On and Beyond Propagation of Singularities

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University of Rome "Tor Vergata"

METE - MATHEMATICS AND ECONOMICS: TRENDS AND EXPLORATIONS
ETH, Zürich June 4-8, 2018

A conference celebrating Mete Soner's 60th birthday and his contributions to Analysis, Control, Finance and Probability

organized by FRANCESCA DA LIO, NICOLE EL KAROUI, MARCEL NUTZ, MARTIN SCHWEIZER, JOSEF TEICHMANN

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A quote

K. KHANIN & A. SOBOLEVSKI, *On Dynamics of Lagrangian Trajectories for Hamilton-Jacobi Equations*, Arch. Rational Mech. Anal. 219 (2016)

> *The evolutionary Hamilton-Jacobi equation,*
>
> $$(HJ) \qquad \frac{\partial \phi}{\partial t} + H(t, x, \nabla \phi) = 0$$
>
> *appears in diverse mathematical models ranging from analytical mechanics to combinatorics, condensed matter, turbulence, and cosmology . . . In many of these applications the objects of interest are described by singularities of solutions, which inevitably appear for generic initial data after a finite time due to the nonlinearity of (HJ). Therefore one of the central issues both for theory and applications is to understand the behaviour of the system after singularities form.*

Overview

$\Omega \subset \mathbb{R}^n$ bounded $\boxed{H(x,u,Du) = 0 \qquad \text{a.e. in } \Omega}$

- $u : \Omega \to \mathbb{R}$ Lipschitz viscosity solution
- $p \mapsto H(x,u,p)$ is convex

The object of our study

$$\mathrm{Sing}(u) = \{x \in \Omega \mid \ \nexists Du(x)\}$$

Examples

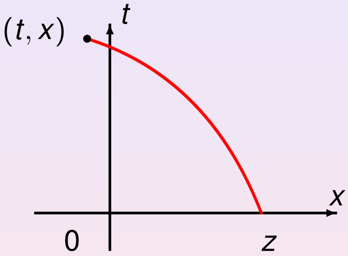
1. Hamilton-Jacobi equation

$$\begin{cases} u_t + H(t,x,D_x u) = 0 &]0,T[\times\mathbb{R}^n \\ u(0,x) = u_0(x) & x \in \mathbb{R}^n \end{cases}$$

2. weak KAM theory $\boxed{\frac{1}{2}|c + Du|^2 + V(x) = \alpha[c] \quad (x \in \mathbb{T}^n)}$

Characteristics

$$\begin{cases} u_t + H(t,x,D_x u) = 0 &]0,T[\times\mathbb{R}^n \\ u(0,x) = u_0(x) & x \in \mathbb{R}^n \end{cases}$$



by using characteristics:
on $[0,T]\times\mathbb{R}^n \setminus \overline{\mathrm{Sing}(u)}$
u is as smooth as the data (maximal regularity)

$$\begin{cases} \dot{x}(t) = D_p H\big(t, x(t), p(t)\big), & x(0) = z \\ \dot{p}(t) = -D_x H\big(t, x(t), p(t)\big), & p(0) = Du_0(z) \end{cases}$$

Calculus of Variations

Denote by $L : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the Legendre transform

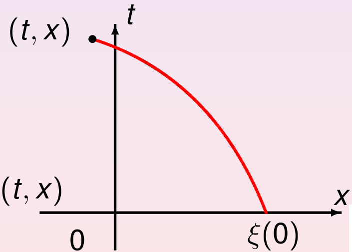
$$L(t, x, q) = \max_{p \in \mathbb{R}^n} \left[\langle q, p \rangle - H(t, x, p) \right]$$

The value function

$$u(t, x) = \inf_{\xi(t)=x} \left\{ \int_0^t L\big(s, \xi(s), \xi'(s)\big)\, dt + u_0\big(\xi(0)\big) \right\}$$

gives the viscosity solution of

$$\begin{cases} u_t(t, x) + H\big(t, x, D_x u(t, x)\big) = 0 \\ u(0, x) = u_0(x) \end{cases}$$



$\exists Du(t, x) \quad \Longleftrightarrow \quad$ unique minimizer at (t, x)

Outline

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1. The beginning
2. Use of (some) geometric measure theory
3. The discovery of singular dynamics
4. From local to global propagation
5. Beyond propagation of singularities
6. Concluding remarks

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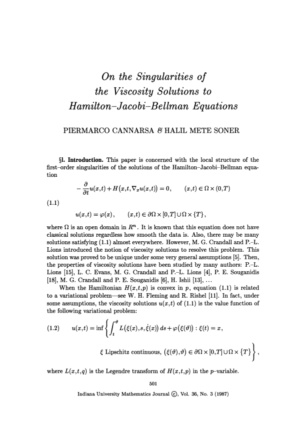
Outline

How this story began



On the Singularities of
the Viscosity Solutions to
Hamilton–Jacobi–Bellman Equations

PIERMARCO CANNARSA & HALIL METE SONER

§l. Introduction. This paper is concerned with the local structure of the first–order singularities of the solutions of the Hamilton–Jacobi–Bellman equation

$$(1.1)\quad \begin{aligned} &-\frac{\partial}{\partial t}u(x,t)+H\big(x,t,\nabla_x u(x,t)\big)=0, \qquad (x,t)\in\Omega\times(0,T) \\ &u(x,t)=\varphi(x), \qquad (x,t)\in\partial\Omega\times[0,T]\cup\Omega\times\{T\}, \end{aligned}$$

where Ω is an open domain in R^n. It is known that this equation does not have classical solutions regardless how smooth the data is. Also, there may be many solutions satisfying (1.1) almost everywhere. However, M. G. Crandall and P.–L. Lions introduced the notion of viscosity solutions to resolve this problem. This solution was proved to be unique under some very general assumptions [5]. Then, the properties of viscosity solutions have been studied by many authors: P.–L. Lions [15], L. C. Evans, M. G. Crandall and P.–L. Lions [4], P. E. Souganidis [18], M. G. Crandall and P. E. Souganidis [6], H. Ishii [13], …

When the Hamiltonian $H(x,t,p)$ is convex in p, equation (1.1) is related to a variational problem—see W. H. Fleming and R. Rishel [11]. In fact, under some assumptions, the viscosity solutions $u(x,t)$ of (1.1) is the value function of the following variational problem:

$$(1.2)\quad u(x,t)=\inf\left\{\int_t^{\vartheta} L\big(\xi(x),s,\dot\xi(x)\big)\,ds+\varphi\big(\xi(\vartheta)\big):\xi(t)=x,\right.$$
$$\left.\xi \text{ Lipschitz continuous, } \big(\xi(\vartheta),\vartheta\big)\in\partial\Omega\times[0,T]\cup\Omega\times\{T\}\right\},$$

where $L(x,t,q)$ is the Legendre transform of $H(x,t,p)$ in the p–variable.

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Indiana University Mathematics Journal ©, Vol. 36, No. 3 (1987)

Figure: how, where, and whom with...

The “discovery” of semiconcave functions

$\Omega \subseteq \mathbb{R}^n$ open
$u : \Omega \to \mathbb{R}$ semiconcave with modulus $\omega : [0, \infty[\to [0, \infty[$ if

$$\lambda u(x) + (1-\lambda)u(y) - u\big(\lambda x + (1-\lambda)y\big) \leqslant \lambda(1-\lambda)|x-y|\omega\big(|x-y|\big)$$

for all x, y such that $[x, y] \subset \Omega$ and $\lambda \in [0, 1]$

Special cases:

- $\omega(s) \equiv 0 \quad \longrightarrow \quad$ concave
- $\omega(s) = Cs \ (C > 0) \quad \longrightarrow \quad$ linearly semiconcave
  In this case, there is a concave function v such that
  $$u(x) = v(x) + \frac{C}{2}|x|^2 \qquad (\star)$$
- $\omega(s) = Cs^{\alpha} \ (C > 0, 0 < \alpha < 1) \ \longrightarrow$ fractionally semiconcave
  In this case, $(\star)$ is no longer valid

Further references on semiconcave functions

- control theory and sensitivity analysis
  Hrustalev 1978, C – Frankowska 1991
  Fleming – McEneaney 2000
  Rifford 2000, 2002
- nonsmooth and variational analysis
  Rockafellar 1982
  Colombo – Marigonda 2006, Colombo – Nguyen 2010
- differential geometry
  Perelman 1995, Petrunin 2007
- monographs
  C – Sinestrari (Birkhäuser 2004)
  Villani (Springer 2009)

Semiconcavity & nonsmooth analysis

For any semiconcave $u : \Omega \to \mathbb{R}$

- the superdifferential at $x \in \Omega$ coincides with Clarke's gradient

$$D^+u(x) = \text{ co } D^*u(x) = \partial u(x)$$

where $D^*u(x) = \left\{ \lim_{i\to\infty} Du(x_i) \;\middle|\; x_i \to x \right\}$ reachable gradients

- $D^+u(x) = \{p\} \quad \Longleftrightarrow \quad u \quad$ differentiable

Semiconcavity & Hamilton-Jacobi equations

For $u : \Omega \to \mathbb{R}$ semiconcave and $H \in \mathcal{C}\big(\Omega \times \mathbb{R} \times \mathbb{R}^n\big)$

- if u is a viscosity solution of $H\big(x, u, Du\big) = 0$ in Ω, then

$$H(x, u(x), p) = 0 \qquad \forall x \in \Omega, p \in D^* u(x)$$

- if $H(x, u, \cdot)$ convex, then

$$H\big(x, u, Du\big) = 0 \ \text{ a.e.} \quad \Longleftrightarrow \quad H\big(x, u, Du\big) = 0 \ \text{ viscosity}$$

- if $H(x, u, \cdot)$ strictly quasi-convex, then

$$x \in \mathrm{Sing}(u) \quad \Longleftrightarrow \quad \min_{p \in D^+ u(x)} H(x, u(x), p) < 0$$

Our first propagation result

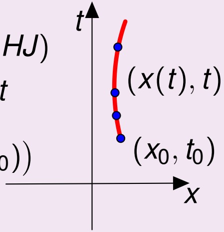
$$u_t + H(t, x, D_x u) = 0 \qquad \text{in } (0, T) \times \mathbb{R}^n \qquad (HJ)$$

Theorem (C – Soner 1987)

Let

- *u be a semiconcave a viscosity solution of (HJ)*
- *$(t_0, x_0) \in (0, T) \times \mathbb{R}^n$ and $\tau > 0$ be such that*

$$u \in \mathcal{C}^1\big(]t_0, t_0 + \tau[\times B_\tau(x_0)\big)$$



Then

$$u \in \mathcal{C}^1\big([t_0, t_0 + \tau[\times B_\tau(x_0)\big)$$

This shows that $(t_0, x_0) \in \mathrm{Sing}(u)$ propagates along a discrete set
Problem: how to connect these singular points with a singular line?

Towards the use of measure theory

- PC & H. M. Soner, On the singularities of viscosity solutions to Hamilton-Jacobi-Bellman equations, *Indiana Univ. Math. J.* 36 (1987), pp.501–524.
- PC & H. M. Soner, Generalized one-sided estimates for solutions of Hamilton-Jacobi equations and applications, *Nonlinear Analysis, Theory, Methods & Applications*, 13 (1989), pp.305– 323.
- L. Ambrosio, PC & H. M. Soner, On the propagation of singularities of semi-convex functions, *Ann. Scuola Norm. Sup. Pisa* 20 (1993), pp.597–616.

Semiconcave functions and rectifiability

$\Omega \subseteq \mathbb{R}^n$ open $u : \Omega \to \mathbb{R}$ semiconcave

Singular set

$$\text{Sing}(u) = \{x \in \Omega \mid \nexists Du(x)\} = \{x \in \Omega \mid \dim D^+u(x) \geqslant 1\}$$

can be stratified by looking at singular magnitude

$$\text{Sing}(u) = \cup_{j=1}^{n} \text{Sing}_j(u) \text{ with } \text{Sing}_j(u) := \{x \in \Omega \mid \dim D^+u(x) = j\}$$

Theorem

$\text{Sing}_j(u)$ countably $(n-j)$-rectifiable
$\text{Sing}(u)$ countably $(n-1)$-rectifiable

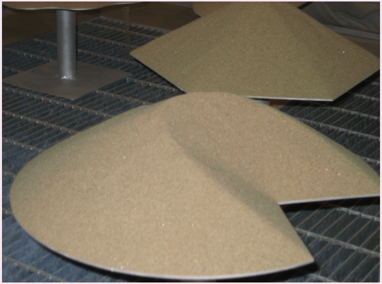
- Zajíček (1978), Veselý (1979) concave functions
- Alberti – Ambrosio – C (1992) general semiconcave functions

Singularities in the real world

The distance function from a set $S \subset \mathbb{R}^n$

$$d_S(x) = \inf_{y \in S} |x - y|$$

is locally semiconcave on $\mathbb{R}^n \setminus \overline{S}$



Closure of the singular set

$u : (0,T) \times \mathbb{R}^n \to \mathbb{R}$ semiconcave

$$\begin{cases} u_t(t,x) + H\big(t,x,D_x u(t,x)\big) = 0 & (t,x) \in (0,T) \times \mathbb{R}^n \\ u(0,x) = u_0(x) & x \in \mathbb{R}^n \end{cases} \qquad (HJ)$$

where, for some $k \geqslant 1$,

- $H = H(t,x,p) \in \mathcal{C}^{k+1}$ strictly convex and superlinear in p
- $u_0 \in \mathcal{C}^{k+1}(\mathbb{R}^n)$

Then $\mathrm{Sing}(u)$ countably n-rectifiable: what about $\overline{\mathrm{Sing}(u)}$?

By characteristics: $u \in \mathcal{C}^{k+1}\big([0,T] \times \mathbb{R}^n \setminus \overline{\mathrm{Sing}(u)}\big)$

Fleming 1969 (by a Sard-type argument)

$$\overline{\mathrm{Sing}(u)} \subseteq \mathrm{Sing}(u) \cup \mathrm{Conj}(u) \text{ and } \mathcal{H}^{n+1/k}\big(\mathrm{Conj}(u)\big) = 0$$

This is not enough to derive n-rectifiability

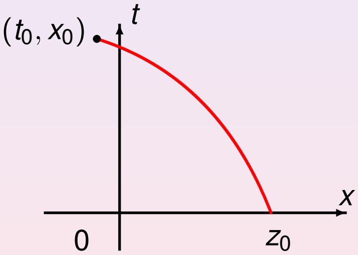
Conjugate points

$$\begin{pmatrix} x(t,z) \\ p(t,z) \end{pmatrix} \quad \begin{cases} \dot{x}(t) = D_pH(t,x(t),p(t)), & x(0)=z \\ \dot{p}(t) = -D_xH(t,x(t),p(t)), & p(0)=Du_0(z) \end{cases}$$

$(t_0, x_0) \in \mathrm{Conj}(u) \quad \Longleftrightarrow \quad \exists z_0$

such that

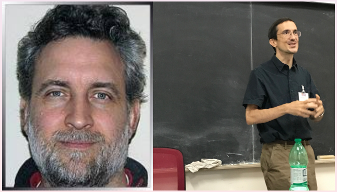
- $x_0 = x(t_0, z_0)$
- $x(\cdot, z_0)$ minimizer at (t_0, x_0)
- $\det \dfrac{\partial x}{\partial z}(t_0, z_0) = 0$



Rectifiability of the cut set

Theorem (C – Mennucci – Sinestrari 1997)

- $\overline{Sing(u)} = Sing(u) \cup Conj(u)$
- $Conj(u)$ *is countably* $\mathcal{H}^n$*-rectifiable (and so is* $\overline{Sing(u)}$*)*
- $\mathcal{H}^{n-1+2/k}\big(Conj(u) \setminus Sing(u)\big) = 0 \quad (k \geqslant 2)$
- $\mathcal{H} - \dim\big(Conj(u) \setminus Sing(u)\big) \leqslant n - 1 \quad (k = \infty)$



A fresh look at propagation of singularities

Do singularities of lower magnitude propagate?

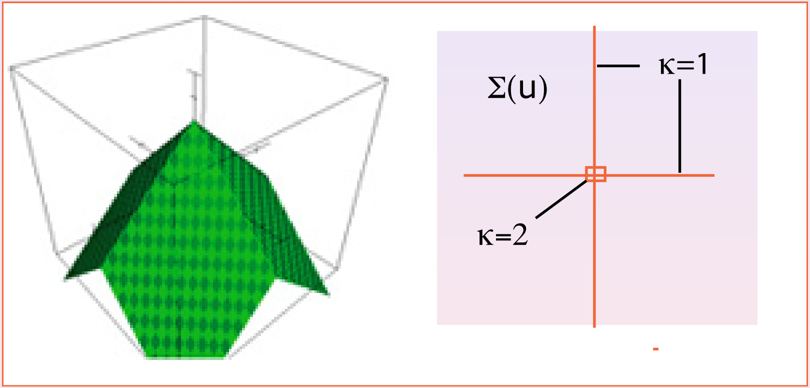


Figure: singularities of magnitude 1 do propagate along straight lines

A counterexample

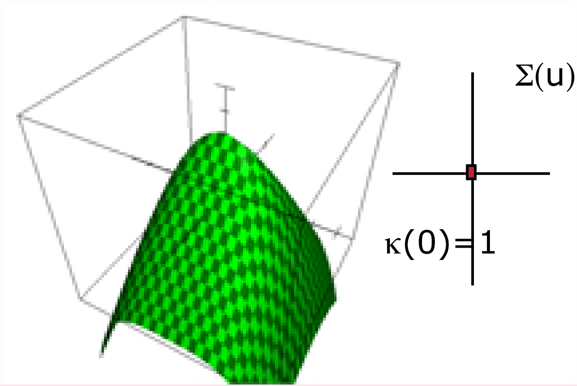


Figure: an isolated singularity of magnitude 1 at the origin

$$u(x,y) = 3 - \sqrt{\left(\frac{3x}{2}\right)^2 + \left(\frac{2y}{3}\right)^4}$$

A closer look at reachable gradients

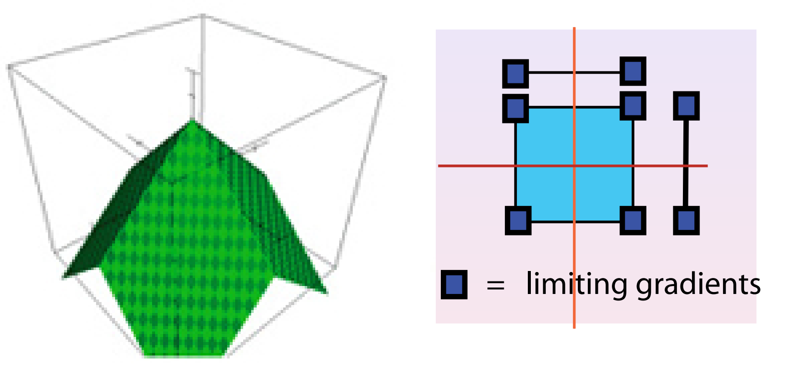


Figure: here $D^{*}u(0,0) \subsetneq \partial D^{+}u(0,0)$

A crucial test

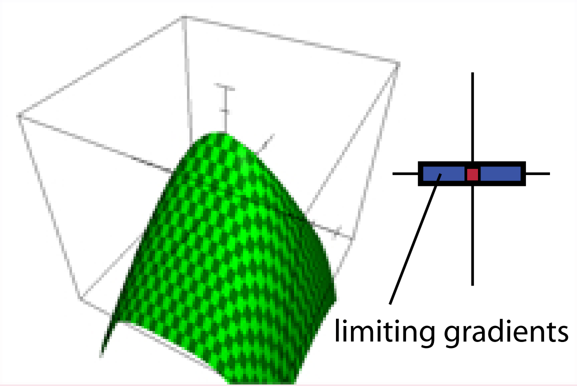


Figure: here $D^*u(0,0) = D^+u(0,0)$

The propagation principle

$\Omega \subseteq \mathbb{R}^n$ open $\quad u : \Omega \to \mathbb{R}$ semiconcave

Theorem (Albano – C 1999)

Let $x_0 \in Sing(u)$ *be such that* $\boxed{\partial D^+u(x_0) \setminus D^*u(x_0) \neq \varnothing}$
Fix any $p_0 \in \partial D^+u(x_0) \setminus D^*u(x_0)$ *and* $q_0 \in \mathbb{R}^n \setminus \{0\}$ *such that*

$$q_0 \cdot (p - p_0) \geqslant 0 \quad \forall p \in D^+u(x_0)$$

Then $\exists\, \xi(\cdot) : [0, \tau] \to \Omega$ *Lipschitz such that*

- $\begin{cases} \dot{\xi}(t) \in q_0 - p_0 + D^+u\big(x(t)\big) & t \in [0,\tau] \text{ a.e.} \\ \xi(0) = x_0 \end{cases}$
- $\xi(t) \in Sing(u) \qquad \forall t \in [0, \tau]$
- $\dot{\xi}^+(0) = q_0$

The role of generalized characteristics

$\xi(\cdot) : [0,\tau[\to \Omega$ $(0 < \tau \leqslant \infty)$ is a generalized characteristic for (u, H)

$$\dot{\xi}(t) \in co\, D_pH\Big(\xi(t), u(\xi(t)), D^+u(\xi(t))\Big) \quad \text{for a.e. } t \in [0,\tau[$$

Theorem (Albano – C 2000, Yu 2006, C – Yu 2009)

- $u : \Omega \to \mathbb{R}$ *semiconcave solution* $H(x, u, Du) = 0$
- $x_0 \in Sing(u)$ *such that* $0 \notin D_pH(x_0, u(x_0), D^+u(x_0))$

Then $\exists \xi : [0,\tau[\to \Omega$ *generalized characteristic for* (u, H) *such that*

$$\begin{cases} \xi(0) = x_0 \\ \xi(t) \in Sing(u) \quad \forall t \in [0,\tau[\\ \dot{\xi}^+(0) = D_pH(x_0, u(x_0), p_0) \\ \text{with } p_0 = \arg \min\limits_{p \in D^+u(x_0)} H(x_0, u(x_0), p) \end{cases}$$

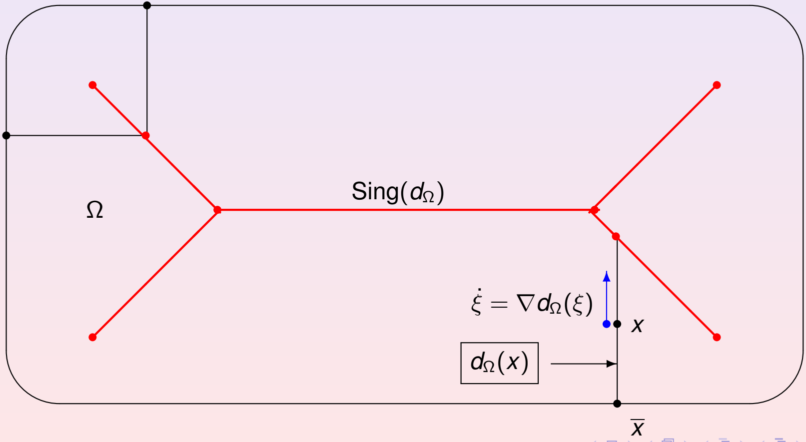
Further references on propagation of singularities

- Albano 2010, 2011, 2014
- Bogaevsky 2006
- Strömberg 2013
- Khanin – Sobolevski 2014
- Strömberg–Ahmadzdeh 2014
- C – Cheng – Zhang 2014

The Euclidean distance function

$\Omega \subset \mathbb{R}^n$ bounded open set $\boxed{d_\Omega(x) = \min_{y \in \partial\Omega} |x - y| \quad (x \in \overline{\Omega})}$

- $\mathrm{Sing}(d_\Omega) = \{x \in \Omega \mid \mathrm{proj}_{\partial\Omega}(x) \text{ multivalued}\} \neq \varnothing$ medial axis



Magic of the eikonal equation

Theorem (Albano – C – Khai T. Nguyen – Sinestrari 2013)

For any given $x_0 \in \Omega$ let $\xi : [0,\infty) \to \Omega$ be the unique solution of

$$\begin{cases} \dot{\xi}(t) \in D^+ d_\Omega\big(\xi(t)\big) & t \in [0,\infty)\ a.e. \\ \xi(0) = x_0 \end{cases}$$

Then

$$x_0 \in Sing(d_\Omega) \quad \Longrightarrow \quad \xi(t) \in Sing(d_\Omega) \quad \forall t \in [0,\infty)$$

A first topological application

Theorem (A. Lieutier, *Computer-Aided Design* 2004)

Ω *has the same homotopy type as* $Sing(d_\Omega)$

F. Wolter (1993): deformation retract technique works if

- $\Omega \subset \mathbb{R}^n$ and $\partial\Omega \in \mathcal{C}^2$
- $\Omega \subset \mathbb{R}^2$ and $\partial\Omega$ is piecewise $\mathcal{C}^2$

Proof.

Use generalized gradient flow $\xi(t,x)$

$$\begin{cases} \dot{\xi}(t) \in D^+ d_\Omega(\xi(t)) & t \in [0,\infty) \ a.e. \\ \xi(0) = x \end{cases}$$

to define homotopy $\mathbb{H} : \Omega \times [0,1] \to \Omega$ by $\mathbb{H}(x,t) = \xi(tT,x)$
where $T > 0$ is such that $\xi(T,x) \in \text{Sing}(d_\Omega)$ $\forall x \in \Omega$ $\square$

Weak KAM solutions on manifolds

M compact connected manifold

$u : M \to \mathbb{R}$ solution of

$$H(x, Du(x)) = 0 \quad (x \in M)$$

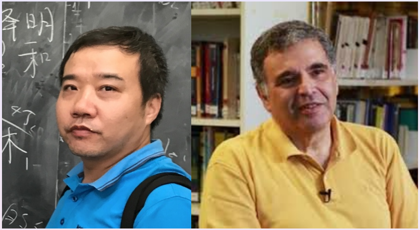


Figure: W. Cheng and A. Fathi

$\gamma : [a, b] \to M$ is u-calibrating if

$$u(\gamma(b)) - u(\gamma(a)) = \int_a^b L(\gamma(s), \dot\gamma(s))ds$$

The Cut and Aubry sets

$\boxed{\mathrm{Cut}(u) = \text{cut set of } u}$ consists of all $x \in M$ such that

$$x \in \gamma([a,b]) \text{ for some } u\text{-calibrating } \gamma \quad \Longrightarrow \quad x = \gamma(b)$$

$\boxed{\mathcal{I}(u) = \text{Aubry set of } u}$ consists of all $x \in M$ such that

$$x = \gamma(0) \text{ for some } u\text{-calibrating } \gamma : \mathbb{R} \to M$$

Observe $\quad \mathrm{Sing}(u) \subseteq \mathrm{Cut}(u) \subseteq \overline{\mathrm{Sing}(u)} \setminus \mathcal{I}(u) \subseteq M \setminus \mathcal{I}(u)$

Topology of singular sets

Theorem (C – Cheng – Fathi 2017)

$u : M \to \mathbb{R}$ solution of $H(x, Du(x)) = 0$
Then all the inclusions

$$Sing(u) \subseteq Cut(u) \subseteq \overline{Sing(u)} \setminus \mathcal{I}(u) \subseteq M \setminus \mathcal{I}(u)$$

are homotopy equivalences

http://dx.doi.org/10.1016/j.crma.2016.12.004

Corollary

For every connected component C of $M \setminus \mathcal{I}(u)$ the sets

$$Sing(u) \cap C, \quad Cut(u) \cap C, \quad \overline{Sing(u)} \cap C$$

are path-connected

The role of Lax-Oleinik operators

Let A_t be the minimal action

$$A_t(x,y) = \inf_{\xi} \Big\{ \int_0^t L\big(\xi(s), \dot{\xi}(s)\big)\,ds \;\Big|\; \xi(0) = x,\ \xi(t) = y \Big\}$$

where $L(x,v) = \max_{p \in T_x^* M} \big\{ \langle p, v \rangle - H(x,p) \big\}$. Then

$$\begin{aligned} T_t^- u(x) &= \inf_{y \in M} \{u(y) + A_t(x,y)\} &\longrightarrow \quad &\text{weak KAM solution} \\ T_t^+ u(x) &= \sup_{y \in M} \{u(y) - A_t(x,y)\} &\longrightarrow \quad &\text{propagation of } \mathrm{Sing}(u) \end{aligned}$$

Three milestones towards global propagation

(a) $\exists\, t_0 > 0$ such that $T_t^+ u \in C^1(M)$ for all $t \in]0, t_0]$ [Bernard 2007]

(b) $\operatorname{argmax}_{y \in M} \{u(y) - A_t(x,y)\} = \{y_x(t)\}\ \forall (t,x) \in [0,t_0] \times M$

Proof. For any $\xi : [0,t] \to M$ action-minimizer with $\xi(0) = x,\ \xi(t) = y$

$$\frac{\partial L}{\partial v}(x, \dot\xi(0)) = D(T_t^+ u)(x) \quad \Longrightarrow \quad \text{maximizer } y \text{ is unique} \qquad \square$$

(c) $x \in \mathrm{Sing}(u) \quad \Longrightarrow \quad y_x(t) \in \mathrm{Sing}(u)$ for all $t \in [0, t_0]$

Proof. For $\xi : [0,t] \to M$ as above $\frac{\partial L}{\partial v}(y_x(t), \dot\xi(t)) \in D^+u(y_x(t))$. So

$$y_x(t) \notin \mathrm{Sing}(u) \quad \Longrightarrow \quad \frac{\partial L}{\partial v}(y_x(t), \dot\xi(t)) = Du(y_x(t)) \qquad \square$$

For u-calibrating $\gamma :]-\infty, 0] \to M$ with $\gamma(0) = y_x(t)$ we have

$$\frac{\partial L}{\partial v}(y_x(t), \dot\gamma(0)) = Du(y_x(t)) = \frac{\partial L}{\partial v}(y_x(t), \dot\xi(t)) \quad \Longrightarrow \quad \xi(s) = \gamma(t-s)\ \forall s \in [0,t]$$

Contradiction: $x = \xi(0)$ and ξ is u-calibrating $\square$

Higher dimensional singular manifolds

$$p_0 \in D^+u(x_0)\,, \quad N_{p_0} = \Big\{ q \in \mathbb{R}^n \;\Big|\; |q| = 1\,,\; q \cdot (p - p_0) \geq 0\,,\; \forall p \in D^+u(x_0) \Big\}$$

Theorem

$u : \Omega \to \mathbb{R}$ *semiconcave* $x_0 \in Sing(u)$

$$\varnothing \neq \partial D^+u(x_0) \setminus D^*u(x_0) \ni p_0$$

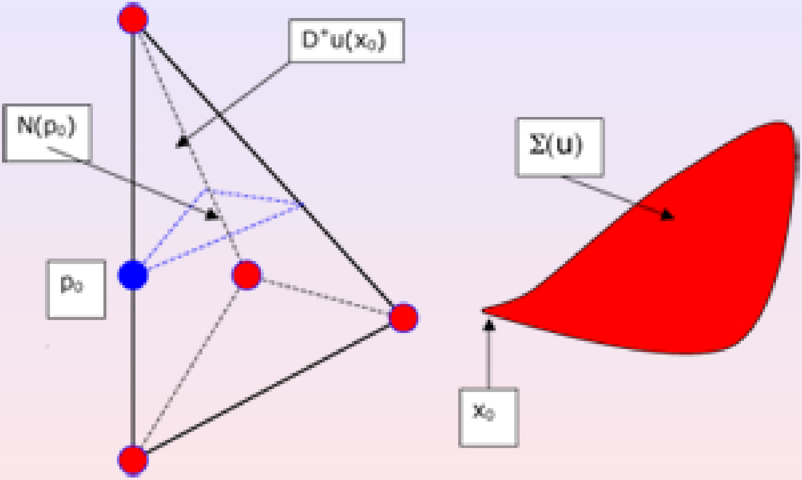
Then $\exists \tau > 0$ & $f : [0, \tau] \times N_{p_0} \to Sing(u)$ *Lipschitz such that*

1. *for all* $q \in N_{p_0}$, $f(\cdot, q)$ *solves*

$$\begin{cases} \partial_s f(s,q) \in q - p_0 + D^+u(f(s,q)) & \text{for a.e } \quad s \in [0,\tau] \\ f(0,q) = x_0 \end{cases}$$

2. $\partial_s^+ f(0,q) = q$
3. *for* $\nu = 1 + \dim_{\mathcal{H}} N_{p_0} = \dim N_{D^+u(x_0)}(p_0)$ *we have that*

$$\liminf_{r \to 0^+} r^{-\nu} \mathcal{H}^\nu \Big(f\big([0,\tau] \times N_{p_0}\big) \cap B_r(x_0) \Big) > 0$$


$D^+u(x_0)$
$N(p_0)$
p_0
$\Sigma(u)$
x_0

Singularities and critical points

For $c \in \mathbb{R}^N$ let u_c be a solution of

$$H(x, c + Du(x)) = \alpha[c] \qquad (x \in \mathbb{T}^N)$$

where

$$H(x,p) = \frac{1}{2}\langle A(x)p, p\rangle + V(x) \qquad (A > 0 \text{ and } \max_{\mathbb{T}^n} V = 0)$$

Define

$$v_c(x) = u_c(x) + \langle c, x\rangle \quad (x \in \mathbb{R}^n)$$

Theorem (C – Cheng 2018)

Any bounded connected component of $Sing(v_c)$ contains a critical point of v_c

Problem: asymptotic behaviour of singular characteristics in connection with relevant invariant sets (Mather and Aubry)

Nonconvex Hamiltonians

It would be extremely interesting to extend part of this theory to nonconvex Hamiltonians

- L. C. EVANS, Envelopes and nonconvex Hamilton-Jacobi equations, *Calc. Var. Partial Differ. Equ.* 50, No. 1-2, 257-282 (2014)
- A. A. MELIKYAN, *Generalized characteristics of first order PDEs. Applications in optimal control and differential games*, Boston, MA: Birkhäuser (1998)

Thank you for your attention and. . .

Happy Birthday Mete!