McKean-Vlasov control problem:

$$\inf_{\alpha \in \mathbb{A}} \mathbb{E}\left[\int_0^T f(t, X_t, \alpha_t, \mathcal{L}(X_t, \alpha_t))\, dt + g(X_T, \mathcal{L}(X_T))\right]$$

$$\text{s.t. } dX_t = b(t, X_t, \alpha_t, \mathcal{L}(X_t, \alpha_t))\, dt + \sigma(t, X_t, \alpha_t, \mathcal{L}(X_t, \alpha_t))\, dW_t.$$

- In the asymptotic formulation of **competitive** equilibria (MFG), we would fix any flow of measures $(\xi_t)_{0 \le t \le T}$, and solve the corresponding control problem, then check fixed point.
- Under suitable conditions, the optimal controls are ϵ-optimal for large systems of players (cf. Lacker 2016).
- For simplicity from now on we remove the dependence on t

Generalized McKean-Vlasov control problem

Classical approaches for MFG and MKV:

$\hookrightarrow$ analytic (by PDEs) $\rightarrow$ HJB equation

$\hookrightarrow$ probabilistic (by BSDEs) $\rightarrow$ stochastic maximum principle

Extensive literature on MFG and MKV:

- Lasry and Lions (2006, 2007)
- Huang, Caines, and Malhamé (2006, 2007)
- Cardaliaguet, Carmona, Delarue, Fischer, Fouque, Lachapelle, Lacker, Lehalle, Pham, Basei, Wei ...

Our contribution: allow dependence on the law of the control and

$\hookrightarrow$ probabilistic approach: develop appropriate Pontryagin maximization principle (N&S) in this general framework;

$\hookrightarrow$ optimal transport approach: use dynamic OT to study a weak formulation of the MKV control problem

Probabilistic approach

Probabilistic approach

Core concepts:

- Hamiltonian: $H : \mathbb{R}^d \times \mathbb{R}^k \times \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^k) \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}$

$$H(x, a, \xi, y, z) = b(x, a, \xi) \cdot y + \sigma(x, a, \xi) \cdot z + f(x, a, \xi)$$

- L-differentiability: differentiability w.r.t. laws via lifting functions ($u : \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^k) \to \mathbb{R}$ L-differentiable at ξ if the lifting function $\tilde{u} : L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; \mathbb{R}^d \times \mathbb{R}^k) \ni (\tilde{X}, \tilde{\alpha}) \mapsto \tilde{u}(\tilde{X}, \tilde{\alpha}) = u(\mathcal{L}(\tilde{X}, \tilde{\alpha}))$ is Fréchet differentiable at some $(\tilde{X}, \tilde{\alpha})$ with $\mathcal{L}(\tilde{X}, \tilde{\alpha}) = \xi$)

- for any admissible $\alpha \in \mathbb{A}$, with $X = X^\alpha$ the corresponding controlled state process, the adjoint processes (Y, Z) satisfy:

$$\begin{cases} dY_t = -\Big[\partial_x H\big(\theta_t, Y_t, Z_t\big) + \tilde{\mathbb{E}}\big[\partial_\nu H\big(\tilde{\theta}_t, \tilde{Y}_t, \tilde{Z}_t\big)(X_t, \alpha_t)\big]\Big] dt + Z_t dW_t, \\ Y_T = \partial_x g\big(X_T, \mathcal{L}(X_T)\big) + \tilde{\mathbb{E}}\big[\partial_\nu g\big(\tilde{X}_T, \mathcal{L}(X_T)\big)(X_T)\big], \end{cases}$$

where $\theta_t = (X_t, \alpha_t, \mathcal{L}(X_t, \alpha_t))$, $\partial_\nu, \partial_\eta$ deriv. w.r.t. $\mathcal{L}(X_t), \mathcal{L}(\alpha_t)$

Probabilistic approach

We need the "usual bunch" of **regularity assumptions**: roughly, b, σ, f, g have continuous and bounded derivatives w.r.t. x, a, ξ.

More precisely:

I. b, σ, f, g differentiable w.r.t. (x, α), for ξ fixed, with $\partial_x, \partial_\alpha$ continuous; and L-differentiable w.r.t. ξ, with ∂_ν continuous.

II. $\partial_x(b, \sigma)$ and $\partial_\alpha(b, \sigma)$ uniformly bounded and $\partial_\nu(b, \sigma)$ has an L^2-norm uniformly bounded in (x, α, ξ). There exists a constant L such that, for any $R \geq 0$ and any (x, α, ξ) s.t. $|x|, |\alpha|, M_2(\xi) \leq R$, it holds that $|\partial_x f(x, \alpha, \xi)| \vee |\partial_x g(x, \nu)| \vee |\partial_\alpha f(x, \alpha, \xi)| \leq L(1 + R)$, and the L^2-norms of $\partial_\nu f$ and $\partial_\nu g$ are bounded by $L(1 + R)$.

Pontryagin: necessary condition

A: convex set where admissible controls take values

Theorem

If α is optimal, with associated X,Y,Z, then $\forall a \in A$, $\mathrm{Leb}_1 \otimes \mathbb{P}$ a.e.,

$$\left(\partial_a H(\theta_t, Y_t, Z_t) + \tilde{\mathbb{E}}\left[\partial_\eta H(\tilde{\theta}_t, \tilde{Y}_t, \tilde{Z}_t)(X_t, \alpha_t)\right]\right) \cdot (\alpha_t - a) \leq 0 \qquad (*)$$

Assume

$$H(x, \alpha', \xi', Y, Z) \geq H(x, \alpha, \xi, Y, Z) + \partial_a H(x, \alpha, \xi, Y, Z) \cdot (\alpha' - \alpha) + \tilde{\mathbb{E}}\left[\partial_\eta H(x, a, \xi, Y, Z)(\tilde{X}, \tilde{\alpha}) \cdot (\tilde{\alpha}' - \tilde{\alpha})\right].$$

Theorem

Then, if α is optimal, for Leb_1-a.e. t, α_t is a minimizer of

$$\inf \left\{ \mathbb{E}\left[H(X_t, \beta, \mathcal{L}(X_t, \beta), Y_t, Z_t)\right] \,:\, \beta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}) \right\} \qquad (**)$$

Pontryagin: necessary condition

Remark:

- In the classical MKV control problem (without dependence on $\mathcal{L}(\alpha)$), the necessary Pontryagin condition reads as

$$H(X_t, \alpha_t, \mathcal{L}(X_t), Y_t, Z_t) \le H(X_t, a, \mathcal{L}(X_t), Y_t, Z_t), \quad \forall a \in A$$

  $\mathrm{Leb}_1 \otimes \mathbb{P}$ a.e..

- In the generalized case, with dependence on $\mathcal{L}(\alpha)$, condition $(**)$ cannot be replaced by a pointwise condition.

Pontryagin: sufficient condition

Assume

$$g(x',\nu') \geq g(x,\nu) + \partial_x g(x,\nu)\cdot(x'-x) + \tilde{\mathbb{E}}\big[\partial_\nu g(x,\nu)(\tilde{X})\cdot(\tilde{X}'-\tilde{X})\big]$$

and

$$\begin{aligned} H(x',\alpha',\xi',Y,Z) \geq H(x,\alpha,\xi,Y,Z) &+ \partial_x H(x,\alpha,\xi,Y,Z)\cdot(x'-x) \\ &+ \partial_a H(x,\alpha,\xi,Y,Z)\cdot(\alpha'-\alpha) \\ &+ \tilde{\mathbb{E}}\big[\partial_\nu H(x,\alpha,\xi,Y,Z)(\tilde{X},\tilde{\alpha})\cdot(\tilde{X}'-\tilde{X})\big], \\ &+ \tilde{\mathbb{E}}\big[\partial_\eta H(x,\alpha,\xi,Y,Z)(\tilde{X},\tilde{\alpha})\cdot(\tilde{\alpha}'-\tilde{\alpha})\big]. \end{aligned}$$

Theorem

Let α be any admissible control, with associated X, Y, Z.
If $(*)$ holds, then α is optimal.

Example: Linear-Quadratic case

Linear drift: $b(x,\alpha,\xi) = b_1 x + b_2\alpha + \bar{b}_1\bar{x} + \bar{b}_2\bar{\alpha}$,

where

$$\bar{x} = \int\int x\xi(dx,d\alpha) \quad \text{and} \quad \bar{\alpha} = \int\int \alpha\xi(dx,d\alpha).$$

Quadratic cost: $g(x,\nu) = \frac{1}{2}\gamma x^2 + \frac{\delta}{2}(x-\rho\bar{x})^2$ and

$$f(x,\alpha,\xi) = \frac{1}{2}\left[qx^2 + \bar{q}(x-s\bar{x})^2 + r\alpha^2 + \bar{r}(\alpha-\bar{s}\bar{\alpha})^2\right].$$

The optimal control is

$$\alpha_t = A_t + B_t X_t + C_t\mathbb{E}[X_t],$$

where A_t, B_t, C_t are solutions of scalar Riccati equations.

Application: Optimal liquidation problem

- Traders have to buy or sell a large amount of shares between time 0 and time T (usually $T = 1$ or $T = 5$, 1 day or 1 week)
- Trades of all market participants reflect on
  - $\longrightarrow$ temporary market impact, influencing the traders' own prices ("cost of liquidity")
  - $\longrightarrow$ permanent market impact, influencing the public price
- Optimal execution: tradeoff between trading fast to reduce the risk of future uncertainty in prices, and trading slowly to reduce market impact (or execution/liquidity cost).

Some literature: Almgren-Chriss (2000), Cartea-Jaimungal (2015), Cardaliaguet-Lehalle (2017), Basei-Pham (2017)

Application: Optimal liquidation problem

Model:

Inventory: $dQ_t^i = \alpha_t^i \, dt, \qquad Q_0^i = q_0^i$: initial inventory of agent i

trading speed

Asset: $dS_t = \lambda \cdot \frac{1}{N} \sum_{i=1}^{N} \alpha_t^i \, dt + \sigma dW_t, \quad S_0 = s_0$

permanent mk impact

Wealth: $dU_t^i = -\alpha_t^i (S_t + k \cdot \alpha_t^i) \, dt, \quad U_0^i = 0$

temporary mk impact

Cost to be minimized:

$$\mathbb{E}\left[-\left(U_T^i + Q_T^i (S_T - A Q_T^i) \right) + \phi \int_0^T (Q_t^i)^2 dt \right]$$

Application: Optimal liquidation problem

Asymptotic formulation:

Asset: $dS_t = \lambda\, \mathbb{E}[\alpha_t]\, dt + \sigma dW_t, \quad S_0 = s_0$

Inventory: $dQ_t = \alpha_t\, dt, \quad Q_0$: random, initial inventory

Wealth: $dU_t = -\alpha_t(S_t + k\alpha_t)\, dt, \quad U_0 = 0$

Problem: $$\inf_{\alpha \in \mathbb{A}} \mathbb{E}\left[-(U_T + Q_T(S_T - AQ_T)) + \phi \int_0^T (Q_t)^2 dt \right]$$

→ Even though this problem does not fall into the above framework, we can still apply our arguments.

→ We can solve the problem and have an explicit formulation of the optimal trading speed and the optimal inventory.

Application: Optimal liquidation problem

Optimal trading speed:

$$\alpha_t = \underset{\substack{\uparrow \\ \text{“follow the crowd”}}}{\mathbb{E}[\alpha_t]} + \underbrace{\varphi_t\,(Q_t - \mathbb{E}[Q_t])}_{},$$

“follow the crowd”

$(\varphi_t \leq 0)$ cf. my inventory with average inventory - “go against the crowd”

→ Cardaliaguet-Lehalle (2017) study the same problem from a competitive point of view, obtaining a similar expression.

→ If agent’s position has opposite sign w.r.t. average population, she trades faster; in the framework in [CL17] she trades slower.

→ $\mathbb{E}[Q_t]$ de/increase slower in our case, i.e. $|\mathbb{E}[\alpha_t]|$ is smaller: buy/sell market orders arrive “at the same time” (smaller permanent market impact in the cooperative framework).

Optimal Transport approach

Weak generalized MKV stochastic control problem

Weak generalized McKean-Vlasov stochastic control problem:

$$\inf_{\mathbb{P},\alpha} \mathbb{E}^{\mathbb{P}}\left[\int_0^T f(t, X_t, \alpha_t, \mathcal{L}_{\mathbb{P}}(X_t, \alpha_t))dt + g(X_T, \mathcal{L}_{\mathbb{P}}(X_T))\right]$$

$$\text{subject to} \quad dX_t = b\,(t, X_t, \alpha_t, \mathcal{L}_{\mathbb{P}}(X_t))\,dt + dW_t^{\mathbb{P}},$$

→ Infimum over filtered probability spaces $(\Omega, \mathbb{F}, \mathbb{P})$ supporting a Wiener process $W^{\mathbb{P}}$, and over α progress. measurable on $(\Omega, \mathbb{F}, \mathbb{P})$.

→ Two simplifications here: no dependence on $\mathcal{L}(\alpha)$ in the drift, and $\sigma \equiv 1$ (or deterministic).

Weak generalized MKV stochastic control problem

Weak generalized McKean-Vlasov stochastic control problem:

$$\inf_{\mathbb{P},\alpha} \mathbb{E}^{\mathbb{P}}\left[\int_0^T f(t, X_t, \alpha_t, \mathcal{L}_{\mathbb{P}}(X_t, \alpha_t))dt + g(X_T, \mathcal{L}_{\mathbb{P}}(X_T))\right]$$

$$\text{subject to} \quad dX_t = b\,(t, X_t, \alpha_t, \mathcal{L}_{\mathbb{P}}(X_t))\,dt + dW_t^{\mathbb{P}},$$

→ Infimum over filtered probability spaces $(\Omega, \mathbb{F}, \mathbb{P})$ supporting a Wiener process $W^{\mathbb{P}}$, and over α progress. measurable on $(\Omega, \mathbb{F}, \mathbb{P})$.

→ Two simplifications here: no dependence on $\mathcal{L}(\alpha)$ in the drift, and $\sigma \equiv 1$ (or deterministic).

Idea: move mass: noise $\longrightarrow$ state

Monge-Kantorovich optimal transport

Classical Optimal Transport: $(\mathcal{X},\mu),(\mathcal{Y},\nu)$ Polish, move the mass from μ to ν minimizing the cost of transportation $c:\mathcal{X}\times\mathcal{Y}\to[0,\infty]$:

$$\inf\left\{\mathbb{E}^{\pi}[c(x,y)] : \pi\in\Pi(\mu,\nu)\right\},$$

$\Pi(\mu,\nu)$: probability measures on $\mathcal{X}\times\mathcal{Y}$ with marginals μ and ν.

Extensive literature on OT:
- Monge (1781)
- Kantorovich (1942, 1948)
- Ambrosio, Brenier, Caffarelli, Figalli, Gigli, McCann, Otto, Santabrogio, Sturm, Villani ...

→ We consider a dynamic setting: we have the time component (mathematically: spaces $\mathcal{X}$ and $\mathcal{Y}$ endowed with filtrations)

→ **Idea**: move the mass in a non-anticipative way: what is transported into the 2^{nd} coordinate at time t, depends on the 1^{st} coordinate only up to t (+ possibly on sth independent)

Causal optimal transport

Definition (Causal transport plans)

$\pi \in \Pi(\mu, \nu)$ s.t. $\forall t, D \in \mathcal{F}_t^{\mathcal{Y}}$, $\mathcal{X} \ni x \mapsto \pi^x(D)$ is $\mathcal{F}_t^{\mathcal{X}}$-measurable.
($\mathcal{F}^{\mathcal{X}}, \mathcal{F}^{\mathcal{Y}}$ canonical filtrations, π^x regular conditional kernel)

Some literature: Yamada-Watanabe (1971), Jacod (1980), Kurtz (2014), Lassalle (2013), Backhoff-Beiglböck-Lin-Zalashko (2016), Acciaio-Backhoff-Zalashko (2016)

Example (Yamada-Watanabe'71)

Assume weak-existence of the solution to the SDE:

$$dY_t = b(Y_t)dt + \sigma(Y_t)dB_t, \quad b, \sigma \text{ Borel measurable.}$$

Then $\mathcal{L}(B, Y)$ causal transport between $(C_0, \mathcal{L}(B))$ and $(C_0, \mathcal{L}(Y))$.

Here $\mathcal{X} = \mathcal{Y} = C_0 := C_0[0, \infty)$ continuous paths starting at zero

McKean-Vlasov control problem and Causal Transport

→ Recall our weak McKean-Vlasov control problem:

$$\inf_{\mathbb{P},\alpha} \mathbb{E}^{\mathbb{P}} \left[\int_0^T f(t, X_t, \alpha_t, \mathcal{L}_{\mathbb{P}}(X_t, \alpha_t)) dt + g(X_T, \mathcal{L}_{\mathbb{P}}(X_T)) \right]$$

$$\text{subject to} \quad dX_t = b\,(t, X_t, \alpha_t, \mathcal{L}_{\mathbb{P}}(X_t))\, dt + dW_t^{\mathbb{P}},$$

→ The joint distribution $\mathcal{L}(W^{\mathbb{P}}, X)$ is a causal transport plan between $(C_0[0,T], \gamma)$ and $(C_0[0,T], \mathcal{L}(X))$, where γ = Wiener measure on $C_0[0,T]$

Assumptions

→ We need some **convexity assumptions:**

- $b(x, ., v)$ injective and convex
- f bdd below, and $f(x, b_t^{-1}(x, ., v)(y), \xi)$ convex in y
- $f(x, a, .)$ is $\prec_{cm}$-monotone

In the case of linear drift:

$$dX_t = (c_1 X_t + c_2 \alpha_t + c_3 \mathbb{E}[X_t])dt + dW_t,$$

$c_i \in \mathbb{R}, c_2 > 0$, our assumptions reduce to: for all x, a, ξ:

- f is bounded from below
- $f(x, ., \xi)$ is convex
- $f(x, a, .)$ is $\prec_{\mathrm{c}}$-monotone

Characterization via causal optimal transport

$\rightarrow$ Here we consider transport problems with $\mathcal{X} = \mathcal{Y} = C_0[0,T]$, with $(\omega, \overline{\omega})$ generic element on $C_0[0,T] \times C_0[0,T]$

Theorem

The weak MKV problem is **equivalent** *to the variational problem*

$$\inf_{\nu \in \mathcal{P}} \inf_{\pi \in \Pi_c(\gamma,\nu)} \mathbb{E}^{\pi}\left[c(\pi, \omega, \overline{\omega})\right]$$

$\mathcal{P}$ is a "good set of measures", $\Pi_c(\gamma,\nu) = \{\pi \in \Pi(\gamma,\nu) : \pi \text{ causal}\}$,

$c(\pi, \omega, \overline{\omega}) = \int_0^T f\left(\overline{\omega}_t, u_t^{\nu}(\omega, \overline{\omega}), p_t\left((\overline{\omega}, u^{\nu})_{\#}\pi\right)\right) dt + g(\overline{\omega}_T, \nu_T)$, with

$u_t^{\nu}(\omega, \overline{\omega}) = b_t^{-1}(\overline{\omega}_t, ., \nu_t)\left((\dot{\widehat{\overline{\omega} - \omega}})_t\right)$ and $\overline{\omega} - \omega = \int_0^{\cdot} (\dot{\widehat{\overline{\omega} - \omega}})_t \, dt$

- when control = drift, and square integrable: $\mathcal{P} = \{\nu \ll \gamma\}$
- in general: $\mathcal{P} = \{\nu \in \mathcal{P}(C_0[0,T]) : \langle\omega\rangle \, \exists \; \nu\text{-a.s., with } \langle\omega\rangle_t = t \; \forall \; t\}$, where $\langle\omega\rangle$ is the pathwise quadratic variation

Characterization via causal optimal transport

'**Equivalence**' means:

- the above variational problem and the weak MKV problem have the same value;
- and the optimizers are related via:
  - $\nu^* = \mathcal{L}(X^*)$
  - $\pi^* \longleftrightarrow \alpha^*$,   with $\pi^* = \mathcal{L}(W^*, X^*)$

Corollary (Weak closed loop)

1. *The infimum can be taken over tuples s.t. α is $\mathcal{F}^X$-measurable (weak closed loop).*
2. *If the infimum is attained, then the optimal control α is in weak closed loop form.*

Characterization via causal optimal transport

→ How to exploit the above characterization?

- Martingale verification result (analogous of (N) Pontryagin)
- Discrete-time approximation: we can define transport problems from $\mathbb{R}^n$ to $\mathbb{R}^n$ (via projection) such that their optimal values converge to the original problem
- New existence and uniqueness results based on tools from optimal transport ...
- Separable costs - Sanov type approximation results ...

Conclusions

We study generalized McKean-Vlasov control problems, where the mean-field dependence is on both state and control.

(I) By **classical probabilistic approach**:

→ Necessary and sufficient Pontryagin conditions in the generalized framework

→ Explicitly solvable cases:

- Linear-Quadratic case
- Optimal liquidation problem

(II) By **optimal transport approach**:

→ Characterization of weak McKean-Vlasov solutions via causal optimal transport

→ Exploitation of the characterization theorem

Thank you for your attention!

&

Happy birthday, Mete!