A new construction of the second order master equation in mean field games with common noise

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A few works of Mete when I started...

- HM Soner (1986), *Optimal control with state-space constraint,* SIAM Journal on Control and Optimization 24 (3), 552-561
- LC Evans, HM Soner, PE Souganidis (1992), *Phase transitions and generalized motion by mean curvature*, Communications on Pure and Applied Mathematics 45 (9), 1097-1123
- HM Soner (1993), *Motion of a set by the curvature of its boundary*, Journal of Differential Equations, 101(2), 313-372.
- L. Ambrosio, P. Cannarsa, HM Soner (1993). *On the propagation of singularities of semi-convex functions.* Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, 20(4), 597-616.
- WH Fleming, HM Soner (1993), CONTROLLED MARKOV PROCESSES AND VISCOSITY SOLUTIONS.

The second order master equation reads :

$$
(\mathbf{M2}) \quad \begin{cases}
-\partial_t U - (1+\beta)\Delta_x U + H(x, D_x U, m) \\
\qquad -(1+\beta)\int_{\mathbb{R}^d} \mathrm{div}_y \left[D_m U\right] \; dm(y) + \int_{\mathbb{R}^d} D_m U \cdot H_p(y, D_x U, m) \; dm(y) \\
\qquad -2\beta \int_{\mathbb{R}^d} \mathrm{div}_x \left[D_m U\right] dm(y) - \beta \int_{\mathbb{R}^{2d}} \mathrm{Tr}\left[D^2_{mm} U\right] \; dm \otimes dm = 0 \\
\qquad\qquad \text{in } [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \\
U(T, x, m) = G(x, m) \qquad \text{in } \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)
\end{cases}
$$

where

- $\beta \geq 0$ is the level of common noise,
- $H = H(x, p, m) : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is a standard Hamiltonian in (x, p), non local and smoothing in m,
- the coupling function $G : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is nonlocal and smoothing.

Some results on the master equation : Lasry-Lions ('13), Buckdahn-Li-Peng-Rainer ('14), Gangbo-Swiech ('14), Bessi ('15), Chassagneux-Crisan-Delarue ('15), C.-Delarue-Lasry-Lions (2015), Lacker-Webster ('15), Ahuja ('16), Carmona-Delarue's monograph (2017),...

Aim of our work : Provide a new construction of solutions for (**M2**).

Outline

1. Interpretation of the master equation
2. Construction of a solution for (**M2**)
3. Uniqueness

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1 Interpretation of the master equation

2 Construction of a solution for (**M2**)

3 Uniqueness

Two approches :

- As limit of Nash equilibria for symmetric $N-$player games,
- Symmetric Nash equilibria in a game with infinitely many players.

First approach : Limit of $N-$player game

- Let $N \in \mathbb{N}^*$ be the (large) number of players.
- Player $i \in \{1, \dots, N\}$ controls a dynamics of the form

$$dX_t^i = \alpha_t^i dt + \sqrt{2} dB_t^i + \sqrt{2\beta} dW_t, \; t \in [0,T] \qquad X_0^i = \bar{X}_0^i$$

where $\bar{X}_0^i$ is fixed, (α^i) is her control and (B^i) and W are i.i.d. BM. She aims at minimizing

$$J^i(\alpha^i, (\alpha^j)_{j \neq i}) = \mathbb{E}\left[\int_0^T L^{N,i}(X_s^i, \alpha_s^i, (X_s^j)_{j \neq i}) ds + G^{N,i}(X_T^i, (X_T^j)_{j \neq i}) \right].$$

- The pair $(\bar{\alpha}^1, \dots, \bar{\alpha}^N)$ is a Nash equilibrium if : $\forall i \in \{1, \dots, N\}$,

$$J^i(\alpha^i, (\bar{\alpha}^j)_{j \neq i}) \geq J^i(\bar{\alpha}^i, (\bar{\alpha}^j)_{j \neq i})$$

for any control α^i.

Limit of $N-$player game (continued)

- A Verification Theorem. Assume that the maps $v^{N,i} : [0,T] \times (\mathbb{R}^d)^N \to \mathbb{R}$ solves

$$\left\{ \begin{array}{l} -\partial_t v^{N,i}(t,\boldsymbol{x}) - \sum_{j=1}^{N} \Delta_{x_j} v^{N,i}(t,\boldsymbol{x}) - \beta \sum_{j,k=1}^{N} \mathrm{Tr} D^2_{x_j,x_k} v^{N,i}(t,\boldsymbol{x}) \\ \qquad +H^{N,i}(x_i, D_{x_i} v^{N,i}(t,\boldsymbol{x}), (x_j)_{-i}) \\ \qquad + \sum_{j\neq i} D_p H^{N,j}(x_j, D_{x_j} v^{N,j}(t,\boldsymbol{x}), (x_k)_{-j}) \cdot D_{x_j} v^{N,i}(t,\boldsymbol{x}) = 0 \qquad \text{in } [0,T] \times (\mathbb{R}^d)^N, \\ v^{N,i}(T,\boldsymbol{x}) = G^{N,i}(x) \qquad \text{in } (\mathbb{R}^d)^N. \end{array} \right.$$

where $H^{N,i}(x,p,z) = \sup_\alpha -\alpha \cdot p - L^{N,i}(x,\alpha,z)$. Then

$$(\bar{\alpha}^1, \dots, \bar{\alpha}^N) := (-D_p H^{N,1}(x_1, Dv^{N,1}, (x_i)_{-1}), \dots, -D_p H^{N,N}(x_N, Dv^{N,N}, (x_i)_{-N}))$$

is a Nash equilibrium.

Limit of $N-$player game (continued)

- Assume that the players are symmetric :

$$H^{N,i}(x_i, p, (x_j)_{-i}) = H(x_i, p, m_{\boldsymbol{x}}^{N,i}), \qquad G^{N,i}(x_i, (x_j)_{-i}) = G(x_i, m_{\boldsymbol{x}}^{N,i})$$

where $m_{\boldsymbol{x}}^{N,i} = \dfrac{1}{N-1} \sum_{j \neq i} \delta_{x_j}$.

- Then $v^{N,i}(t, x_i, (x_j)_{-i}) = V^N(t, x_i, m_{\boldsymbol{x}}^{N,i})$ and, if $V^N \to U$, U formally solves

$$(\mathbf{M2}) \left\{ \begin{array}{l} -\partial_t U - (1+\beta)\Delta_x U + H(x, D_x U, m) \\ \qquad -(1+\beta) \displaystyle\int_{\mathbb{R}^d} \mathrm{div}_y \left[D_m U \right] \, dm(y) + \int_{\mathbb{R}^d} D_m U \cdot H_p(y, D_x U, m) \, dm(y) \\ \qquad -2\beta \displaystyle\int_{\mathbb{R}^d} \mathrm{div}_x \left[D_m U \right] dm(y) - \beta \int_{\mathbb{R}^{2d}} \mathrm{Tr} \left[D^2_{mm} U \right] \, dm \otimes dm = 0 \\ \qquad\qquad \text{in } [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \\ U(T, x, m) = G(x, m) \qquad \text{in } \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \end{array} \right.$$

Limit of $N-$player game (continued)

Comparison between the Nash system :

$$\left\{\begin{array}{l} -\partial_t v^{N,i}(t,\boldsymbol{x}) - \sum_{j=1}^{N} \Delta_{x_j} v^{N,i}(t,\boldsymbol{x}) - \beta \sum_{j,k=1}^{N} \mathrm{Tr} D^2_{x_j,x_k} v^{N,i}(t,\boldsymbol{x}) \\ \qquad +H^{N,i}(x_i, D_{x_i} v^{N,i}(t,\boldsymbol{x}), (x_j)_{-i}) \\ \quad + \sum_{j\neq i} D_p H^{N,j}(x_j, D_{x_j} v^{N,j}(t,\boldsymbol{x}), (x_k)_{-j}) \cdot D_{x_j} v^{N,i}(t,\boldsymbol{x}) = 0 \qquad \text{in } [0,T]\times(\mathbb{R}^d)^N, \\ v^{N,i}(T,\boldsymbol{x}) = G^{N,i}(x) \qquad \text{in } (\mathbb{R}^d)^N. \end{array}\right.$$

and the master equation (for $v^{N,i}(t, x_i, (x_j)_{-i}) \simeq U(t, x_i, m^{N,i}_{\boldsymbol{x}})$) :

$$(\mathbf{M2}) \left\{\begin{array}{l} -\partial_t U - (1+\beta)\Delta_x U + H(x, D_x U, m) \\ \qquad -(1+\beta)\int_{\mathbb{R}^d} \mathrm{div}_y\,[D_m U]\; dm(y) + \int_{\mathbb{R}^d} D_m U \cdot H_p(y, D_x U, m)\; dm(y) \\ \qquad -2\beta \int_{\mathbb{R}^d} \mathrm{div}_x\,[D_m U]\, dm(y) - \beta \int_{\mathbb{R}^{2d}} \mathrm{Tr}\left[D^2_{mm} U\right]\; dm\otimes dm = 0 \\ \qquad\qquad \text{in } [0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d) \\ U(T,x,m) = G(x,m) \qquad \text{in } \mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d) \end{array}\right.$$

Limit of $N-$player game (end)

- Key difficulty : Not enough estimates on the $(v^{N,i})$ to justify the limit.
- Rigorous proof of the convergence : Build a solution to **(M2)** and use it to justify the limit (C.-Delarue-Lasry-Lions)

Second approach : Nash equilibria in the infinite player game

- We now directly consider a game with infinitely many (infinitesimal, symmetric) players.
- The dynamics of each player is

$$dX_t = \alpha_t dt + \sqrt{2}dB_t + \sqrt{2\beta}dW_t,\ \ t \in [0,T] \qquad X_0 = \bar{X}_0,$$

  where B and W are indep. (B being the individual noise and W the common noise).
  The individual cost is of the form

$$J(\alpha,(m_t)) = \mathbb{E}\left[\int_0^T L(X_s,\alpha_s,m_s)ds + G(X_T,m_T)\right],$$

  where (m_t) is the (random) distribution of all players at time t (anticipated by the players and adapted to W).
- The value function of the small player is

$$u_t(x) = \inf_{\alpha} \mathbb{E}\left[\int_t^T L(X_s,\alpha_s,m_s)ds + G(X_T,m_T) \mid (W_s)_{s\le t}\right]$$

  where

$$dX_s = \alpha_s ds + \sqrt{2}dB_s + \sqrt{2\beta}dW_s,\ \ s \in [t,T] \qquad X_t = x.$$

The MFG system (continued)

- The optimal feedback of each player is then

$$\alpha^*(t,x) = -D_pH(x, Du_t(x), m_t),$$

so that the optimal dynamic of the player solves

$$dX_s = -D_pH(X_s, Du_t(X_s), m_s)ds + \sqrt{2}dB_s + \sqrt{2\beta}dW_s,\ \ s \in [t,T] \qquad X_0 = \bar{X}_0.$$

- (By mean field argument), the distribution of the players is then $\tilde{m}_t = [X_t|W]$.

- An equilibrium configuration is obtained when $\tilde{m} = m$.

The MFG system (end)

- The PDE formulation : The pair (u, m) formally solves

$$(MFGs)\quad \begin{cases} d_t u_t = \left\{-(1+\beta)\Delta u_t + H(x, Du_t, m_t) - \sqrt{2\beta}\,\mathrm{div}(v_t)\right\}dt \\ \qquad\qquad + v_t \cdot dW_t \qquad \text{in } [0,T]\times\mathbb{R}^d, \\ d_t m_t = \left[(1+\beta)\Delta m_t + \mathrm{div}\left(m_t D_p H(x, Du_t, m_t)\right)\right]dt - \sqrt{2\beta}\,\mathrm{div}(m_t dW_t) \\ \qquad\qquad \text{in } [0,T]\times\mathbb{R}^d \\ m_0 = [\bar{X}_0],\ u_T(x) = G(x, m_T) \qquad \text{in } \mathbb{R}^d. \end{cases}$$

where (v_t) is a vector field which ensures (u_t) to be adapted to $(W_s)_{s\leq t}$.

- Link with the master equation : Let U solves **M2**. Let m solve the stochastic McKean-Vlasov equation :

$$\begin{cases} d_t m_t = \left[(1+\beta)\Delta m_t + \mathrm{div}\left(m_t D_p H(x, D_x U(t,x,m_t), m_t)\right)\right]dt - \sqrt{2\beta}\,\mathrm{div}(m_t dW_t) \\ \qquad\qquad \text{in } [0,T]\times\mathbb{R}^d \\ m_0 = [\bar{X}_0] \qquad \text{in } \mathbb{R}^d. \end{cases}$$

and $u_t(x) := U(t, x, m_t)$. Then the pair (u, m) solves (MFGs).

Motivations to study (**M2**)

- Allows to pass to the limit in the $N-$player problem,
- Allows to build easily a solution to the stochastic MFG system

Need of a new construction for the solution of (**M2**) :

- So far the construction of solutions to (**M2**) relies on the method of characteristics...
- ... which are the solution to the stochastic MFG system.
- But the stochastic MFG system is heavy to manipulate.

Outline

1 Interpretation of the master equation

2 Construction of a solution for (**M2**)

3 Uniqueness

Derivatives in the space of measures

We denote by $\mathcal{P}_2(\mathbb{R}^d)$ the set of Borel probability measures on $\mathbb{R}^d$ with finite second order moment, endowed for the Wasserstein distance

$$\mathbf{d}_2^2(m, m') = \inf_{\pi} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, d\pi(x, y),$$

where the infimum is taken over coupling between m and m'.

Derivatives

A map $U : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is $\mathcal{C}^1$ if there exists a continuous and bounded map $\frac{\delta U}{\delta m} : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ such that, for any $m, m' \in \mathcal{P}_1(\mathbb{R}^d)$,

$$U(m') - U(m) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}((1-s)m + sm', y) d(m' - m)(y) ds.$$

We set

$$D_m U(m, y) := D_y \frac{\delta U}{\delta m}(m, y).$$

- Note that $\frac{\delta U}{\delta m}$ is defined up to an additive constant. We adopt the normalization convention

$$\int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m,y)dm(y) = 0.$$

- $D_m U$ corresponds to the derivative in the space of measures as introduced by Ambrosio-Gigli-Savaré.
- $D_m U$ controls the Lipschitz norm of U :

$$|U(m_1) - U(m_2)| \leq \sup_{\mu \in \mathcal{P}_1(\mathbb{R}^d)} \|D_m U(\mu, \cdot)\|_{L^2_\mu} \mathbf{d}_2(m_1, m_2) \qquad \forall m_1, m_2 \in \mathcal{P}_1(\mathbb{R}^d).$$

Method of proof for the existence of a solution of (**M2**)

- We see the second order master equation

$$(\mathbf{M2})\quad \begin{cases} -\partial_t U - (1+\beta)\Delta_x U + H(x, D_x U, m) \\ \qquad -(1+\beta)\int_{\mathbb{R}^d} \mathrm{div}_y\left[D_m U\right]\, dm(y) + \int_{\mathbb{R}^d} D_m U \cdot H_p(y, D_x U, m)\, dm(y) \\ \qquad -2\beta \int_{\mathbb{R}^d} \mathrm{div}_x\left[D_m U\right] dm(y) - \beta \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{Tr}\left[D^2_{mm} U\right]\, dm\otimes dm = 0 \\ \qquad\qquad \text{in } [0,T]\times\mathbb{R}^d\times\mathcal{P}_1(\mathbb{R}^d) \\ U(T,x,m) = G(x,m) \qquad \text{in } \mathbb{R}^d\times\mathcal{P}_1(\mathbb{R}^d) \end{cases}$$

- as the superposition of (**M1**) and (**L2**)

$$(\mathbf{M1})\quad \begin{cases} -\partial_t U - (1+\beta)\Delta_x U + H(x, D_x U, m) \\ \qquad -(1+\beta)\int_{\mathbb{R}^d} \mathrm{div}_y\left[D_m U\right]\, dm(y) + \int_{\mathbb{R}^d} D_m U \cdot H_p(y, D_x U, m)\, dm(y) \\ \qquad -2\beta \int_{\mathbb{R}^d} \mathrm{div}_x\left[D_m U\right] dm(y) - \beta \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{Tr}\left[D^2_{mm} U\right]\, dm\otimes dm = 0 \\ \qquad\qquad \text{in } [0,T]\times\mathbb{R}^d\times\mathcal{P}_1(\mathbb{R}^d) \\ U(T,x,m) = G(x,m) \qquad \text{in } \mathbb{R}^d\times\mathcal{P}_1(\mathbb{R}^d) \end{cases}$$

Method of proof for the existence of a solution of (**M2**)

- We see the second order master equation

$$(\mathbf{M2}) \quad \begin{cases} -\partial_t U - (1+\beta)\Delta_x U + H(x, D_x U, m) \\ \qquad -(1+\beta)\int_{\mathbb{R}^d} \mathrm{div}_y\,[D_m U]\; dm(y) + \int_{\mathbb{R}^d} D_m U \cdot H_p(y, D_x U, m)\; dm(y) \\ \qquad -2\beta \int_{\mathbb{R}^d} \mathrm{div}_x\,[D_m U]\, dm(y) - \beta \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{Tr}\left[D^2_{mm} U\right]\; dm \otimes dm = 0 \\ \qquad\qquad \text{in } [0,T]\times\mathbb{R}^d\times\mathcal{P}_1(\mathbb{R}^d) \\ U(T,x,m) = G(x,m) \qquad \text{in } \mathbb{R}^d\times\mathcal{P}_1(\mathbb{R}^d) \end{cases}$$

- as the superposition of (**M1**) and (**L2**)

$$(\mathbf{L2}) \quad \begin{cases} -\partial_t U - (1+\beta)\Delta_x U + H(x, D_x U, m) \\ \qquad -(1+\beta)\int_{\mathbb{R}^d} \mathrm{div}_y\,[D_m U]\; dm(y) + \int_{\mathbb{R}^d} D_m U \cdot H_p(y, D_x U, m)\; dm(y) \\ \qquad -2\beta \int_{\mathbb{R}^d} \mathrm{div}_x\,[D_m U]\, dm(y) - \beta \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{Tr}\left[D^2_{mm} U\right]\; dm \otimes dm = 0 \\ \qquad\qquad \text{in } [0,T]\times\mathbb{R}^d\times\mathcal{P}_1(\mathbb{R}^d) \\ U(T,x,m) = G(x,m) \qquad \text{in } \mathbb{R}^d\times\mathcal{P}_1(\mathbb{R}^d) \end{cases}$$

The first order master equation (**M1**)

It is the backward equation

$$(\mathbf{M1})\qquad \left\{\begin{array}{l} -\partial_t U(t,x,m) - \Delta_x U(t,x,m) + H\big(x, D_x U(t,x,m), m\big) \\ \qquad - \displaystyle\int_{\mathbb{R}^d} \mathrm{div}_y \left[D_m U\right]\big(t,x,m,y\big)\, dm(y) \\ \qquad + \displaystyle\int_{\mathbb{R}^d} D_m U\big(t,x,m,y\big) \cdot D_p H\big(y, D_x U(t,y,m), m\big)\, dm(y) = 0 \\ U(T,x,m) = G(x,m), \qquad \text{for } (x,m) \in \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \end{array}\right.$$

Theorem (Chassagneux-Crisan-Delarue)

Under the suitable assumptions, there exists $T > 0$ such that the first order master equation (**M1**) has a unique classical solution on $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$.

See also C.-Delarue-Lasry-Lions for a PDE construction.

- The proof of Theorem 1 relies on the method of characteristics in infinite dimension.

- Given $(t_0, m_0) \in [0, T) \times \mathcal{P}_2(\mathbb{R}^d)$, let $(u, m) = (u(t, x), m(t, x))$ be the solution of the MFG system :

$$(MFG) \qquad \left\{ \begin{array}{l} -\partial_t u - \Delta u + H(x, Du, m(t)) = 0 \text{ in } [t_0, T] \times \mathbb{R}^d \\ \partial_t m - \Delta m - \mathrm{div}(mD_pH(x, Du, m(t))) = 0 \text{ in } [t_0, T] \times \mathbb{R}^d \\ u(T, x) = G(x, m(T)), \;\; m(t_0, \cdot) = m_0 \text{ in } \mathbb{R}^d \end{array} \right.$$

- If $T > 0$ is small or under some monotonicity assumptions on F and G, the (MFG) system is well-posed. (Lasry-Lions, 2007)

- We *define* U by

$$U(t_0, \cdot, m_0) := u(t_0, \cdot)$$

Claim : U is a solution to the first order master equation.

- Note that, for any $h \in [0, T - t_0]$, $u(t_0 + h, x) = U(t_0 + h, x, m(t_0 + h))$.
- So

$$\begin{aligned}
\partial_t u(t_0, x) &= \partial_t U(t_0, x, m_0) + \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(t_0, x, m_0, y)\partial_t m(t_0, y)dy \\
&= \partial_t U(t_0, x, m_0) + \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m_0, y)\left(\Delta m_0 + \mathrm{div}(m_0 D_p H(x, Du, m_0))\right) dy \\
&= \partial_t U(t_0, x, m_0) + \int_{\mathbb{R}^d} \Delta_y \left[\frac{\delta U}{\delta m}\right](m_0, y)m_0(y)dy \\
&\qquad - \int_{\mathbb{R}^d} D_y \left[\frac{\delta U}{\delta m}\right](m_0, y) \cdot D_p H(x, Du, m_0))m_0(y)dy \\
&= \partial_t U(t_0, x, m_0) + \int_{\mathbb{R}^d} \mathrm{div}_y \left[D_m U\right](m_0, y)m_0(y)dy \\
&\qquad - \int_{\mathbb{R}^d} D_m U(m_0, y) \cdot D_p H(x, Du, m_0)m_0(y)dy
\end{aligned}$$

- Then U satisfies (**M1**) because

$$\begin{aligned}
\partial_t u(t_0, x) &= -\Delta u + H(x, Du, m_0) \\
&= -\Delta_x U(t_0, x, m_0) + H(x, D_x U(t_0, x, m_0), m_0).
\end{aligned}$$

- In the actual proof, one has to show that U is regular in m : this relies on linearizations of the MFG system.

Remarks for the second order master equation (**M2**) :

- The same principle applies, but the system of characteristics becomes the stochastic MFG system

$$(MFGs)\quad \begin{cases} d_t u_t = \left\{-(1+\beta)\Delta u_t + H(x, Du_t, m_t) - \sqrt{2\beta}\mathrm{div}(v_t)\right\}dt \\ \qquad\qquad + v_t \cdot dW_t \qquad \text{in } [t_0, T] \times \mathbb{R}^d, \\ d_t m_t = \left[(1+\beta)\Delta m_t + \mathrm{div}\big(m_t D_p H(x, Du_t, m_t)\big)\right] dt - \sqrt{2\beta}\mathrm{div}(m_t dW_t) \\ \qquad\qquad \text{in } [t_0, T] \times \mathbb{R}^d \\ m_{t_0} = m_0,\ u_T(x) = G(x, m_T) \qquad \text{in } \mathbb{R}^d. \end{cases}$$

  where (v_t) is a vector field which ensures (u_t) to be adapted to the filtration $(\mathcal{F}_t)_{t\in[t_0,T]}$ generated by the M.B. $(W_t)_{t\in[0,T]}$.

- Intermediate result : well-posedness of (MFGs).

- Proof much more difficult than for the case $\beta = 0$ (see C-Delarue-Lasry-Lions and Carmona-Delarue).

The linear second order equation (**L2**)

Let $\Gamma = \Gamma(t,x)$ be the heat kernel. For a map $G : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ of class C^2, we set

$$U(t,x,m) = \int_{\mathbb{R}^d} G(\xi, (id - x + \xi)\sharp m)\Gamma(T-t, x-\xi)d\xi.$$

Proposition

The map U solves the second order equation

$$(\mathbf{L2}) \left\{ \begin{array}{l} -\partial_t U - \Delta U - \int_{\mathbb{R}^d} \mathrm{div}_y [D_m U] dm - 2\int_{\mathbb{R}^d} \mathrm{Tr}[D^2_{xm} U] dm \\ \qquad - \int_{\mathbb{R}^{2d}} \mathrm{Tr}[D^2_{mm} U] dm dm = 0 \qquad \text{in } (0,T) \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \\ U(T,x,m) = G(x,m) \qquad \text{in } \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \end{array} \right.$$

Proof : Computation.

The short time existence for (**M2**)

Let

- $(\mathcal{S}_t^1)$ be the backward semi-group associated with (**M1**) ,
- $(\mathcal{S}_t^2)$ be the backward semi-group associated with (**L2**).

For $h > 0$ small and $T - t = 2kh$ ($k \in \mathbb{N}$), we set

$$\mathcal{S}^h_{T-t} := (\mathcal{S}^1_h \circ \mathcal{S}^2_h)^k.$$

Theorem

For $M > 0$ there exists $T_M > 0$ such that, if $T \leq T_M$ and

$$\|D^2_{xx}G\|_\infty \leq M \qquad \text{and} \qquad \|D^2_{xm}G\|_\infty \leq M,$$

then $(\mathcal{S}^h_t G)_{t\in[0,T]}$ converges to a solution of (**M2**) on $[0, T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d)$.

Remarks :

- The above Theorem gives the existence of a solution on a short time interval.
- The length of the interval depends on $\|D^2_{xx}G\|_\infty$ and $\|D^2_{xm}G\|_\infty$ only

Idea of proof : relies of the estimates.

- For (M1) : Fix $M > 0$ and $n \geq 2$. There exists $C_{M,n} > 0$ and $T_{M,n} > 0$ such that, if

$$\|D^2_{xx} G\|_\infty \leq M, \qquad \|D^2_{xm} G\|_\infty \leq M \qquad \text{and} \qquad T \in (0, T_{M,n}],$$

then the solution $U := (\mathcal{S}^1_t G)_{t \in [0,T]}$ to (**M1**) satisfies

$$\sup_{t\in[0,T]} \left(\|U(t)\|_{n+1} + \left\| \frac{\delta U}{\delta m}(t) \right\|_n + \left\| \frac{\delta^2 U}{\delta m^2}(t) \right\|_{n-1} + \mathrm{Lip}_{n-2}\left(\frac{\delta^2 U}{\delta m^2}(t) \right) \right)$$
$$\leq \left(\|G\|_{n+1} + \left\| \frac{\delta G}{\delta m} \right\|_n + \left\| \frac{\delta^2 G}{\delta m^2} \right\|_{n-1} + \mathrm{Lip}_{n-2}\left(\frac{\delta^2 G}{\delta m^2} \right) \right) (1 + C_{M,n} T) + C_{M,n} T.$$

- For (L2) : Similar estimes for (**L2**) are straightforward.

Outline

1 Interpretation of the master equation

2 Construction of a solution for (**M2**)

3 Uniqueness

- **Goal :** prove the uniqueness by PDE arguments,
- by using a maximum principle.
- Difficulty : (**M2**) is a nonlocal equation, without maximum principle.

Optimality conditions

Let $U : \times\mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be smooth and have a local maximum point at $(\hat{x},\hat{m}) \in \times\mathcal{P}_2(\mathbb{R}^d)$. We have

- $D_x U(\hat{x},\hat{m}) = 0$,
- $\frac{\delta U}{\delta m}(\hat{x},\hat{m},y) \leq 0, \ \forall y \in \mathbb{R}^d$, and $\frac{\delta U}{\delta m}(\hat{x},\hat{m},y) = 0, \ \hat{m} - \text{a.e. } y \in \mathbb{R}^d$,
- for any $(v,\phi) \in \times L^2_{\hat{m}}(\mathbb{R}^d,)$,

$$D^2_{xx}U(\hat{x},\hat{m})v\cdot v + 2\int_{\mathbb{R}^d} D^2_{xm}U(\hat{x},\hat{m},y)\phi(y)\cdot v d\hat{m}(y) + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} D^2_{mm}U(\hat{x},\hat{m},y,z)\phi(y)\cdot\phi(z)d\hat{m}(y)d\hat{m}(z) \leq 0.$$

In particular :

- $D_m U(\hat{x},\hat{m},y) = 0, \qquad D^2_{ym}U(\hat{x},\hat{m},y) \leq 0 \qquad \hat{m} - \text{a.e. } y \in \mathbb{R}^d$,
- and

$$\Delta_x U(\hat{x},\hat{m}) + 2\int_{\mathbb{R}^d} \mathrm{Tr}[D^2_{xm}U](\hat{x},\hat{m},y)d\hat{m}(y) + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{Tr}[D^2_{mm}U](\hat{x},\hat{m},y,z)d\hat{m}(y)d\hat{m}(z) \leq 0.$$

A maximum principle

Let $W = W(t,x,m)$ satisfy in $(0,T)\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$ the backward inequality :

$$\begin{array}{l}\mathcal{L}(W) := -\partial_t W - (1+\beta)\Delta_x W + v_1(t,x,m)\cdot D_x W \\ \quad -(1+\beta)\int_{\mathbb{R}^d} \mathrm{div}_y\left[D_m W\right]\ m(dy) + \int_{\mathbb{R}^d} D_m W\cdot v_2(t,y,m)\ m(dy) \\ \quad -2\beta\int_{\mathbb{R}^d}\mathrm{div}_x\left[D_m W\right] dm(y) - \beta\int_{\mathbb{R}^{2d}}\mathrm{Tr}\left[D^2_{mm}W\right]\ dm\otimes dm \leq f(t,x,m) \\ \qquad \text{in } [0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\end{array}$$

Proposition

Assume that v_1 and v_2 are continuous and bounded vector fields and f is continuous and bounded. If W is bounded, then

$$W \leq \sup_{x,m}|W(T,x,m)| + T\|f\|_\infty .$$

Uniqueness for (**M2**)

Theorem

(**M2**) has at most one classical solution.

Remarks

- Standard maximum principle cannot work because of the nonlocal term

$$\int_{\mathbb{R}^d} D_m U(t,x,m) \cdot H_p(y, D_x U(t,y,m), m) m(dy)$$

- Usual proof by methods of characteristics (C.-Delarue-Lasry-Lions, Carmona-Delarue)

Sketch of proof : Let U_1 and U_2 be two solutions.

- Key step : show that $D_x U_1 = D_x U_2$ by using Bernstein method.
- Indeed, $V = |D_x(U_1 - U_2)|^2$ satisfies $\mathcal{L}(V) \leq C\|V\|_\infty$.
- Equality $U_1 = U_2$ then follows again by maximum principle.

Conclusion

In this work :

- We understood how to build a short time solution of the second order master equation with general Hamiltonians,
- obtained uniqueness results without the use of characteristics,

by purely PDE methods.

Extensions :

- diffusions terms depending on (x, m),
- major/minor MFG problem.

Open problem :

- Existence on large time intervals.
- Regularizing effects of the equation.

Thank you...

.... and Happy Birthday, Mete !