Soner and Touzi's Stochastic Target Approach

Application to almost sure hedging with price impact

B. Bouchard

Ceremade - Univ. Paris-Dauphine, PSL University

Based on Soner and Touzi's discoveries and joint works with G. Loeper (Monash Univ.), M. Soner (ETH Zürich), C. Zou (NUS) and Y. Zou (ex Paris-Dauphine)

Motivation

- BS and local (stochastic) vol models :
  - Are useful because they provide a clear hedging rule
  - Disregard frictions because do not work at high frequency
  - Taking costs into account would lead to useless degenerate prices/strategies (in theory) and is helpless

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- However :
  - Do not take price impact and illiquidity into account
  - Problematic when large positions (possibly shared) or illiquid underlying (may run after the delta)
- Question : Can we built a model which
  - Takes price impact and illiquidity into account
  - Leads to a clear hedging and pricing rule
  - Does not have embedded hidden transaction costs (otherwise the super-hedging price would be degenerate)

Some references

□ Many works on hedging with illiquidity or impact : Sircar and Papanicolaou 98, Schönbucher and Wilmot 00, Frey 98, Liu and Yong 05, Cetin, Jarrow and Protter 04, Cetin, Soner and Touzi 09, Almgren and Li 13, Millot and Abergel 11, Guéant and Pu 13,...

□ Illiquidity + impact + perfect hedging : Loeper 14 (updated in 16, verification arguments).

□ Past and ongoing related works by T. Bilarev and D. Becherer.

Impact rule and continuous time trading dynamics

Impact rule

□ Basic rule (only permanent for the talk) : an order of δ units moves the price by

$$X_{t-} \longrightarrow X_t = X_{t-} + \delta f(X_{t-}),$$

and costs

$$\delta X_{t-} + \frac{1}{2}\delta^2 f(X_{t-}) = \delta \frac{X_{t-} + X_t}{2}.$$

□ We just model the curve around $\delta = 0$ as will pass to continuous time (infinitesimal) rebalancements (could be more general away from 0).

□ One could add a resilience effect in the following.

Trading signal and discrete trading dynamics

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- The stock price evolves according to

$$X = X_{t_i^n} + \int_{t_i^n}^{\cdot} \sigma(X_s) dW_s$$

between two trades (can add a drift or be multivariate).

- The corresponding dynamics are

$$Y_t^n := \sum_{i=0}^{n-1} Y_{t_i^n} \mathbf{1}_{\{t_i^n \le t < t_{i+1}^n\}} + Y_T \mathbf{1}_{\{t=T\}} \ , \ \delta_{t_i^n}^n = Y_{t_i^n}^n - Y_{t_{i-1}^n}^n$$

$$X^n = X_0 + \int_0^\cdot \sigma(X_s^n) dW_s + \sum_{i=1}^n \mathbf{1}_{[t_i^n, T]} \delta_{t_i^n}^n f(X_{t_i^n -}^n),$$

$$V^n = V_0 + \int_0^\cdot Y_{s-}^n dX_s^n + \sum_{i=1}^n \mathbf{1}_{[t_i^n, T]} \frac{1}{2} (\delta_{t_i^n}^n)^2 f(X_{t_i^n -}^n),$$

where

$$V^n = \text{ cash part } + Y^n X^n = \text{“portfolio value”}.$$

□ Passing to the limit $n \to \infty$, it converges in $\mathbf{S}_2$ to

$$
\begin{aligned}
Y &= Y_0 + \int_0^{\cdot} b_s ds + \int_0^{\cdot} a_s dW_s \\
X &= X_0 + \int_0^{\cdot} \sigma(X_s) dW_s + \underbrace{\int_0^{\cdot} f(X_s) dY_s + \int_0^{\cdot} a_s(\sigma f')(X_s) ds}_{(Y^n_{t^n_i} - Y^n_{t^n_{i-1}}) f(X^n_{t^n_i -})} \\
V &= V_0 + \int_0^{\cdot} Y_s dX_s + \frac{1}{2} \underbrace{\int_0^{\cdot} a_s^2 f(X_s) ds}_{(Y^n_{t^n_i} - Y^n_{t^n_{i-1}})^2 f(X^n_{t^n_i -})} ,
\end{aligned}
$$

at a speed $\sqrt{n}$.

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□ See B., Loeper and Zou 2016 for the un-covered case : surprisingly (?) the picture is very different !

Let us assume that we use the delta-hedging rule (as in Black and Scholes) :

$$V = \mathrm{v}(\cdot, X) \quad , \quad Y = \partial_x \mathrm{v}(\cdot, X).$$

Then, equating the dt terms implies

$$\frac{1}{2} a^2 f(X) = \partial_t \mathrm{v}(\cdot, X) + \frac{1}{2} (\sigma + af)^2 (X) \partial^2_{xx} \mathrm{v}(\cdot, X),$$

and applying Itô's Lemma to $Y - \partial_x \mathrm{v}(\cdot, X)$ leads to

$$\frac{a}{\sigma + fa} = \partial^2_{xx} \mathrm{v}(\cdot, X).$$

By a little bit of algebra :

$$\left[-\partial_t \mathrm{v} - \frac{1}{2} \frac{\sigma^2}{(1 - f\partial^2_{xx}\mathrm{v})} \partial^2_{xx} \mathrm{v} \right] (\cdot, X) = 0.$$

The pricing pde should be

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\begin{aligned}
-\partial_t \mathrm{v} - \frac{1}{2}\frac{\sigma^2}{(1 - f\partial^2_{xx}\mathrm{v})}\partial^2_{xx}\mathrm{v} = 0 \quad \text{on } [0, T) \times \mathbb{R}, \\
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- One possibility : add a gamma constraint $\partial^2_{xx}\mathrm{v} \leq \bar{\gamma}$ with $f\bar{\gamma} < 1$.

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Singular pde : take care of $1 \neq f\partial^2_{xx}\mathrm{v}$
- One possibility : add a gamma constraint $\partial^2_{xx}\mathrm{v} \leq \bar{\gamma}$ with $f\bar{\gamma} < 1$.
- A constraint of the form $f\partial^2_{xx}\mathrm{v} > 1$ does not make sense.

Hedging with a gamma contraint

Reformulation of the dynamics

$$dY = \gamma^a(X)dX + \mu_Y^{a,b}(X)dt \;\text{ and }\; dX = \sigma^a(X)dW + \mu_X^{a,b}(X)dt.$$

□ We now define v with respect to the gamma constraint

$$\gamma^a(X) \leq \bar{\gamma}(X)$$

with

$$f\bar{\gamma} \leq 1 - \varepsilon, \;\; \varepsilon > 0.$$

Pricing pde :

$$\min\left\{-\partial_t \mathrm{v} - \frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{v})}\partial^2_{xx}\mathrm{v}\ ,\ \bar{\gamma}-\partial^2_{xx}\mathrm{v}\right\} = 0 \quad \text{on } [0,T)\times\mathbb{R}.$$

Propagation of the gamma contraint at the boundary :

$$\mathrm{v}(T-,\cdot) = \hat{g} \quad \text{on } \mathbb{R}$$

with $\hat{g}$ the smallest (viscosity) super-solution of

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□ Perfect hedging : Smooth solution under additional conditions, leading to perfect hedging by following $Y = \partial_x \mathrm{v}(\cdot, X)$.

Super-solution property

Use a weak formulation approach and results on small time behavior of double stochastic integrals, see Soner and Touzi 00 and Cheridito, Soner and Touzi 05.

It is based on the Geometric DPP (Soner and Touzi) :
if

$$V_0 > \mathrm{v}(0, X_0)$$

then we can find (a, b, Y_0) such that

$$V_\theta \geq \mathrm{v}(\theta, X_\theta)$$

for any stopping time θ with values in $[0, T]$.

Sub-solution property

□ Main difficulty : can not establish the reverse Geometric DPP, i.e.

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□ Problem :
- at θ we have a position Y_θ that may not match with the position $\hat{Y}_\theta$ associated to $\mathrm{v}(\theta, X_\theta)$. Can not jump from Y_θ to $\hat{Y}_\theta$...
- can neither go smoothly to it as it will move X because of the impact, and therefore $\hat{Y}$ (sort of fixed point problem), compare with Cheridito, Soner, and Touzi 2005.

The smoothing approach

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Step 1. Using Perron's method + comparison, construct a (bounded) viscosity solution w^{ι} of

$$\min \left\{ -\partial_t \varphi - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial^2_{xx}\varphi)} \partial^2_{xx}\varphi \ , \ \bar{\gamma} - \partial^2_{xx}\varphi \right\} = 0 \quad \text{on } [0, T) \times \mathbb{R},$$

with terminal condition

$$\mathrm{w}^{\iota}(T, \cdot) = \hat{g} + \iota \quad \text{on } \mathbb{R}$$

with $\iota > 0$.

Step 2. Up to replacing w^{ι} by an approximating sequence of quasi-concave functions (by quadratic inf-convolution), we can assume that w^{ι} is quasi-concave

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with $\partial^2_{xx}\mathrm{w}^\iota$ the density of the absolute continuous part of the second order derivative measure

See Jensen 88.

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with $\partial^2_{xx}\mathrm{w}^\iota$ the density of the absolute continuous part of the second order derivative measure, and

$$\mathrm{w}^\iota(T,\cdot) \geq \hat{g} + \iota/2.$$

See Jensen 88.

Step 3. Consider a (non-negative) smooth kernel ψ with support $[-1, 0] \times [-1, 1]$, take a window size $\delta > 0$, and set

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$$0\le \min\left\{-\partial_t\mathrm{w}^\iota-\frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w}^\iota)}\partial^2_{xx}\mathrm{w}^\iota\ ,\ \bar\gamma-\partial^2_{xx}\mathrm{w}^\iota\right\}\star\psi_\delta$$

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The pde operator is concave

$$\begin{aligned}
0 &\le \min\left\{-\partial_t\mathrm{w}^\iota-\frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w}^\iota)}\partial^2_{xx}\mathrm{w}^\iota\ ,\ \bar\gamma-\partial^2_{xx}\mathrm{w}^\iota\right\}\star\psi_\delta\\
&\le \min\left\{-\partial_t\mathrm{w}^\iota\star\psi_\delta-\frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w}^\iota\star\psi_\delta)}\partial^2_{xx}\mathrm{w}^\iota\star\psi_\delta,\bar\gamma-\partial^2_{xx}\mathrm{w}^\iota\star\psi_\delta\right\}
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The pde operator is concave decreasing, and $\partial^2_{xx}\mathrm{w}^\iota_\delta \le \partial^2_{xx}\mathrm{w}^\iota \star \psi_\delta$ (by quasi-concavity),

$$\begin{aligned}
0 &\le \min\left\{-\partial_t \mathrm{w}^\iota - \frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w}^\iota)}\partial^2_{xx}\mathrm{w}^\iota \;,\; \bar{\gamma} - \partial^2_{xx}\mathrm{w}^\iota\right\}\star\psi_\delta \\
&\le \min\left\{-\partial_t \mathrm{w}^\iota\star\psi_\delta - \frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w}^\iota\star\psi_\delta)}\partial^2_{xx}\mathrm{w}^\iota\star\psi_\delta, \bar{\gamma} - \partial^2_{xx}\mathrm{w}^\iota\star\psi_\delta\right\} \\
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\end{aligned}$$

while, for δ small with respect to ι,

$$\mathrm{w}^\iota_\delta(T,\cdot)\ge\hat{g}.$$

Step 4. We have produced a smooth function satisfying

$$\min\left\{-\partial_t \mathrm{w}_\delta^\iota - \frac{1}{2}\frac{\sigma^2}{(1-f\partial_{xx}^2 \mathrm{w}_\delta^\iota)}\partial_{xx}^2 \mathrm{w}_\delta^\iota \ , \ \bar{\gamma} - \partial_{xx}^2 \mathrm{w}_\delta^\iota\right\} \geq 0$$

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Taking

$$V=\mathrm{w}_\delta^\iota(\cdot,X)\quad\text{and}\quad Y=\partial_x\mathrm{w}_\delta^\iota(\cdot,X),$$

we obtain

$$V_T\geq \hat{g}(X_T)\geq g(X_T).$$

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This implies that $\mathrm{v} \leq \mathrm{w}_\delta^\iota \to \mathrm{w}^\iota$, as $\delta \to 0$.

Step 5. Since w^ι is solution of

$$\min\left\{-\partial_t \mathrm{w}^\iota - \frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{w}^\iota)}\partial^2_{xx}\mathrm{w}^\iota\ ,\ \bar{\gamma}-\partial^2_{xx}\mathrm{w}^\iota\right\}=0$$

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with

$$\mathrm{w}(T,\cdot) = \hat{g}.$$

It satisfies $\mathrm{w} \leftarrow \mathrm{w}^\iota \geq \mathrm{v}$.

Step 6. But v is a super-solution of the same equation : $\mathrm{w} \leq \mathrm{v}$ by comparison, and therefore $\mathrm{w} = \mathrm{v}$ by the above.

To sum up :

$$\underbrace{\mathrm{v}}_{\text{super-solution}} \geq \underbrace{\mathrm{w}}_{\text{solution}} \underbrace{\longleftarrow}_{\delta,\iota\to 0} \underbrace{\mathrm{w}_{\delta}^{\iota}}_{\text{super-hedging}} \geq \mathrm{v}$$

Extensions with Mete

□ A general impact function :

$$
\begin{aligned}
X &= x + \int_t^{\cdot} \mu(s, X_s, \gamma_s, b_s)ds + \int_t^{\cdot} \sigma(s, X_s, \gamma_s)dW_s \\
Y &= y + \int_t^{\cdot} b_s ds + \int_t^{\cdot} \gamma_s dX_s \\
V &= v + \int_t^{\cdot} F(s, X_s, \gamma_s)ds + \int_t^{\cdot} Y_s dX_s
\end{aligned}
$$

□ A general impact function :

$$X = x + \int_t^{\cdot} \mu(s, X_s, \gamma_s, b_s) ds + \int_t^{\cdot} \sigma(s, X_s, \gamma_s) dW_s$$

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$$V = v + \int_t^{\cdot} F(s, X_s, \gamma_s) ds + \int_t^{\cdot} Y_s dX_s$$

□ Relaxation of the gamma constraint. Can be as close as one wants to the singularity :

$$\min\{-\partial_t \mathrm{v} - \bar{F}(\cdot, \partial^2_{xx}\mathrm{v}) \, , \; \bar{\gamma} - \partial^2_{xx}\mathrm{v}\} = 0 \text{ on } [0, T) \times \mathbb{R},$$

where

$$\bar{F}(t, x, z) := \frac{1}{2}\sigma(t, x, z)^2 z - F(t, x, z)$$

and

$$\{\bar{F} < \infty\} = \{F < \infty\} = \{(t, x, z) : z < \bar{\gamma}(t, x)\}.$$

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□ For this, we need a-priori estimates : If u with $\partial^2_{xx} u < \bar{\gamma}$ solves the PDE, then $w := \bar{F}(\cdot, \partial^2_{xx} u)$ solves

$$\partial_t w + \partial_z \bar{F}(\cdot, \partial^2_{xx} u) \partial^2_{xx} w = \frac{\partial_t \bar{F}(\cdot, \partial^2_{xx} u)}{\bar{F}(\cdot, \partial^2_{xx} u)} w.$$

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Then,

$$w(t, x) = \mathbb{E}[w(T, \bar{X}^{t,x}_T) e^{-\int_t^T (\partial_t \bar{F}(\cdot, \partial^2_{xx} u) / \bar{F}(\cdot, \partial^2_{xx} u))(s, \bar{X}^{t,x}_s) ds}]$$

where $\bar{X} = x + \int_t^{\cdot} (2 \partial_z \bar{F}(\cdot, \partial^2_{xx} u)(s, \bar{X}_s))^{\frac{1}{2}} dW_s$.

Provides a uniform bound if $\partial^2_{xx} u(T, \cdot) \leq \bar{\gamma} - \iota$ with $\iota > 0$.

Expansion around 0 impact

- Scaling :

$$X = x + \int_t^{\cdot} \mu(s, X_s, \epsilon\gamma_s, b_s)ds + \int_t^{\cdot} \sigma(s, X_s, \epsilon\gamma_s)dW_s$$

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- Expansion performed around the solution v^0 of $(\partial_z \bar{F}(\cdot, 0) =: \partial_z \bar{F}_0)$

$$\partial_t \mathrm{v}^0 + \partial_z \bar{F}(\cdot, 0) \partial^2_{xx} \mathrm{v}^0 = 0 \text{ on } [0, T) \times \mathbb{R} \text{ and } \mathrm{v}^0(T, \cdot) = \hat{g} \text{ on } \mathbb{R}.$$

Expansion around 0 impact

□ **Proposition :**

$$
\begin{aligned}
\mathrm{v}^{\epsilon}(0,x) =& \mathrm{v}^0(0,x) + \frac{\epsilon}{2}\mathbb{E}\left[\int_0^T [\partial^2_{zz}\bar{F}_0|\partial^2_{xx}\mathrm{v}^0|^2](s,\tilde{X}^0_s)ds\right] + o(\epsilon)\\
=& \mathrm{v}^0(0,x) + \frac{\epsilon}{2}\,\mathbb{E}\left[\partial_x\hat{g}(T,\tilde{X}^0_T)\tilde{Y}_T\right] + o(\epsilon)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{X}^z &= x + \int_t^{\cdot}(2\partial_z\bar{F}(\cdot,z\partial^2_{xx}\mathrm{v}^0))^{\frac{1}{2}}(s,\tilde{X}^z_s)dW_s,\\
\tilde{Y} &= \frac{1}{\sqrt{2}}\int_t^{\cdot}\frac{\partial_x\partial_z\bar{F}_0(s,\tilde{X}^0_s)\tilde{Y}_s + \partial^2_{zz}\bar{F}_0\partial^2_{xx}\mathrm{v}^0(s,\tilde{X}^0_s)}{\sqrt{\partial_z\bar{F}_0(s,\tilde{X}^0_s)}}dW_s.
\end{aligned}
$$

Expansion around 0 impact

□ **Proposition :**

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where

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□ The leading order term allows for super-hedging with $\mathbf{L}^{\infty}$-error controlled by ϵ^2.

Dual formulation

- In the concave case :

$$\mathrm{v}(t,x) = \sup_{\mathfrak{s}} \mathbb{E}\left[\hat{g}(X_T^{t,x,\mathfrak{s}}) - \int_t^T \bar{F}^*(s, X_s^{t,x,\mathfrak{s}}, \mathfrak{s}_s^2)ds\right]$$
$$= \sup_{\mathfrak{s}} \mathbb{E}\left[g(X_T^{t,x,\mathfrak{s}}) - \int_t^T \bar{F}^*(s, X_s^{t,x,\mathfrak{s}}, \mathfrak{s}_s^2)ds\right]$$

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□ In the previous model :

$$\bar{F}^*(t,x,s^2) = \frac{1}{2}\frac{(s-\sigma_\circ(t,x))^2}{f(x)}, \quad \text{for } s \geq 0.$$

□ Under additional smoothness, the optimum is achieved by

$$\hat{\mathfrak{s}}_{t,x} := (2\partial_z \bar{F}(\cdot, \partial_x^2 \mathrm{v})(\cdot, X^{t,x,\hat{\mathfrak{s}}_{t,x}}))^{\frac{1}{2}}$$

and

$$g(X_T^{t,x,\hat{\mathfrak{s}}_{t,x}}) = \hat{g}(X_T^{t,x,\hat{\mathfrak{s}}_{t,x}}).$$

Open problems :

No constraint at all on the gamma ?

Existence/stability of FBSDE with impact ?

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Thank you !

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