Approximation in Lusin's sense of Sobolev functions by Lipschitz functions[1]

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[1] Joint work with E.Brué, D.Trevisan

My three papers with Mete

Structure of the singular set $\Sigma^k(u) = \{x : \dim(\partial^+ u(x)) \geq k\}$ of (semi)concave functions (with Piermarco Cannarsa, Ann. SNS '93)

Level set method for codimension $k \in [1, n-1]$ mean curvature flow in $\mathbb{R}^n$ (JDG, '96)

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{n-k} \lambda_i \bigl(P_{\nabla u} \nabla^2 u P_{\nabla u}\bigr).$$

Convergence of reaction-diffusion equation (for $u : (0, \infty) \times \mathbb{R}^n \to \mathbb{R}^2$)

$$\frac{\partial u}{\partial t} - \Delta u = \frac{1}{\epsilon} u\bigl(1 - |\nabla u|^2\bigr), \quad \mu^\epsilon(B) = \frac{1}{\log(1/\epsilon)} \int_B \frac{|\nabla u|^2}{2} + \frac{(1 - |u|^2)^2}{\epsilon^2}\, dx$$

to Brakke's mean curvature flow in codimension 2 (Ann. SNS '97).

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Plan

1. Setting of the Lusin-Lipschitz approximation problem
2. Some classical applications
3. The Gaussian and RCD cases
4. An application to the theory of flows

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Lusin-type Lipschitz approximation

In a metric measure space $(X, \mathrm{d}, \mathrm{m})$, a function $f : X \to \mathbb{R}$ is said to be approximable in Lusin's sense by Lipschitz functions on $A \in \mathscr{B}(X)$ if for all $\epsilon > 0$ there exists $C \in \mathscr{B}(X)$ such that

$$\mathrm{m}(A \setminus C) < \epsilon \qquad \text{and} \qquad f|_C \text{ is d-Lipschitz.}$$

This property implies not only m-measurability of $f|_A$, but also a weak differentiability property (differentiability in measure), in sufficiently "nice" spaces, for instance:

Theorem. *If $X = \mathbb{R}^n$, d is the Euclidean distance and $\mathrm{m} = \mathscr{L}^n$, then f is approximable in Lusin's sense by Lipschitz functions on A if and only if for $\mathscr{L}^n$-a.e. $x \in A$ one has*

$$\int_{B_r(x)} \min\left\{\frac{|f(y) - f(x) - L(y-x)|}{|y-x|}, 1\right\} dy = o(r^n)$$

for some $L = L_x : \mathbb{R}^n \to \mathbb{R}$ linear.

Lusin-type Lipschitz approximation

We are interested in the quantitative version of this statement, in particular to

$$|f(x) - f(y)| \leq \mathrm{d}(x, y)(g(x) + g(y)) \qquad \forall x,\, y \in X \setminus N$$

with $\mathrm{m}(N) = 0$, for some $g \in L^p(X, \mathrm{m})$. It implies that one can take $C = \{g \leq M\} \setminus N$, so that $\mathrm{Lip}(f|_C) \leq 2M$ and

$$\mathrm{m}(X \setminus C) \leq M^{-1/p} \int_{\{g>M\}} |g|^p \, d\mathrm{m} = o(M^{-1/p})$$

by Markov inequality. In Euclidean and other "nice" spaces, the decay property $\mathrm{m}(X \setminus C) = o\big((\mathrm{Lip}(f|_C))^{-1/p}\big)$ *characterizes* $W^{1,p}$ Sobolev spaces for $p \geq 1$ (the so-called Hajlasz-Sobolev spaces).

An analogous weak L^1 estimate (with O in place of o) holds for BV functions, but it does not characterize the BV property.

Some classical applications

The Lusin-Lipschitz property has a variety of applications, let's see a few of them.

Lower semicontinuity in vectorial Calculus of Variations: *If* $p > 1$, $(u_h) \subset W^{1,p}(\mathbb{R}^n, \mathbb{R}^m)$ *weakly converge to* u, *and if* $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \to [0, \infty)$ *is a continuous Lagrangian, quasi-convex with respect to the last variable and with p-growth, one has*

$$\int_{\mathbb{R}^n} L(x, u(x), \nabla u(x))\, dx \leq \liminf_{h\to\infty} \int_{\mathbb{R}^n} L(x, u_h(x), \nabla u_h(x))\, dx.$$

The case $p = \infty$ is a classical result by Morrey in the 50's, the case $p < \infty$ was only achieved in '81 by Acerbi-Fusco.

Related applications are in the theory of elliptic PDE's (Diening, Duzaar, Mingione, Stroffolini, Verde,....).

Some classical applications: currents

Theory of currents. *If T is a normal (i.e. with finite mass and boundary with finite mass) k-dimensional current in a metric space E, and if the mass measure $\|T\|$ is concentrated on a set L σ-finite with respect to $\mathscr{H}^k$, then T is a rectifiable current.*
Equivalently, there exists a countably $\mathscr{H}^k$-rectifiable set S such that

$$\|T\|(E \setminus S) = 0.$$

The classical proofs, by Federer-Fleming '60 (later on simplified by White '89) work only in Euclidean spaces.

The metric proof (A-Kirchheim '00) exploits the Lusin-Lipschitz property of the slicing operator $x \ni \mathbb{R}^k \mapsto \langle T, \pi, x\rangle \in \mathbf{M}_0(E)$ for $\pi \in \mathrm{Lip}\,(E, \mathbb{R}^k)$ (first observed by Jerrard-Soner '99), of class BV with an appropriate choice of metrics, and it works in full generality.

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Some classical applications: flow of vector fields

The DiPerna-Lions theory provides existence, uniqueness and stability to the flow $\boldsymbol{X}(t,x)$ of a large class of vector fields $\boldsymbol{b}_t(x)$, $t \in (0,T)$, including Sobolev vector fields. We follow the axiomatization of A. '04, based on the concept of *Regular Lagrangian Flow*.

Definition. We say that $\boldsymbol{X}(t,x)$ is a regular lagrangian flow associated to $\boldsymbol{b}_t$ if:

(i) $\boldsymbol{X}(\cdot,x)$ is an absolutely continuous solution in $[0,T]$ to the ODE for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$;

(ii) for some constant $L \geq 0$, called compression constant, one has $\boldsymbol{X}(t,\cdot)_\#\mathscr{L}^n \leq L\mathscr{L}^n$ for all $t \in [0,T]$.

Can we obtain *quantitative* stability/uniqueness results? The answer is yes (Crippa-De Lellis '08), and the proof requires once more the Lusin-Lipschitz property.

Recently, using notions from Γ-calculus, the DiPerna-Lions theory has been extended in A.-Trevisan to a large class of metric measure structures, including all $\mathrm{RCD}(K,\infty)$ metric measure spaces.

A more recent applications: the matching problem

Rate of convergence in the matching problem. *Assume that* $(X_i)_{i\geq 1}$ *are independent and identically distributed on* $[0,1]^2$*, with uniform law* m*, and let* $\mu^N(\omega)$ *be the (random) induced empirical measures*

$$\mu^N(\omega) := \frac{1}{N}\sum_{i=1}^{N}\delta_{X_i(\omega)}.$$

Then for all $p \geq 1$ *there exist constants* $c_2(p) \geq c_1(p) > 0$ *such that*

$$c_1(p)\frac{(\log N)^{p/2}}{N^{p/2}} \leq \mathbb{E}\big[W_p^p(\mu^N, \mathrm{m})\big] \leq c_2(p)\frac{(\log N)^{p/2}}{N^{p/2}},$$

where W_p *denotes the Wasserstein distance with cost* $c = d^p$*.*

The original proof is due to Ajtai-Komlos-Tusnady '84, see also Talagrand's recent monograph.

A more recent application: the matching problem

In this case one can use (A.-Stra-Trevisan) the Lusin-Lipschitz property to build "good" 1-Lipschitz functions in Kantorovich's duality formula.

More precisely, on the basis of an ansatz recently proposed by Caracciolo, Lucibello, Parisi, Sicuro) we first consider the random Poisson's equation

$$-\Delta f_\omega = \mu^N(\omega) - \mathrm{m}$$

(arising from the linearization of Monge-Ampére equation).

Then, we build the "test" 1-Lipschitz function in Kantorovich duality starting from the Lusin-Lipschitz property of f_ω. When $p = 2$, with appropriate error estimates and regularizations, this "PDE" approach to the matching problem provides also the formula

$$\lim_{N\to\infty} \frac{N}{\log N}\mathbb{E}\left[W_2^2(\mu^N, \mathrm{m})\right] = \frac{1}{4\pi}.$$

Strategies of proof of Lusin-Lip: Euclidean case

For $f \in W^{1,p}(X, d, \mathrm{m})$, $p > 1$, we want to find $g \in L^p(X, \mathrm{m})$ and a m-negligible set N such that

$$|f(x) - f(y)| \leq d(x, y)(g(x) + g(y)) \qquad \forall x,\, y \in X \setminus N.$$

In Euclidean spaces the proof of can be achieved writing f as a singular integral

$$f(x) = -\int \langle \nabla f(y), \nabla_x G(x, y) \rangle \, dy$$

with G fundamental solution of Laplace's operator Δ, in the end a suitable g is proportional to $M(|\nabla f|)$, namely

$$M(|\nabla f|)(x) := \sup_{r>0} \fint_{B_r(x)} |\nabla f| \, dy.$$

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Strategies of proof of Lusin-Lip: PI metric measure spaces

In more general metric measure structures, we can compare f with a regularization f_r, for instance $f_r(x) = \fint_{B_r(x)} f \, d\mathrm{m}$. Choosing $r \sim d(x,y)$, $f_r(x)$ is comparable to $f_r(y)$ (with an estimate $\sim r \fint_{B_{Cr}(x)} |\nabla f| \, d\mathrm{m}$) and the problem reduces to the pointwise estimate of $f(x) - f_r(x)$.

This estimate involves once more Hardy-Littlewood's maximal function $M(|\nabla f|)(x)$.

However, these strategies seem to fail when either m is not doubling or the local Poincaré inequality fails. This happens for instance for Gaussian spaces (even when they are topologically finite-dimensional), for the Wiener space and for $\mathrm{RCD}(K,\infty)$ spaces.

Our method covers all these important cases, and builds upon another powerful maximal theorem.

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New strategy of proof

Theorem. (Rota) *For $p \in (1,\infty]$ and for the m-a.e. continuous version of a Markov semigroup R_t one has*

$$\|\sup_{t>0} R_t f\|_p \leq C_p \|f\|_p \qquad \forall f \in L^p(X,\mathrm{m}).$$

In addition, for all $f \in L^p(X,\mathrm{m})$, one has $R_t f \to f$ m-a.e. as $t \to 0^+$.

Then, our method uses the semigroup R_t associated to the Sobolev class $W^{1,2}$ instead of the inversion of Laplace's operator: the regularization is $f_t = R_t f$, now with $t \sim d^2(x,y)$.

It follows that we need to estimate

$$|f(x) - f(y)| \leq |f(x) - R_t f(x)| + |R_t f(x) - R_t f(y)| + |R_t f(y) - f(y)|.$$

Roughly speaking the estimates of all terms involve $|\nabla f|$, but while the estimate of the oscillation $|R_t(x) - R_t(y)|$ involves mostly the curvature properties of the metric measure space, the estimate of $f - R_t f$ is more related to the regularity of the transition probabilities $p_t(x,y)$ of R_t.

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New strategy of proof

We need also an extra ingredient, not present in the "finite dimensional" versions, namely the operator $\sqrt{-\Delta}$ as a replacement for the modulus of the gradient. To illustrate the necessity of this, we estimate:

$$\begin{aligned} R_t f(x) - f(x) &= \int_0^t \Delta R_s f(x)\,ds = \int_0^t \int_X p_s(x,y)\Delta f(y)\,\mathrm{dm}(y)\,ds \\ &= -\int_0^t \int_X \nabla f \cdot \nabla p_s(x,\cdot)\,\mathrm{dm}\,ds \\ &= -2\int_0^t \int_X \nabla f \cdot \nabla \sqrt{p_s}(x,\cdot)\sqrt{p_s}(x,\cdot)\mathrm{dm}ds \\ &\le 2\int_0^t \left(R_s|\nabla f|^2\right)^{1/2}\left(\int_X |\nabla\sqrt{p_s}(x,\cdot)|^2\,\mathrm{dm}\right)^{1/2} ds. \end{aligned}$$

Even if we knew that $s \mapsto \left(\int |\nabla\sqrt{p_s}(x,\cdot)|^2\,\mathrm{dm}\right)^{1/2}$ is integrable in $(0,t)$, this would give an estimate with $g \sim \sup_{s>0}\sqrt{R_s|\nabla f|^2}$ which would not be enough to deal with $W^{1,2}$ functions, because weak-L^1 estimates are not available in the setting of Rota's theorem.

New strategy of proof

We modify this approach using the "fractional representation" formula

$$R_t f - f = \int_0^\infty K(s,t) R_s \sqrt{-\Delta} f \, ds \qquad \forall f \in D(\sqrt{-\Delta}), \quad \forall t \geq 0$$

for a suitable (explicitly computable) kernel $\mathscr{K}(s,t)$ with

$$\int_0^\infty |\mathscr{K}(s,t)| \, ds = \frac{4}{\sqrt{\pi}} \sqrt{t} \qquad \forall t > 0.$$

This formula provides the correct integrability estimates, at the price of working with the nonlocal operator $\sqrt{-\Delta}$.

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Statement of main result in $\mathrm{RCD}(K,\infty)$ spaces

I state first the result in $\mathrm{RCD}(K,\infty)$ spaces and for $p=2$.

Recall that a metric measure space (X,d,m) is said to be $\mathrm{RCD}(K,\infty)$ if it is $\mathrm{CD}(K,\infty)$ according to Lott-Villani and Sturm, i.e.

$$\mathrm{Ent}_{\mathrm{m}} \text{ is } K\text{-convex along geodesics of } (\mathscr{P}_2(X), W_2)$$

and Cheeger's energy

$$\mathrm{Ch}(f) := \inf\left\{\liminf_{h\to\infty}\int_X |\nabla f_h|^2\,d\mathrm{m}:\ f_h\in\mathrm{Lip}(X,\mathrm{d}),\ \|f_h-f\|_2\to 0\right\}$$

is a quadratic form in $L^2(X,\mathrm{m})$.

By now this class of spaces, and the smaller class $\mathrm{RCD}(K,N)$ is well understood and characterized in many ways, after the work of many authors (A., Bolley, Gentil, Gigli, Guillin, Kuwada, Mondino, Savaré, Sturm,..), via properties of the heat flow H_t, gradient contractivity properties, or suitable Bochner inequalities (Bakry-Emery).

Statement of main result in $\mathrm{RCD}(K,\infty)$ spaces

Theorem. *For all* $f \in W^{1,2}(X,d,\mathrm{m})$ *and all* $\alpha \in (1,2)$ *there exists a* m*-negligible set* $N \subset X$ *such that*

$$|f(x) - f(y)| \le C_{\alpha,K} \mathsf{d}(x,y)(g(x) + g(y)) \qquad \forall x,\ y \in X \setminus N,$$

with

$$g := \left(\sup_{t>0} H_t |\nabla f|^\alpha\right)^{1/\alpha} + \sup_{t>0} |H_t \sqrt{-\Delta} f| \in L^2(X,\mathrm{m}).$$

In addition, this property characterizes the space $W^{1,2}(X,d,\mathrm{m})$.

The $H_t\sqrt{-\Delta}f$ term in the definition of g is due to the fractional representation

$$H_t f(x) - f(x) = \int_0^\infty \mathscr{K}(s,t) H_s \sqrt{-\Delta} f(x)\, ds.$$

Statement of main result in $\mathrm{RCD}(K,\infty)$ spaces

On the other hand, the term $(H_t|\nabla f|^\alpha)^{1/\alpha}$ in the definition of g is due to the estimate of

$$|H_t f(x_0) - H_t f(x_1)| \le C_{\alpha,K} \mathrm{d}(x_0, x_1) \left(\sup_{t>0} H_t|\nabla f|^\alpha(x_0)\right)^{1/\alpha}$$

when $t = \mathrm{d}^2(x_0, x_1)$. It can be obtained from

$$|H_t f(x_0) - H_t f(x_1)| \le e^{-Kt} \int_0^1 H_t|\nabla f|(x_s)\, ds$$

by applying Wang's Harnack-estimate

$$(H_t g)^\alpha(x) \le H_t g^\alpha(y) \exp\left(\frac{\alpha d^2(x,y)}{2\sigma_K(t)(\alpha-1)}\right) \qquad \forall x,\, y \in X$$

with $x = x_0$, $y = x_s$, $g = |\nabla f|$, $\sigma_K(t) = (e^{2Kt} - 1)/K$.

The Gaussian case

The result covers also Sobolev spaces $W^{1,2}_{\mathcal{E}}(H,\mathrm{m})$, with H separable Hilbert and m Gaussian and non-degenerate, induced by the Dirichlet form

$$\mathcal{E}(f) := \int_H |\nabla f|^2 \, d\mathrm{m}.$$

They are indeed particular cases of $\mathrm{RCD}(K,\infty)$ spaces, but in this case Wang's estimate is not needed and g takes the simpler form

$$g := \sup_{t>0} P_t|\nabla f| + \sup_{t>0} |P_t\sqrt{-\Delta_{\mathcal{E}}}f| \in L^2(H,\mathrm{m})$$

for all $f \in W^{1,2}_{\mathcal{E}}(H,\mathrm{m})$, where P_t is the standard Markov semigroup associated to $\mathcal{E}$ and $\Delta_{\mathcal{E}}$ the infinitesimal generator.

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The Wiener case

Still on Gaussian spaces (H, m) one can consider the covariance operator

$$\langle Qx, y\rangle := \int_H \langle x, u\rangle\langle y, u\rangle \, d\mathrm{m}(u)$$

and the new Dirichlet form

$$\mathcal{E}_Q(f) := \int_H |Q^{1/2}\nabla f|^2 \, d\mathrm{m}.$$

Recall that

$$\mathcal{H} := Q^{1/2}H$$

is the Cameron-Martin subspace of H, corresponding to the quasi-invariant directions of the Gaussian measure m, with Hilbert norm $|h|_{\mathcal{H}} := |Q^{-1/2}h|$.

The Sobolev space $W^{1,2}_{\mathcal{E}_Q}(H, \mathrm{m})$ corresponding to the Dirichlet form $\mathcal{E}_Q$ is larger than $W_{\mathcal{E}}(H, \mathrm{m})$, and related to differentiability along Cameron-Martin directions only.

The Wiener case

In this case, the Ornstein-Uhlenbeck semigroup T_t naturally associated to $\mathcal{E}_Q$, is given explicitly by Mehler's formula

$$T_t f(x) = \int_H f(e^{-t}x + \sqrt{1 - e^{-2t}}y)\, d\mathrm{m}(y),$$

for all $f \in W^{1,2}_{\mathcal{E}_Q}(H, \mathrm{m})$. Then, the Lusin-Lipschitz estimate has the form:

$$|f(x+h) - f(x)| \leq C|h|_{\mathcal{H}}(g(x+h) + g(x)) \qquad \forall h \in \mathcal{H},$$

with $g := \sup_{t>0} T_t|\nabla f|_{\mathcal{H}} + \sup_{t>0} |T_t\sqrt{-L}f| \in L^2(H, \mathrm{m})$, L being the infinitesimal generator of T_t.

Obviously this gives the Lusin-Lipschitz property in quantitative form (and, as usual, characterizes $W^{1,2}_{\mathcal{E}_Q}(H, \mathrm{m})$ functions) provided the Lipschitz property is understood with pairs $(x, y) \in H \times H$ with $x - y \in \mathcal{H}$, and with the norm $|x - y|_{\mathcal{H}}$.

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Extension to $p \neq 2$

By standard arguments, the estimate can be extended to the case $p \in (1, \infty)$, with

$$g = \sup_{t>0} P_t|\nabla f| + \sup_{t>0} |P_t\sqrt{I - L}f| \in L^p(H, \mathrm{m}),$$

provided one has the Riesz inequalities

$$\|\nabla f\|_p \leq c_p\|\sqrt{I - L}f\|_p, \qquad \|\sqrt{I - L}f\|_p \leq \tilde{c}_p\|f\|_{W^{1,p}},$$

relating the generator L to the corresponding gradient structure.

Riesz inequalities are known to hold in Gaussian spaces and in the Wiener spaces.

Open problem. We expect that the proof by Γ-calculus of Riesz inequalities should be applicable also to $\mathrm{RCD}(K, \infty)$ spaces, but for the moment these are known only for $\mathrm{RCD}(K, N)$ spaces, $N < \infty$.

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Vector-valued maps

For E-valued maps f, with E Hilbert space, and for $p = 2$, the results immediately extend arguing componentwise, so that (e.g. in the Wiener case)

$$|f(x+h) - f(x)|_E \leq C|h|_{\mathcal{H}}(g(x+h) + g(x)) \qquad \forall h \in \mathcal{H},$$

with $g := \sup_{t>0} T_t|\nabla f|_{\mathcal{H}} + \sup_{t>0} |T_t\sqrt{-L}f| \in L^2(H, \mathrm{m})$.

The case when $p \neq 2$ and f is E-valued requires once more the vector-valued version of Riesz's inequalities, available for the moment only in the Gaussian and Wiener settings.

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An application to the theory of flows

Let's see how the Crippa-De Lellis stability estimates, relative to flows $\boldsymbol{X}^1$, $\boldsymbol{X}^2$ with bounded compression of Sobolev vector fields $\boldsymbol{b}^1$, $\boldsymbol{b}^2$ can be extended to these infinite-dimensional settings. The non-quantitative version, via commutator estimates and renormalization, goes back to A.-Figalli '09.

In the Gaussian case (the Wiener case is similar), denoting by L a compression constant for both flows $\boldsymbol{X}^i$, one defines

$$\Phi(t) := \int_H \log(1 + \frac{|\boldsymbol{X}^1 - \boldsymbol{X}^2|}{\delta})\, d\mathrm{m},$$

so that $\Phi(0) = 0$, and tries to estimate its derivative

$$\Phi'(t) = \int_H \frac{\langle \boldsymbol{b}^1_t(\boldsymbol{X}^1) - \boldsymbol{b}^2_t(\boldsymbol{X}^2), \boldsymbol{X}^1 - \boldsymbol{X}^2\rangle}{(\delta + |\boldsymbol{X}^1 - \boldsymbol{X}^2|)|\boldsymbol{X}^1 - \boldsymbol{X}^2|}\, d\mathrm{m} \le \int_H \frac{|\boldsymbol{b}^1_t(\boldsymbol{X}^1) - \boldsymbol{b}^2_t(\boldsymbol{X}^2)|}{\delta + |\boldsymbol{X}^1 - \boldsymbol{X}^2|}\, d\mathrm{m}.$$

An application to the theory of flows

Adding an subtracting $\boldsymbol{b}_t^1(\boldsymbol{X}^2)$, we need to estimate

$$\int_H \frac{|\boldsymbol{b}_t^1(\boldsymbol{X}^1) - \boldsymbol{b}_t^1(\boldsymbol{X}^2)|}{\delta + |\boldsymbol{X}^1 - \boldsymbol{X}^2|}\, d\mathrm{m}$$

and

$$\int_H \frac{|\boldsymbol{b}_t^1(\boldsymbol{X}^2) - \boldsymbol{b}_t^2(\boldsymbol{X}^2)|}{\delta + |\boldsymbol{X}^1 - \boldsymbol{X}^2|}\, d\mathrm{m}.$$

The former can be estimated, neglecting δ, with the Lusin-Lip property of $\boldsymbol{b}_t^1$, the latter can be estimated with $L\delta^{-1}\|\boldsymbol{b}_t^1 - \boldsymbol{b}_t^2\|_1$, using the bounded compression property, to get

$$\Phi'(t) \leq 2L\|g_t\|_p + \frac{L}{\delta}\|\boldsymbol{b}_t^1 - \boldsymbol{b}_t^2\|_1, \qquad g_t \sim M(|\nabla \boldsymbol{b}_t^1|).$$

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An application to the theory of flows

By integration, for all $t \in [0, T]$ we have

$$\int_H \log(1+\frac{|\boldsymbol{X}^1(t,\cdot)-\boldsymbol{X}^2(t,\cdot)|}{\delta})\,d\mathrm{m} \le 2L\int_0^T \|g_t\|_p\,dt + \frac{L}{\delta}\|\boldsymbol{b}^1-\boldsymbol{b}^2\|_{L^1((0,T)\times H)}.$$

When $\boldsymbol{b}^1 = \boldsymbol{b}^2$, this immediately gives $\boldsymbol{X}^1 = \boldsymbol{X}^2$, since we can choose δ arbitrarily small.

In general, optimization w.r.t. δ gives a logarithmic modulus of continuity on the dependence of $\boldsymbol{X}$ on $\boldsymbol{b}$.

Crippa-De Lellis in a geometric setting

Finally, a challenging question is the Lusin-Lipschitz regularity of the flow map $\boldsymbol{X}(t,\cdot)$, when the flow is provided by Sobolev vector fields $\boldsymbol{b}_t$ in a $\mathrm{RCD}(K,N)$ space, $N<\infty$.

Theorem. (Bruè-Semola '18) *Under the assumptions*

$$|\boldsymbol{b}_t|+|\mathrm{div}\,\boldsymbol{b}_t|\in L^1(L^\infty(X,\mathrm{m})),\qquad \int_0^T\int_X|\nabla\boldsymbol{b}_t|^2\,d\mathrm{m}\,dt<\infty$$

one has that $\boldsymbol{X}(t,\cdot)$ is Lusin-Lipschitz w.r.t. the quasi-distance

$$\hat{\mathrm{d}}(x,y):=\frac{1}{G(x,y)}\qquad (G=\textit{Green function of }(X,\mathrm{d},\mathrm{m})).$$

In particular, this new *intrinsic* apriori estimate can be applied to the family of geodesics joining any two parts of X, to obtain that $\mathrm{RCD}(K,N)$ spaces have a unique "essential dimension" $k\in[1,N]$.

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Happy birthday Mete!

Slides available upon request