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D MATH
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Mod-gaussian convergence for trigonometric sums and analogues

ETH-Zürich Mete's birthday party,

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Outline

1 Authors

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2 Trigonometric sums, results of Salem and Zygmund

$[0,1]$ unit interval with Lebesgue measure m.

We consider a lacunary sequence $(n_k)_k$ of integers: there is $q>1$ with $\frac{n_{k+1}}{n_k} \geq q$. The trigonometric functions $cos(2\pi n_k x) \, : \, x \in [0,1]$, “almost" behave like independent random variables. Consider now

$$\sum_{k=1}^{k=N_l} a_{k,l} \, cos(2\pi n_k x)$$

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If $\sum_k a_{k,l}^2 = 2$ and $\max(|a_{k,l}|; k = 1, \dots N_l) \to 0$ then the above sums converge in law to a standard gaussian probability distribution. That means for $l \to \infty$:

$$m\left[x \mid \sum_{k=1}^{k=N_l} a_{k,l} \cos(2\pi n_k x) \leq \alpha \right] \to \Phi(\alpha)$$

where

$$\Phi(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} \exp\left(-\frac{y^2}{2}\right) dy.$$

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3 A local convergence theorem

Given a sequence of random variables ξ_l with characteristic functions φ_l. Suppose that $\mathbb{E}[\xi_l] = 0$ and $\mathbb{E}[\xi_l^2] = \sigma_l^2 \to +\infty$. We say that the sequence converges mod-gaussian if

1. $\varphi_l(t/\sigma_l) \to \exp(-t^2/2)$, i.e. ξ_l (when normalised) converge in law to a standard normal,
2. for each $K > 0$, the sequence $\varphi_l(t/\sigma_l)\mathbf{1}_{|t|\le K\sigma_l}$ is uniformly integrable

In that case we have

$$\sigma_l \, \mathbb{P}[\xi_l \in [a, b]] \to \frac{1}{\sqrt{2\pi}}(b - a).$$

This is related to the standard local convergence known from statistics.

Of course this implies for $\lambda_l \to +\infty$ with $\frac{\sigma_l}{\lambda_l} \to +\infty$:

$$\frac{\sigma_l}{\lambda_l} \, \mathbb{P}[\xi_l \in [\lambda_l a, \lambda_l b]] \to \frac{1}{\sqrt{2\pi}}(b - a)$$

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4 Trigonometric sums

Again we consider a sum $\sum_{k=1}^{k=N_l} a_{k,l}\, cos(2\pi n_k x)$, with $A_l^2 = \sum_k a_{k,l}^2 \to +\infty$ and $d_l = \max\{|a_{k,l}| \mid k = 1 \ldots N_l\}$ is small i.e there is $\varepsilon > 0$ with

1. for $1 < q < 2$ we have $N_l^{1+\varepsilon} d_l^3 \to 0$
2. for $2 \leq q$ we have $N_l^{1+\varepsilon} d_l^4 \to 0$.

In these cases we have a mod-gaussian convergence.

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5 Further Notation

$f\colon [0,1] \to \mathbb{R}$ is an L^2 function with $\int_0^1 f = 0$. We extend f periodically to $\mathbb{R}$. (only for notational reasons)

We are interested in the limit behaviour of

$$\frac{1}{\sqrt{n}}\left(f(t) + f(2t) + \ldots .f(2^{n-1}t)\right)$$

6 The Result of Mark Kac

Annals of Mathematics, vol 47, 1946.

If f is Hölder continous then the sums converge in law to a normal distribution with σ^2 where

$$\sigma^2 = \lim_n \left\| \frac{1}{\sqrt{n}} \left(f(t) + f(2t) + \ldots . f(2^{n-1}t) \right) \right\|_2^2,$$

provided this limit is not zero.

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7 Some Related Work

If $\sigma = 0$ then Fortet proved (with Doeblin) that under some boundedness conditions f is of the form $f(t) = g(t) - g(2t)$.

This functional equation was later investigated by I. Berkes.

Stationary sequences were analysed by
e.g. Ibragimov, Bolthausen, ...

Erdös and Fortet gave another example where for $f(t) = \cos(2\pi t) + \cos(4\pi t)$ the sums

$$\frac{1}{\sqrt{n}}\Big(f((1+1)t) + f((2+1)t) + \dots . f((2^{n-1}+1)t)\Big)$$

converge in distribution but the limit is not normal (gaussian). This was never published and the closest reference is in a paper by Kac.

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8 The Proof of Kac

The proof of Kac is based on a Fourier expansion of f and an estimate of the Fourier coefficients, possible because f is supposed to be Hölder continuous. The probabilistic ingredient is a central limit theorem for m-dependent random variables. This theorem goes back to Markoff ($\leq$ 1912) but the formulation is imprecise (lack of good definitions of independence). A better reference is Diananda (Proc. Camb. Phil. Soc. 1955).

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9 The CLT for m-dependent Variables

Let $(X_n)_n$ be a sequence of random variables $X_n \in L^2$, all having the same distribution. Suppose that there is m such that for $k \geq 1$ and indices $i_1 < \ldots i_k < j_1 < \ldots j_k$, where $j_1 - i_k > m$, the vectors

$$(X_{i_1}, \ldots, X_{i_k}) \text{ and } (X_{j_1}, \ldots, X_{j_k})$$

are independent. Under these hypotheses the sequence satisfies a CLT. Later more extensive work is e.g. by Ibragimov.

10 Less Complicated Approach

For each n let $\mathcal{D}_n$ be the $\sigma-$algebra generated by the intervals $(\frac{k}{2^n}, \frac{k+1}{2^n}]$. Martingale theory (or some real analysis) shows that for $f \in L^2$

$$f_n = \mathbb{E}[f \mid \mathcal{D}_n] \to f \text{ in } L^2 \text{ and almost surely.}$$

Let $\phi_n = f - f_n$. We suppose that

$$\sum_n \|\phi_n\|_2 < \infty.$$

We see that for each $r \geq 1$ the sequence

$$\left(f_r(2^k t)\right)_{k\geq 0}$$

is $r-$dependent.

Markoff's CLT then shows that

$$\frac{1}{\sqrt{n}}\left(f_r(t) + \cdots + f_r(2^{n-1}t)\right)$$

converges to a normal distribution.

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We have for each $r \geq 1$ and each $n \geq 1$:

$$\frac{1}{n}\left\|\phi_r(t) + ... + \phi_r(2^{n-1}t)\right\|_2^2$$

$$\leq \|\phi_r\|_2^2 + 2\|\phi_r\|_2 \left(\sum_{s\geq r+1} \|\phi_s\|_2\right).$$

This can be made arbitrary small by the approximation hypothesis.

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11 Hölder Continuity

If f is Hölder continuous with exponent β we have

$$\|\phi_n\|_2 \le \|\phi_n\|_\infty \le C\,2^{-n\beta}.$$

The approximation hypotheses is therefore satisfied. For the case of Hölder continuous functions we could prove some mod-gaussian results.

We wish Mete all the best for his 50th birthday

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50 and a couple of months