"FLUCTUATIONS IN FINITE STATE MANY PLAYER GAMES"

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(joint with Asaf Cohen)

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METE

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Contents

Contents

Literature

Mean-field games:
(1) J.-M. Lasry and P.-L. Lions. *Jeux à champ moyen. I. Le cas stationnaire.* C. R. Math. Acad. Sci. Paris, 2006..
(2) J.-M. Lasry and P.-L. Lions. *Jeux à champ moyen. II. Horizon ni et contrôle optimal.* C. R. Math. Acad. Sci. Paris, 2006.
(3) J.-M. Lasry and P.-L. Lions. *Mean field games.* Jpn. J. Math., 2007.
(4) M. Huang, P. E. Caines, and R. P. Malhame. *The Nash certainty equivalence principle and Mckean-Vlasov systems: An invariance principle and entry adaptation.* Decision and Control, IEEE, 2007.
(5) M. Huang, R. P. Malhame, and P. E. Caines. *Large population stochastic dynamic games: Closed loop McKean-Vlasov systems and the Nash certainty equivalence principle.* Commun. Inf. Syst., 2006.
(6) P. Cardaliaguet. *Notes on mean field games*. 2013.
(7) D. A. Gomes, J. Mohr, and R. R. Souza. *Continous time finite state mean field games.* AMO, 2013.
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Literature – contd.

The master equation in mean-field games:

(1) D. A. Gomes, R. M. Velho, and M.-T. Wolfram. *Socio-economic applications of finite state mean field games.* Phil. Trans. R. Soc. A, 2014.

(2) R. Carmona and F. Delarue. *The master equation for large population equilibriums.* Stoch. Analysis & App., 2014.

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(4) R. Carmona and P. Wang. *Finite state mean field games with major and minor players. arXiv,* 2016.

(5) P. Cardaliaguet, F. Delarue, J.-M. Lasry, and P.-L. Lions. *The master equation and the convergence problem in mean field games. arXiv,* 2015

(6) F. Delarue, D. Lacker, and K. Ramanan. *From the master equation to mean field game limit theory: A central limit theorem. arXiv,* 2018.

(7) F. Delarue, D. Lacker, and K. Ramanan. *From the master equation to mean field game limit theory: Large deviations and concentration of measure. arXiv,* 2018.

(8) A. Cecchin and G. Pelino. *Convergence, Fluctuations and Large Deviations for finite state Mean Field Games via the Master equation. arXiv,* 2017.

The talk is based on the paper:

(1) E. Bayraktar and A. Cohen. *Analysis of a finite state many player game using its master equation,* arXiv, July, 2017.

The *n*-Player Game

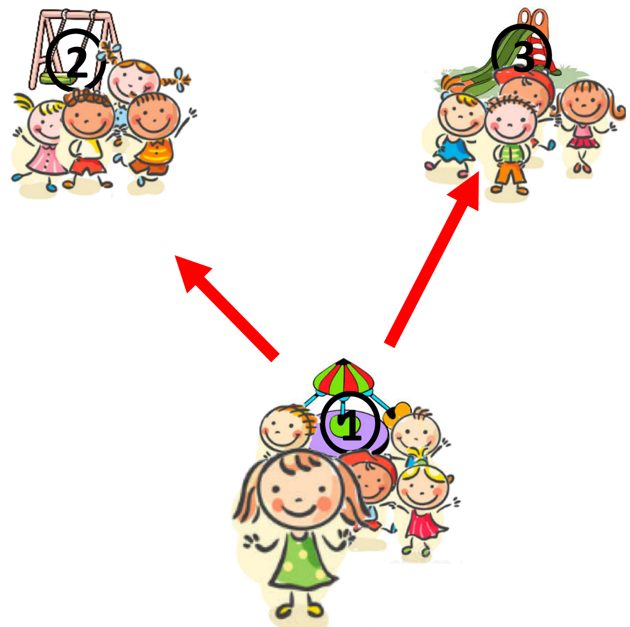
- *{1,...,d}* – states
- *{1,...,n}* – players
- *T* – time horizon

- $X^i(s)$ – private state

- $\mu^{-i}(s) := \frac{1}{n-1} \sum_{j, j \neq i} \delta_{X^j(s)}$

- $a_z^i(s, X^i(s), \mu^{-i}(s)), \; z \in \{1, \dots, d\}$

  rate of moving from $X^i(s)$ to z.



Private cost:

$$J^{n,i}(t, x, \eta, a) := \mathbb{E}\Big[\int_t^T f(X^i(s), \mu^{-i}(s), a^i(s, X^i(s), \mu^{-i}(s)))ds + g(X^i(T), \mu^{-i}(T)) \Big],$$

$\longrightarrow$ MIN

Goal: find the *fluctuations* for the empirical measure μ^n under equilibrium

The Hamilton–Jacobi–Bellman equation

Fix a strategy profile β^{-i} for players $n \setminus \{i\}$ and a strategy a^i for player i

$$0 = \partial_t V^n(s,x,\eta) + \min_a \Big\{ f(x,\eta,a) + \sum_{z, z\neq x} [V^n(s,z,\eta) - V^n(s,x,\eta)]\, a_z \Big\}$$

$$+ \sum_{y,z} \eta_y (n-1) \Big[V^n\Big(s, x, \eta + \tfrac{1}{n-1}(e_z - e_y)\Big) - V^n(s,x,\eta) \Big] \beta_{yz}(s)$$

$$V^n(t, X^i(t), \mu^{-i}(t))$$

$$\leq J^n(t, X^i(t), \mu^{-i}(t); [a^i, \beta^{-i}])$$

$$= \mathbb{E}\Big[g(X^i(T), \mu^{-i}(T)) + \int_t^T f(X^i(s), \mu^{-i}(s), a^i(s, X^i(s), \mu^{-i}(s)))ds \Big]$$

$$= \mathbb{E}\Big[V^n(T, X^i(T), \mu^{-i}(T)) + \int_t^T f(X^i(s), \mu^{-i}(s), a^i(s, X^i(s), \mu^{-i}(s)))ds \Big]$$

$$= V^n(t, X^i(t), \mu^{-i}(t))$$

$$+ \mathbb{E} \int_t^T \Big\{ \partial_t V^n(s, X^i(s), \mu^{-i}(s)) + f(X^i(s), \mu^{-i}(s), a^i(s, X^i(s), \mu^{-i}(s)))$$

$$+ \sum_{z \neq X^i(s)} \Big[V^n(s, z, \mu^{-i}(s)) - V^n(s, X^i(s), \mu^{-i}(s)) \Big] a^i_z(s, X^i(s), \mu^{-i}(s))$$

$$+ \sum_{y,z} \mu_y^{-i}(s)(n-1) \Big[V^n(s, X^i(s), \mu^{-i}(s) + \tfrac{1}{n-1}(e_z - e_y)) - V^n(s, X^i(s), \mu^{-i}(s)) \Big] \beta_{yz}(s) \Big\} ds$$

$0 \leq$

$0 \overset{\text{opt.}}{=}$

The Hamilton–Jacobi–Bellman equation

Fix a strategy profile β^{-i} for players $n \setminus \{i\}$ and a strategy a^i for player i

$$0 = \partial_t V^n(s,x,\eta) + \min_a \Big\{ f(x,\eta,a) + \Delta_x V^n(t,\cdot,\eta)\cdot a \Big\}$$

$$+ \sum_{y,z} \eta_y (n-1)\left[V^n\left(s,x,\eta+\tfrac{1}{n-1}(e_z-e_y)\right) - V^n(s,x,\eta)\right] \beta_{yz}(s)$$

$$0 = \partial_t V^n(s,x,\eta) + f(x,\eta,a^*(x,\eta,\Delta_x V^n(s,\cdot,\eta))) + \Delta_x V^n(s,x,\eta)\cdot a^*(x,\eta,\Delta_x V^n(s,\cdot,\eta))\Big\} \qquad \text{(HJB)}$$

$$+ \sum_{y,z} \eta_y (n-1)\left[V^n\left(s,x,\eta+\tfrac{1}{n-1}(e_z-e_y)\right) - V^n(s,x,\eta)\right] a^*_z(y,\eta+O(\tfrac{1}{n}),\Delta_y V^n(t,\cdot,\eta+O(\tfrac{1}{n}))$$

$$\approx \partial_{\eta_z} V^n(t,x,\eta) - \partial_{\eta_y} V^n(t,x,\eta)$$

$$a^*_x(t,x,\eta) := -\sum_{y,y\neq x} a^*_y(t,x,\eta)$$

another player's point of view

symmetric equilibrium, take β^{-i} *to be* a^*

$$\begin{cases} 0 = \partial_t U(t,x,\eta) + H(x,\eta,\Delta_x U(t,\cdot,\eta)) + \sum_{y,z} \eta_y \partial_{\eta_z} U(t,x,\eta)\, a^*_z(y,\eta,\Delta_y U(t,\cdot,\eta)) \\ U(T,x,\eta) = g(x,\eta) \end{cases} \qquad \text{(ME)}$$

What can we do with the *Master Equation*?

Assumptions

$$H(x,\eta,p) = \min_a \left\{ f(x,\eta,a) + \Delta_x p \cdot a \right\} = \min_a \left\{ f_1(x,a) + \Delta_x p \cdot a \right\} + f_2(x,\eta)$$

Assumptions

a) [Control set] A compact set bounded away from 0.

b) [Separation] $f(x,\eta,a) = f_1(x,a) + f_2(x,\eta)$.

c) [Lipschitz] $\|\nabla_\eta g(x,\eta) - \nabla_\eta g(x,\eta')\| + \|\nabla_\eta f_2(x,\eta) - \nabla_\eta f_2(x,\eta')\| \le c_L \|\eta - \eta'\|$.

d) [Monotonicity]

$$\sum_{x\in[d]} (\eta_x - \eta'_x)(g(x,\eta) - g(x,\eta')) \ge 0,$$

$$\sum_{x\in[d]} (\eta_x - \eta'_x)(f_2(x,\eta) - f_2(x,\eta')) \ge 0.$$

e) [Concavity] on compact subsets $\partial^2_{p_y,p_z} H_1(x,p) \le -C$.

As a consequence

$$a^*_y(x,\eta,p) = a^*_y(x,p) = \partial_{p_y} H_1(x,p)$$

the assumptions in most parts of the paper are less restrictive and a^* *directly depends on* η

Regularity of the solution of the Master Equation

The Assumptions imply that $p \mapsto a^*_y(x,p)$ is regular. Specifically, $p \mapsto \nabla_p a^*_y(x,p)$ is Lipschitz continuous.

Following PDE techniques (linearizing a forward-backward system) developed for the diffusion case by

P. Cardaliaguet, F. Delarue, J.-M., and Lasry, P.-L. Lions. *The master equation and the convergence problem in mean field games.* *arXiv,* 2015

we show that $\eta \mapsto \nabla_\eta U(t,x,\eta)$ is Lipschitz. This property is essential for the forthcoming results.

Contents

Approximating the value function

$$\begin{cases} 0 = \partial_t U(t,x,\eta) + H(x,\eta,\Delta_x U(t,\cdot,\eta)) + \sum_{y,z} \eta_y \partial_{\eta_z} U(t,x,\eta) a_z^*(y,\Delta_y U(t,\cdot,\eta)) \\ U(T,x,\eta) = g(x,\eta) \end{cases} \qquad \text{(ME)}$$

What can we do with the *Master Equation*?

Proposition 1 (Bayraktar and Cohen)

$$\sup_{t\in[0,T]} \mathbb{E}\left[\left|(V^n - U)\left(t, X^i(t), \mu^{-i}(t)\right)\right|^2\right] + \int_0^T \mathbb{E}\left[\left\|\Delta_{X^{n,i}(s)}(V^n - U)\left(s, X^i(s), \mu^{-i}(s)\right)\right\|^2\right] ds \le \frac{C}{n^2}.$$

Proof. Itô' lemma, Grönwall's inequality, the structure of (HJB) and (ME)…

Approximating the empirical distribution

$$\begin{cases} 0 = \partial_t U(t,x,\eta) + H(x,\eta,\Delta_x U(t,\cdot,\eta)) + \sum_{y,z} \eta_y \partial_{\eta_z} U(t,x,\eta) a_z^*(y,\Delta_y U(t,\cdot,\eta)) \\ U(T,x,\eta) = g(x,\eta) \end{cases} \quad \text{(ME)}$$

What can we do with the *Master Equation*?

Consider the processes $Y^1,\dots,Y^n$ whose transition rates are given by

$a_y^*(x,\Delta_x U(t,\cdot,\nu^{-i}(t)))$, where $\nu^{-i}(t) := \frac{1}{n-1}\sum_{j,j\neq i} \delta_{Y^j(t)}$

Proposition 2 (Bayraktar and Cohen)

There is a coupling between $X^1,\dots,X^n$ and $Y^1,\dots,Y^n$ such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|\mu^{-i}(t) - \nu^{-i}(t)\|\Big] \leq \frac{C}{n}$$

Approximating the empirical distribution

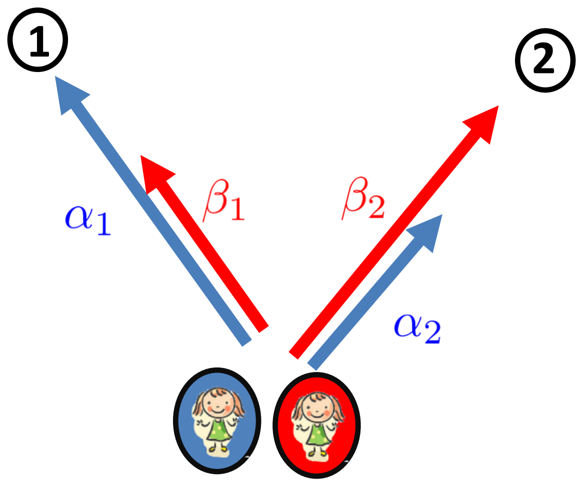
$Y^1, \ldots, Y^n$

rates: $a_y^*(x, \Delta_x U(t, \cdot, \nu^{-i}(t)))$,
empirical measure: $\nu^{-i}(t)$

$X^1, \ldots, X^n$

rates: $a_y^*(x, \Delta_x V^n(t, \cdot, \mu^{-i}(t)))$,
empirical measure: $\mu^{-i}(t)$

Initially, for every i, $Y^i(0) = X^i(0)$.



The processes $(Y^i(t), X^i(t))$ jumps with rate
rate $= \max\{\alpha_1 + \alpha_2, \beta_1 + \beta_2\}$
to:

$$\begin{cases} (1,1) & \text{w.p.} \quad \beta_1/\text{rate} \\ (1, X^i(t)) & \text{w.p.} \quad (\alpha_1 - \beta_1)/\text{rate} \\ (2,2) & \text{w.p.} \quad \alpha_2/\text{rate} \\ (Y^i(t), 2) & \text{w.p.} \quad (\beta_2 - \alpha_2)/\text{rate} \end{cases}$$

The coupling of pair i stops at the first time they split

$$\tau^{n,i} := \inf\{t \geq 0 : X^{n,i}(t) \neq Y^{n,i}(t)\} \wedge T$$

Then, they move independently.

A key observation:
The *splitting rate* is
$(\alpha_1 - \beta_1) + (\beta_2 - \alpha_2)$

Approximating the empirical distribution

Proof of Proposition 2: $\mathbb{E}\Big[\sup_{t\in[0,T]}\|\mu^{-i}(t)-\nu^{-i}(t)\|\Big]\leq \frac{C}{n}$

$$\Big\|\tfrac{1}{n-1}\sum_{j,\neq i} e_{x_j}-\tfrac{1}{n-1}\sum_{j,j\neq i} e_{y_j}\Big\|\leq \tfrac{2}{n-1}\sum_{j,j\neq i}|x_j-y_j|$$

$$\mathbb{E}\Big[\sup_{s\in[0,t]}\|\mu^{-i}(s)-\nu^{-i}(s)\|\Big]$$
$$\leq 2\mathbb{E}\Big[\sup_{s\in[0,t]}\|X^i(s)-Y^i(s)\|\Big]$$
$$\leq 2(d-1)\mathbb{P}\Big(\tau^{n,i}\leq t\Big)$$
$$=2(d-1)\left(1-\mathbb{E}\Big[e^{-\sum_{z\in[d]}\int_0^t\left|a_z^*(X^i(s),\Delta_x V^n(s,\cdot,\mu^{-i}(s)))-a_z^*(Y^i(s),\Delta_x U(s,\cdot,\nu^{-i}(s)))\right|ds}\Big]\right)$$
$$\leq 2(d-1)\sum_{z\in[d]}\int_0^t\mathbb{E}\left[\left|a_z^*(X^i(s),\Delta_x V^n(s,\cdot,\mu^{-i}(s)))-a_z^*(Y^i(s),\Delta_x U(s,\cdot,\nu^{-i}(s)))\right|\right]ds.$$

A key observation: The *splitting rate* is $(\alpha_1-\beta_1)+(\beta_2-\alpha_2)$

The Assumptions implies that a^* and U are Lipschitz.

$$\leq C\sum_{z\in[d]}\int_0^t\mathbb{E}\left[\left\|\mu^{-i}(s)-\nu^{-i}(s)\right\|\right]ds+\tfrac{C}{n}.$$

Proposition 1 (Bayraktar and Cohen)

$$\int_0^T\mathbb{E}\left[\left\|\Delta_{X^{n,i}(s)}(V^n-U)\left(s,X^i(s),\mu^{-i}(s)\right)\right\|^2\right]ds\leq\frac{C}{n^2}.$$

Grönwall's inequality.... □

Contents

Theorem 2 (Bayraktar and Cohen)

Set $\alpha_{xy}(t,\eta) = a^*_y(x,\eta,\Delta_x U(t,\cdot,\eta))$ and $\frac{d}{dt}\mu(t) = \mu(t)\alpha(t,\mu(t)), \quad \mu(0)=\mu_0.$

Assuming that $\exists \psi_0 := \lim_{n\to\infty} \sqrt{n}(\mu^n(0)-\mu(0))$

Then $\exists \psi := \lim_{n\to\infty} \sqrt{n}(\mu^n - \mu)$,

Matrix whose xy term is $\psi(t)\cdot\nabla_\eta \alpha_{xy}(t,\eta)$

where,
$$d\psi(t) = \left[(\alpha(t,\mu(t)))^\top \psi(t) + \left(\psi(t)\otimes\nabla_\eta \alpha(t,\mu(t))\right)^\top \mu(t)\right]dt + \Sigma(t)dB(t),$$

$$(\Sigma^2)_{xy}(t) = -\mu_y(t)\alpha_{yz}(t,\mu(t)) - \mu_x(t)\alpha_{xy}(t,\mu(t)), \qquad x\neq y$$
$$(\Sigma^2)_{xx}(t) = \sum_{z,z\neq x}\mu_z(t)\alpha_{zx}(t,\mu(t)) + \mu_x(t)\sum_{z,z\neq x}\alpha_{xz}(t,\mu(t)).$$

Let's focus on the diffusion structure
$$d\psi(t) = b(t)dt + \Sigma(t)dB(t)$$
rather than on the explicit expressions

$$\mu^n(t) = \frac{1}{n}\sum_{i=1}^n \delta_{X^i(t)}$$
$$= \mu^{-i}(t) + O(\tfrac{1}{n}) = \nu^{-i}(t) + O(\tfrac{1}{n}) = \nu^n(t) + O(\tfrac{1}{n})$$

Key idea:

$$\lim_{n\to\infty}\sqrt{n}(\mu^n-\mu) = \lim_{n\to\infty}\sqrt{n}(\nu^n-\mu)$$

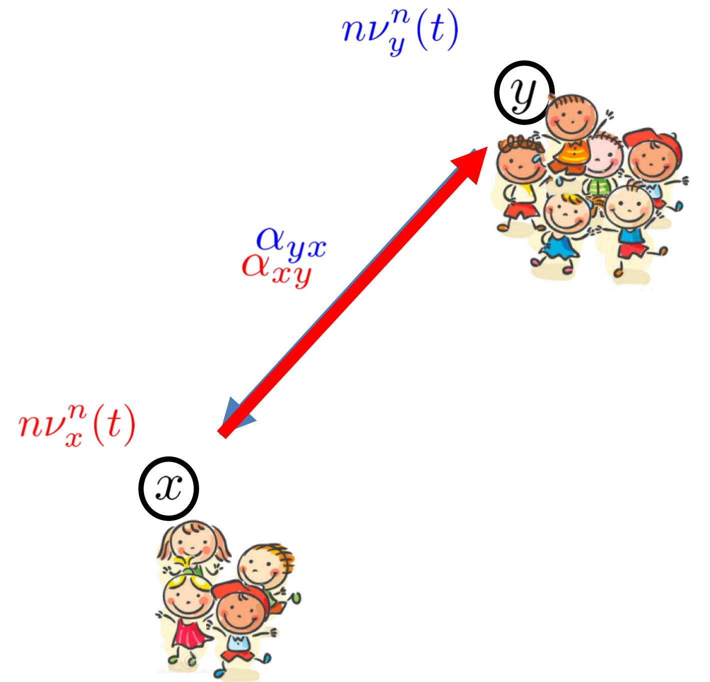
Proof of Theorem 2 - Notation

$A_x^n(t) =$ # Players moved into state x until time t.
The rate is $n\lambda_x^n(t)$, where

$$\lambda_x^n(t) := \sum_{y, y \neq x} \nu_y^n(t) \alpha_{yx}(s, \nu^n(t) + \text{~~}O(\tfrac{1}{n})\text{~~})$$

$S_x^n(t) =$ # Players moved from state x until time t.
The rate is $n\sigma_x^n(t)$, where

$$\sigma_x^n(t) := \nu_x^n(t) \sum_{y, y \neq x} \alpha_{xy}(s, \nu^n(t) + \text{~~}O(\tfrac{1}{n})\text{~~})$$



$$\nu^n(t) = \nu^n(0) + \frac{1}{n}(A^n(t) - S^n(t))$$

$$\tilde{A}_x^n(t) := \frac{A_x^n(t) - n\lambda_x^n(t)}{\sqrt{n}} \qquad \tilde{S}_x^n(t) := \frac{S_x^n(t) - n\sigma_x^n(t)}{\sqrt{n}}$$

$$\nu^n(t) = \nu^n(0) + \frac{1}{\sqrt{n}}(\tilde{A}^n(t) - \tilde{S}^n(t)) + \int_0^t [\nu^n(s)\alpha(s, \nu^n(s)) \cdot ds$$

Proof of Theorem 2 – Fluid Limit $\sup_{s\in[0,t]}\|\nu^n(s)-\mu(s)\| \overset{Prob.}{\to} 0$

$$\nu^n(t)-\mu(t)=\nu^n(0)-\mu(0)+\frac{1}{\sqrt{n}}(\tilde{A}^n(t)-\tilde{S}^n(t)) + \int_0^t [\nu^n(s)\alpha(s,\nu^n(s))-\mu(s)\alpha(s,\mu(s))]\,ds$$

$$\sup_{s\in[0,t]}\|\nu^n(s)-\mu(s)\| = \|\nu^n(0)-\mu(0)\| + \sup_{s\in[0,t]}\frac{1}{\sqrt{n}}\|\tilde{A}^n(t)-\tilde{S}^n(t)\| + \int_0^t \underbrace{\|\nu^n(s)\alpha(s,\nu^n(s))-\mu(s)\alpha(s,\mu(s))\|}_{\le C\|\nu^n(s)-\mu(s)\|}\,ds$$

α is Lipschitz and bounded

$$\left\langle \tilde{A}^n_x(t)-\tilde{S}^n_x(t)\right\rangle = \frac{1}{n}\left\langle A^n_x(\cdot)-\int_0^{\cdot} n\lambda^n_x(s)ds\right\rangle(t) + \frac{1}{n}\left\langle S^n_x(\cdot)-\int_0^{\cdot} n\sigma^n_x(s)ds\right\rangle(t)$$

$$= \int_0^t [\lambda^n_x(s)+\sigma^n_x(s)ds < C$$

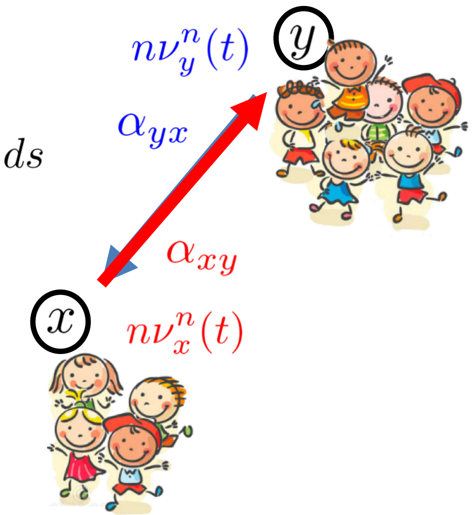
So, $\sup_{s\in[0,t]}\left\langle \frac{1}{\sqrt{n}}(\tilde{A}^n_x(t)-\tilde{S}^n_x(t))\right\rangle \le \frac{C}{n}$ and BDG (with •) imply

$$\mathbb{E}\Big[\sup_{s\in[0,t]}\|\nu^n(s)-\mu(s)\|\Big] \le \|\nu^n(0)-\mu(0)\| + \frac{C}{n} + \int_0^t \mathbb{E}\,\|\nu^n(s)-\mu(s)\|\,ds$$

Grönwall's and Markov's inequalities….

Proof of Theorem 2 – Diffusion Limit

$$\nu^n(t) - \mu(t) = \nu^n(0) - \mu(0) + \frac{1}{\sqrt{n}}(\tilde{A}^n(t) - \tilde{S}^n(t)) + \int_0^t [\nu^n(s)\alpha(s,\nu^n(s)) - \mu(s)\alpha(s,\mu(s))]\, ds$$



Setting $\psi^n(t) := \sqrt{n}(\nu^n(t) - \mu(t))$, and rearranging the above,

$$\psi^n(t) - \psi^n(0) = \underbrace{(\tilde{A}^n(t) - \tilde{S}^n(t))}_{C\text{-tight}} + \int_0^t b^n(s)ds$$

$$\left\langle \tilde{A}^n_x(t) - \tilde{S}^n_x(t), \tilde{A}^n_y(t) - \tilde{S}^n_y(t) \right\rangle = \begin{cases} -\int_0^t \nu^n_y(s)\alpha_{yx}(s,\nu^n(s)) - \nu^n_x(s)\alpha_{xy}(t,\nu^n(s)), & x \neq y \\ \int_0^t [\lambda^n_x(s) + \sigma^n_x(s)]ds, & x = y \end{cases}$$

Martingale CLT implies $\tilde{A}^n - \tilde{S}^n \Rightarrow \int_0^\cdot \Sigma(s)dB(s)$

Some technicalities: C-tightness arguments, together with the fluid limit, boundedness and Lipschitz continuity of α^*, $\nabla_\eta a^*$, Grönwall's….

$$\int_0^\cdot b^n(s)ds \Rightarrow \text{ required drift}$$

THANK YOU!

NICE YILLARA METE!

Large Deviation Principle

Set $\mu^n(t) := \frac{1}{n}\sum_i \delta_{X^i(t)}$.

Theorem 1 (Bayraktar and Cohen)

Assume that $\exists \lim_{n\to\infty} \mu^n(0)$. The empirical distribution μ^n satisfies a sample path *large deviation principle* with a good rate function I (with explicit form).
That is, for every

closed set F of flow of measures $\quad \liminf_n \frac{1}{n} \log \mathbb{P}(\mu^n \in F) \le -\inf_{\gamma\in F} I(\gamma)$

open set E of flow of measures $\quad \liminf_n \frac{1}{n} \log \mathbb{P}(\mu^n \in E) \ge -\inf_{\gamma\in E} I(\gamma)$

Proof. Each player uses the control a^*, which is regular. The proof now follows from the results about interacting particles given in

P. Dupuis, K. Ramanan, and W. Wu. *Large deviation principle for finite-state mean field interacting particle systems. arXiv,* 2016

□