Duality and Convergence for Super–Hedging with Price Impact

Yan Dolinsky

Hebrew University and Monash University

METE, ETH Zurich, 5.6.18

Based on joint work with Mete Soner, Peter Bank and Selim Gokay

Setup

- Financial market with one risky asset $S = \{S_t\}_{t=0}^{T}$.
- Linear price impact:

$$S \to S + 2\Lambda\nu.$$

- Wealth process:

$$V_t = V_0 + \int_0^t \gamma_u dS_u - \Lambda\langle\gamma\rangle_t.$$

Bank and Baum Density Result

- **Theorem:** Let S be a continuous semi–martingale. For any $\epsilon > 0$ and $\gamma = \{\gamma_t\}_{t=0}^T$ there exists a continuous process of bounded variation $\delta = \{\delta_t\}_{t=0}^T$ such that

$$\sup_{0 \leq t \leq T} \left| \int_0^t \gamma_u dS_u - \int_0^t \delta_u dS_u \right| < \epsilon \ \ \textbf{a.s.}$$

- In particular

$$\langle \delta \rangle \equiv 0.$$

- **Corollary: In continuous time there is no liquidity premium.**

Two Approaches two deal with this "Paradox"

- First approach: (**Cetin, Soner and Touzi 2010**). Putting Gamma constraints on portfolio strategies.
- Second approach: (**Gokay and Soner 2012**). Consider binomial models with quadratic costs and no constraints on the trading strategies.
- Both approaches lead to the same liquidity premium and were solved for Markovian payoffs by using PDE techniques.
- $$\phi_t(t,s) + \frac{\sigma^2 s^2}{2}(1 + 4\Lambda\phi_{ss}(t,s))\phi_{ss}(t,s) = 0.$$

The Stochastic Control Problem

- $$\phi(0,s) = \sup_{\nu \geq 0} \mathbb{E}\left(f(S^{(\nu)}_T) - \frac{1}{16\Lambda\sigma^2}\int_0^1 [(\nu^2-\sigma^2)S^{(\nu)}_t]^2 dt \right)$$

  where the above control problem defined on a Brownian probability space and supremum is taken over all adapted processes, and bounded processes $\nu \geq 0$. The process $S^{(\nu)}$ is the Doleans–Dade exponential

  $$S^{(\nu)}_t = s\exp\left(\int_0^t \nu_u dW_u - \frac{1}{2}\int_0^t \nu_u^2 du\right).$$

Binomial Model

- Time horizon: $T = 1$.
- Number of time steps: $n \in \mathbb{N}$.
- Market active at times $0, \frac{1}{n}, ..., 1$.
- Risky asset given by

$$S_k^{(n)} = s \exp\left(\frac{\sigma}{\sqrt{n}} \sum_{i=1}^{k} \xi_i\right)$$

where $\xi_1, ..., \xi_n = \pm 1$.

Super–replication with Quadratic Costs

- Portfolio value at maturity:

$$V_n^{\gamma} = V_0 + \sum_{i=0}^{n-1} \gamma_i (S_{i+1} - S_i) - \Lambda \sum_{i=1}^{n} |\gamma_i - \gamma_{i-1}|^2$$

where $\gamma_n = \gamma_0 \equiv 0$.

- The super–replication price of a European contingent claim

$$X = f(S_1, ..., S_n)$$

defined by

$$V_0 = \inf\{V_0 : \exists \gamma \text{such that } V_n^{\gamma} \geq X\}.$$

Duality

- **Theorem: (Soner & D)**
  The super–replication price given by

$$P = \sup_{(\mathbb{Q},M)} \mathbb{E}_{\mathbb{Q}} \left(X - \frac{1}{4\Lambda} \sum_{k=0}^{n} |M_k - S_k|^2 \right)$$

  where $M = (M_0, ..., M_n)$ is martingale with respect to $\mathbb{Q}$ and the filtration generated by S.
- M can be viewed as a shadow price.

Asymptotic Behaviour

- In the n–step model we consider a European claim

$$X_n = F\left(\mathcal{W}_n(S_0, ..., S_n)\right)$$

where $F : C[0,1] \to \mathbb{R}_+$ is a continuous function and

$$\mathcal{W}_n : \mathbb{R}^{n+1} \to C[0,1]$$

is the linear interpolation operator.

Main Result

- **Theorem: (Soner & D)**

$$\liminf V_n \geq \sup_{\nu \geq 0} \mathbb{E}\left(F(S^{(\nu)}) - \frac{1}{16\Lambda\sigma^2}\int_0^1 [(\nu^2-\sigma^2)S_t^{(\nu)}]^2 dt\right)$$

  where the above control problem defined on a Brownian probability space and supremum is taken over all adapted processes, and bounded processes $\nu \geq 0$. The process $S^{(\nu)}$ is the Doleans–Dade exponential

$$S_t^{(\nu)} = s\exp\left(\int_0^t \nu_u dW_u - \frac{1}{2}\int_0^t \nu_u^2 du\right).$$

- In fact for the case where F satisfy some growth conditions we can prove that the above right hand side is also the upper bound.

Intuition

- Main idea goes back to **Kusuoka 1995**.
- Shadow price:

$$M_k^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} \xi_i + \frac{\alpha \xi_k}{\sqrt{n}}.$$

Martingale measure: $(\alpha > -1/2)$

$$\mathbb{E}_Q[\xi_k | \xi_1, ..., \xi_{k-1}] = \frac{\alpha}{1+\alpha} \xi_{k-1}.$$

$$\mathbb{E}_Q \left((M_{k+1} - M_k)^2 | \xi_1, ..., \xi_k \right) = \frac{1+2\alpha}{n}.$$

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \Rightarrow \sqrt{1+2\alpha} W.$$

The Setup

- Volatility uncertainty interval

$$I = [\underline{\sigma}, \overline{\sigma}].$$

- Time horizon: $T = 1$.
- Number of time steps: $n \in \mathbb{N}$.
- Market active at times $0, \frac{1}{n}, ..., 1$.
- Risky asset give by

$$S_k^{(n)} = s \exp\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{k} X_i\right)$$

where $X_1, ..., X_n$ random variables such that

$$|X_i| \in I.$$

Super–replication with Quadratic Costs

- Portfolio value at maturity:

$$V_n^\gamma = V_0 + \sum_{i=0}^{n-1} \gamma_i (S_{i+1} - S_i) - \Lambda \sum_{i=1}^{n} |\gamma_i - \gamma_{i-1}|^2$$

where $\gamma_n = \gamma_0 \equiv 0$.

- The super–replication price of a European contingent claim

$$X = f(S_1, ..., S_n)$$

defined by

$$V_0 = \inf\{V_0 : \exists \gamma \text{ such that } V_n^\gamma \geq X \textbf{ for } \textbf{ all } X_1, ..., X_n\}.$$

Duality Result

- Let
$$K=\{(x_1,\ldots,x_n):|x_i|\in I\}\subset\mathbb{R}^n.$$
- **Theorem: (Bank, D & Gokay)**
The super–replication price given by
$$P=\sup_{(\mathbb{Q},M)}\mathbb{E}_{\mathbb{Q}}\left(X-\frac{1}{4\Lambda}\sum_{k=0}^{n}|M_k-S_k|^2\right)$$
where $M=(M_0,\ldots,M_n)$ is martingale with respect to $\mathbb{Q}$ and the filtration generated by S.
- M can be viewed as the shadow price.
- There is no a reference measure.

Asymptotic Behaviour

- In the n–step model we consider a European claim

$$X_n = F\left(\mathcal{W}_n(S_0, ..., S_n)\right)$$

where $F : C[0,1] \to \mathbb{R}_+$ is a continuous function and

$$\mathcal{W}_n : \mathbb{R}^{n+1} \to C[0,1]$$

is the linear interpolation operator.

Main Result

- **Theorem: (Bank, D & Gokay)**

$$\liminf V_n \geq \sup_{\nu\geq 0} \mathbb{E}\Bigg(F(S^{(\nu)}) -$$
$$\mathbb{I}_{\nu>\overline{\sigma}} \frac{1}{16\Lambda\overline{\sigma}^2} \int_0^1 [(\nu^2 - \overline{\sigma}^2) S^{(\nu)}_t]^2 dt -$$
$$\mathbb{I}_{\nu<\underline{\sigma}} \frac{1}{16\Lambda\underline{\sigma}^2} \int_0^1 [(\nu^2 - \underline{\sigma}^2) S^{(\nu)}_t]^2 dt \Bigg)$$

- The process $S^{(\nu)}$ is the Doleans–Dade exponential

$$S^{(\nu)}_t = s \exp\left(\int_0^t \nu_u dW_u - \frac{1}{2}\int_0^t \nu^2_u du \right).$$

Bank & Voss Model

- $\gamma = \{\gamma_t\}_{t=0}^{T}$ trading strategy.
- Price Impact

$$dA_t^\gamma = dP_t + \eta d\gamma_t^+ - \tfrac{\kappa}{2}(A_{t-}^\gamma - B_{t-}^\gamma)dt$$
$$dB_t^\gamma = dP_t - \eta d\gamma_t^- + \tfrac{\kappa}{2}(A_{t-}^\gamma - B_{t-}^\gamma)dt.$$

- P is the exogenous fundamental random shock.
- $\frac{1}{\eta}$ is the market depth.
- κ is the resilience rate.

Portfolio Value

- The spread

$$\zeta_t^\gamma = A_t^\gamma - B_t^\gamma$$

is given by

$$d\zeta_t^\gamma = \eta|d\gamma_t| - \kappa\zeta_{t-}^\gamma dt$$

and satisfies

$$\zeta_t^\gamma = \eta e^{-\kappa t}\int_0^t e^{\kappa u}|d\gamma_u|.$$

- Portfolio value: (we require $\gamma_0 = \gamma_T = 0$)

$$Z_T^\gamma = -\int_0^T P_t d\gamma_t - \frac{\kappa}{2\eta}\int_0^T |\zeta_{t-}^\gamma|^2 dt - \frac{1}{4\eta}|\zeta_T^\gamma|^2.$$

Duality Theory

- European contingent claim $X \in L^1(\mathcal{F}_T, \mathbb{P})$.
- The super–replication price

$$V := \inf\{x : \exists \gamma \ x + Z^{\gamma}_T \geq X \text{ a.s.}\}.$$

- In continuous time, duality for non linear friction was only studied in **Guasoni & Rasonyi 2015**. They considered a penalty of the form

$$\Lambda \int_0^T |\dot{\gamma}|^p dt.$$

Main Result

- **Theorem: (Bank & D)**
  Let $\mathcal{A}$ be the set of all triples $(\mathbb{Q}, Y, S = M + A)$ such that S–is a semi–martingale with

$$S_t \geq e^{-\kappa t} |\mathbb{E}_{\mathbb{Q}}(Y|\mathcal{F}_t) - P_t|, \quad \forall t.$$

  Then super–replication price given by

$$V = \sup_{(\mathbb{Q}, Y, S)} \mathbb{E}_{\mathbb{Q}} \left(X - \frac{1}{2\kappa\eta} \int_0^T e^{2\kappa t} \left(\frac{dA_t}{dt} \right)^2 dt - \frac{e^{2\kappa T}}{\eta} S_T^2 \right).$$

Intuition about the Duality

- Choose a triple $(\mathbb{Q}, Y, S)$ and a trading strategy γ. Then

$$V+\epsilon \geq \mathbb{E}_{\mathbb{Q}}\left(X+\int_0^T P_t d\gamma_t+\frac{\kappa}{2\eta}\int_0^T|\zeta^{\gamma}_{t-}|^2dt+\frac{1}{4\eta}|\zeta^{\gamma}_T|^2\right)=$$

$$\mathbb{E}_{\mathbb{Q}}\left(X+\int_0^T(P_t-\mathbb{E}_{\mathbb{Q}}(Y|\mathcal{F}_t))d\gamma_t+\frac{\kappa}{2\eta}\int_0^T|\zeta^{\gamma}_{t-}|^2dt+\frac{1}{4\eta}|\zeta^{\gamma}_T|^2\right)$$

$$\geq \mathbb{E}_{\mathbb{Q}}\left(X-\int_0^T S_t e^{\kappa t}|d\gamma_t|+\frac{\kappa}{2\eta}\int_0^T|\zeta^{\gamma}_{t-}|^2dt+\frac{1}{4\eta}|\zeta^{\gamma}_T|^2\right)=$$

$$\mathbb{E}_{\mathbb{Q}}\Bigg(X+\int_0^T\frac{dA_t}{dt}e^{\kappa t}|d\gamma_t|-S_T\int_0^T e^{\kappa t}|d\gamma_t|+$$

$$\frac{\kappa}{2\eta}\int_0^T|\zeta^{\gamma}_{t-}|^2dt+\frac{1}{4\eta}|\zeta^{\gamma}_T|^2\Bigg)$$

$$\mathbb{E}_{\mathbb{Q}}\left(X-\frac{1}{2\kappa\eta}\int_0^T e^{2\kappa t}\left(\frac{dA_t}{dt}\right)^2dt-\frac{e^{2\kappa T}}{\eta}S_T^2\right).$$

Upper Bound

- Separation argument yields that $\exists \mathbb{Q} \sim \mathbb{P}$ such that

$$V \leq \mathbb{E}_{\mathbb{Q}}[X] +$$

$$\inf_{\gamma, \gamma_0 = \gamma_T = 0} \mathbb{E}_{\mathbb{Q}} \left(\int_0^T P_t d\gamma_t + \frac{\kappa}{2\eta} \int_0^T |\zeta^{\gamma}_{t-}|^2 dt + \frac{1}{4\eta} |\zeta^{\gamma}_T|^2 \right).$$

- $$\zeta^{\gamma}_t = \eta e^{-\kappa t} \int_0^t e^{\kappa u} |d\gamma_u|.$$

Continuation

$$V \leq \mathbb{E}_{\mathbb{Q}}[X] + \sup_{Y \in L^{\infty}} \inf_{\gamma, \gamma_0 = 0} \mathbb{E}_{\mathbb{Q}} \bigg(\int_{(0,T]} (P_t - \mathbb{E}_{\mathbb{Q}}(Y|\mathcal{F}_t)) d\gamma_t$$

$$+ \frac{\kappa}{2\eta} \int_0^T |\zeta^{\gamma}_{t-}|^2 dt + \frac{1}{4\eta} |\zeta^{\gamma}_T|^2 \bigg) =$$

$$E_{\mathbb{Q}}[X] + \sup_{Y \in L^{\infty}} \inf_{\gamma, \gamma_0 = 0} \mathbb{E}_{\mathbb{Q}} \bigg(- \int_{(0,T]} |P_t - \mathbb{E}_{\mathbb{Q}}(Y|\mathcal{F}_t)||d\gamma_t|$$

$$+ \frac{\kappa}{2\eta} \int_0^T |\zeta^{\gamma}_{t-}|^2 dt + \frac{1}{4\eta} |\zeta^{\gamma}_T|^2 \bigg)$$

Auxiliary result

- **Lemma:** For a given $\mathbb{Q}$ and

$$X_t = e^{-\kappa t}|\mathbb{E}_{\mathbb{Q}}(Y|\mathcal{F}_t) - P_t|, \ t \in [0, T]$$

we have

$$\inf_{\theta \text{ increasing }, \theta_0=0} \mathbb{E}_{\mathbb{Q}}\Bigg(-\int_{(0,T]} X_t d\theta_t + \frac{\kappa\eta}{2}\int_0^T e^{-2\kappa t}\theta_t^2 dt + \frac{\eta e^{-2\kappa T}\Theta_T^2}{4}\Bigg)$$
$$= \sup_{S=M+A\geq X} \mathbb{E}_{\mathbb{Q}}\left(\frac{e^{2\kappa T}}{\eta}S_T^2 + \frac{1}{2\kappa\eta}\int_0^T e^{2\kappa t}\left(\frac{dA_t}{dt}\right)^2 dt\right).$$

- Proved by applying stochastic representation theorem **P.Bank and N.E.Karoui**.

Main Idea

- Extending X beyond T by $X_t = e^{-\kappa(t-T)} X_T$.
- $\mu(dt) = \kappa e^{-2\kappa t} dt$.
- Stochastic representation theorem: $\exists L$

$$X_t = \mathbb{E}_{\mathbb{Q}} \left(\int_{(t,\infty)} \sup_{t \le v < s} L_v d\mu(s) | \mathcal{F}_t \right).$$

- Set: $\theta_t := \frac{1}{\eta} \sup_{0 \le v < t} L_v \vee 0, \ t \ge 0.$

$$S_t = \eta \mathbb{E}_{\mathbb{Q}} \left(\int_{(t,\infty)} \theta_s d\mu(s) | \mathcal{F}_t \right), \quad t \ge 0.$$

- Then θ is the minimizer and $S \ge X$ is the maximizer and they achieve the same value given by $\frac{\eta}{2} \mathbb{E}_{\mathbb{Q}} \left(\int_{(0,\infty)} \theta_u^2 d\mu(u) \right)$.

Formulation of the problem

- **In continuous time super–replication leads to buy–and–hold strategies.**
- Time horizon: $T = 1$.
- Number of time steps: $n \in \mathbb{N}$.
- Market active at times $0, \frac{1}{n}, ..., 1$.
- Price Impact:

$$A_k^{(n)} = A_{k-1}^{(n)} - \frac{\kappa}{2}(A_{k-1}^{(n)} - B_{k-1}^{(n)}) + \eta(\gamma_k - \gamma_{k-1})^+ + \frac{\sigma\xi_k}{\sqrt{n}}$$

$$B_k^{(n)} = B_{k-1}^{(n)} + \frac{\kappa}{2}(A_{k-1}^{(n)} - B_{k-1}^{(n)}) - \eta(\gamma_k - \gamma_{k-1})^- + \frac{\sigma\xi_k}{\sqrt{n}}$$

$$\xi_i = \pm 1.$$

- $\kappa \in (0, 1]$.
- **The resilience is scaled.**

Main Result

- In the n–step model we consider a European claim

$$X_n = F(P_n).$$

- **Theorem: (Bank & D)**

$$\lim V_n = \sup_{\nu \geq 0} \mathbb{E}_{\mathbb{P}_W} \left(F\left(\Pi_0 + \int_0^1 \nu_t dW_t \right) - \frac{\kappa}{4\eta(2-\kappa)} \int_0^1 \left(\frac{\nu_t^2}{\sigma} - \sigma \right)^2 \right).$$

- We get the same limit as we get with quadratic costs

$$\nu \to \frac{\eta(2-\kappa)}{4\kappa} \nu^2.$$

Happy Birthday Mete