Itô's Calculas under Sublinear Expectations via Regularity of PDEs and Rough Path

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Based on the joint work with Chen Pan (NUS)

Work related to Mete

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Outline

Linearity in probability theory

- 1-1 correspondence between linear expectation and additive probability measure,

$$P(X \in A) = E[1_A].$$

Sublinear Expectation $\tilde{E}$

Given χ (e.g. all bounded measurable random variables), $\tilde{E} : \chi \to R$ is sublinear iff

- (a) Monotonicity: If $X \leq Y$, then $\tilde{E}[X] \leq \tilde{E}[Y]$
- (b) Constant preserving: $\tilde{E}[X + c] = \tilde{E}[X] + c$
- (c) Sublinearity: $\tilde{E}[X + Y] \leq \tilde{E}[X] + \tilde{E}[Y]$.
- (d) Positive homogeneity: $\tilde{E}[\lambda X] = \lambda \tilde{E}[X]$ for all $\lambda > 0$

No more 1-1 correspondence between $\tilde{E}$ and $\tilde{P}$

Clearly

$$\tilde{P}(A) = \tilde{E}[1_A] = \tilde{E}_f[1_A]$$

for all f continuous and strictly increasing, $f(x) = x$ for $x \in [0, 1]$, where $\tilde{E}_f[X] = f^{-1}(\tilde{E}[f(X)])$.

Linear and sublinear expectations (Denis, Hu and Peng (2011))

There exists a weakly compact family of probability measures $\mathcal{P}$ such that

$$\tilde{E}[X] = \max_{P \in \mathcal{P}} E^P[X],$$

where E^P is the linear expectation with respect to P, for a proper class of random process X.

Sublinear expectation and model uncertainty

Sublinear expectation "measures" the model uncertainty, the bigger the expectation $\tilde{E}$, the more the uncertainty.

$$\tilde{E}_1[X] \leq \tilde{E}_2[X] \;\text{ iff }\; \mathcal{P}_1 \subset \mathcal{P}_2$$

Sublinear expectation and risk measure

Let

$$\rho(X) = \tilde{E}[-X]$$

Then we get a coherent risk measure $\rho : \chi \to R$

- (a) Monotonicity: If $X \geq Y$, then $\rho(X) \leq \rho(Y)$.
- (b) Constant translatability: $\rho(X + c) = \rho(X) - c$
- (c) Convexity: $\rho(\alpha X + (1 - \alpha)Y) \leq \alpha\rho(X) + (1 - \alpha)\rho(Y)$, $\alpha \in [0, 1]$.
- (d) Positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$ for all $\lambda > 0$.

G- expectation theory (Peng (2005))

- G-normal distribution $N(0 \times [\underline{\sigma}^2, \overline{\sigma}^2])$, characterized by the G-heat equation
$$\partial_t u - G(D^2 u) = 0, \quad u|_{t=0} = \phi.$$
Here $G(\cdot) : \mathbb{R} \to \mathbb{R}$ is a monotonic, sublinear function, with
$$G(\gamma) = \frac{1}{2} \sup_{\alpha \in [\underline{\sigma}^2, \overline{\sigma}^2]} \gamma\alpha,$$
- $G-$ Brownian Motion, via the notion of G-independence
- Itô's calculus developed under G-Itô's isometry

Link between PDEs and probability

- Independence: probability= measure theory + notion of independence
- Regularity: the subject of PDEs/Control
- Stroock and Varadhan (1979) explored the regularity of the solution to a linear PDE for the uniqueness of the solution to a martingale problem

Our work

- Propose and study the martingale problem in a sublinear expectation space in the spirit of Stroock and Varadhan (1979)
- Build Itô's calculus for the canonical process in this sublinear expectation space

Key ideas: Regularity of fully nonlinear PDEs and rough path theory

Martingale problem

Find a family of operators $\{\mathcal{E}_t\}_{t\geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H})$ such that

$$\varphi(X_t) - \int_0^t G(X_\theta, \varphi_x(X_\theta), \varphi_{xx}(X_\theta))\, d\theta, t \geq 0$$

is an $\{\mathcal{E}_t\}$-martingale for all $\varphi \in C_0^\infty(\mathbb{R}^d)$.

- $G : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d \to \mathbb{R}$ continuous with desirable properties
- $\Omega = C_{x_0}([0, \infty); \mathbb{R}^d)$ and $X_t(\omega) = \omega(t), \omega \in \Omega$.

Our martingale problems vs classical martingale problems

• **Classical M.P.'s**	• **Our M.P.'s**
to find a probability measure P on $(\Omega, \mathcal{F})$	to find a sublinear expectation $\mathcal{E}$ on $(\Omega, \mathcal{H})$
$X_0 = x$ P-a.s.	$X_0 = x$ in $L^1_{\mathcal{E}}$ or $L^2_{\mathcal{E}}$
$\varphi(X_t) - \int_0^t L_\theta \varphi(X_\theta)\, d\theta$ is a P-martingale for $\forall \varphi \in C_0^\infty$	$\varphi(X_t) - \int_0^t G(X_\theta, \varphi_x(X_\theta), \varphi_{xx}(X_\theta)) d\theta$ is an $\mathcal{E}$-martingale for $\forall \varphi \in C_0^\infty$
$L_\theta = \frac{1}{2} \sum a^{ij}(\theta, \cdot) \frac{\partial^2}{\partial x_i \partial x_j} + \sum b^i(\theta, \cdot) \frac{\partial}{\partial x_i}$ is a linear differential operator	Nonlinear PDE associated with $G : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d \to \mathbb{R}$ is a continuous function with some properties

Choice of G

A function $G : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d \to \mathbb{R}$

$$G = G(x, p, A) = \sup_{\gamma \in \Gamma} \left\{ \frac{1}{2} tr[a(x, \gamma)A] + (b(x, \gamma), p) \right\},$$

- Γ given compact metric space
- $a(x, \gamma) = \sigma(x, \gamma)\sigma'(x, \gamma)$ positive semidefinite
- $\sigma, b \in C(\mathbb{R}^d \times \Gamma)$, and σ and b uniformly bounded and uniformly Lipschitz continuous with respect to x

This function $G:\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^d\to\mathbb{R}$ satisfies

A. Subadditivity $G(x,p+\bar{p},A+\bar{A})\leq G(x,p,A)+G(x,\bar{p},\bar{A})$;

B. Positive Homogeneity $G(x,\lambda p,\lambda A)=\lambda G(x,p,A)$;

C. Monotonicity $G(x,p,A)\leq G(x,p,A+\tilde{A})$;

D. G is uniformly Liptschitz continuous with respect to x.

PDEs associated with G

For a given G, the associated state-dependent parabolic PDE

$$\begin{cases} \partial_t u(t,x) - G(x, Du(t,x), D^2 u(t,x)) = 0, \ (t,x) \in (0,\infty] \times \mathbb{R}^d, \\ u(0,x) = \varphi(x), \ x \in \mathbb{R}^d. \end{cases}$$

- $a(\cdot,\cdot) \equiv I$ and $b(\cdot,\cdot) \equiv 0$, this PDE (P) is the heat equation
- In general, this type of PDE (P) corresponds to the HJB equation from stochastic control

Known results about related PDEs

Fleming and Soner (1992)

There exists a unique viscosity solution for this PDE (P) with polynomial growth, assuming $\varphi(x) \in C(R^d)$ with a polynomial growth.

Evans-Krylov (1982)

For a class of convex, fully nonlinear PDEs, if $\varphi(x) \in C_b([0,T] \times \mathbb{R}^d)$, then the solution u possess the following properties:

I) $u \in C_b([0,T] \times \mathbb{R}^d)$;

II) there exists a constant $\alpha_0 \in (0,1)$ only depending on d, K, ε such that for each $\kappa > 0$, $\|u\|_{C^{2+\alpha_0}([\kappa,T]\times\mathbb{R}^d)} < \infty$.

Furthermore, if $\varphi \in C^{2+\alpha_1}(\mathbb{R}^d)$ and bounded, then $u \in C^{2+\alpha}([0,T] \times \mathbb{R}^d)$ for $\alpha = \alpha_0 \wedge \alpha_1 \in (0,1)$.

Properties of such PDEs

By the property of G, the stability of the viscosity solution and Evans-Krylov (1982),

Smooth approximation of viscosity solutions

If $\varphi(x) \in C_0^\infty(R^d)$ The unique solution of the PDE can be approximated by $C^{2+\alpha}$ solutions of the same type of PDEs on compact subsets.

Properties of solutions to the PDEs

Let $u^\varphi \in C([0,T] \times \mathbb{R}^d)$ denote the unique solutions of (P) with polynomial growth, respectively. Then

$$u^{\varphi+c} = u^\varphi + c,$$
$$u^\varphi - u^\phi \leq u^{\varphi-\phi},$$
$$u^{\lambda\varphi} = \lambda u^\varphi,\ \lambda \geq 0.$$

with $c \in \mathbb{R}$ a constant, and φ, ϕ continuous with polynomial growth.

Constructing conditional expectations

- Finite dimensional construction via backward induction, in the spirit of BSDE by using the unqiue solution of the PDEs
- Extend to a Banach space under the norm $\|\cdot\|_1 = \mathcal{E}[|\cdot|]$ or $\|\cdot\|_2 = \sqrt{\mathcal{E}[|\cdot|^2]}$.

The finite dimensional construction

- $\Omega = C_{x_0}([0,\infty);\mathbb{R}^d)$, $X_t(\omega) = \omega_t \in \Omega$ is canonical
- Define an operator $\mathcal{T}_t[\varphi(\cdot)](x) = u(t,x)$, where u is the unique viscosity solution of the PDE
- Take φ in a proper function space, say $\mathcal{C}_{l,Lip}((\mathbb{R}^d)^N)$ where $C_{l.Lip}(\mathbb{R}^n)$ is the space of real-valued continuous functions defined on $\mathbb{R}^n$ such that for all $\varphi \in C_{l.Lip}(\mathbb{R}^n)$,

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \forall\, x, y \in \mathbb{R}^n$$

for some $C > 0, m \in \mathbb{N}$ depending on φ.

- Set $\xi(\omega) = \varphi(X_{t_1}, \cdots, X_{t_N})$, $0 = t_0 \leq t_1 \leq \cdots \leq t_N \leq T$,
- Define $\mathcal{E}_t$ by

$$\mathcal{E}_t[\xi] = \varphi_{N-j}(\omega_{t_1}, \cdots, \omega_{t_j}), \text{ if } t = t_j, 0 \leq j \leq N,$$

Here

$$\begin{aligned} \varphi_1(x_1, \cdots, x_{N-1}) &= \mathcal{T}_{t_N - t_{N-1}}[\varphi(x_1, \cdots, x_{N-1}, \cdot)](x_{N-1}), \\ &\cdots \\ \varphi_{N-j}(x_1, \cdots, x_j) &= \mathcal{T}_{t_{j+1} - t_j}[\varphi_{N-j-1}(x_1, \cdots, x_j, \cdot)](x_j), \\ &\cdots \\ \varphi_{N-1}(x_1) &= \mathcal{T}_{t_2 - t_1}[\varphi_{N-2}(x_1, \cdot)](x_1), \\ \varphi_N &= \mathcal{T}_{t_1}[\varphi_{N-1}(\cdot)](x_0), \end{aligned}$$

Sublinear expectation space $(\Omega, \mathcal{H}, \mathcal{E})$

- $(\Omega_T, \mathcal{H}_T, \mathcal{E})$ is a sublinear expectation space, with $\Omega_T := \{\omega_{\cdot \wedge T}; \omega \in \Omega\}$ and $\mathcal{H}_T := L_{ip}(\Omega_T) = \{\varphi(X_{t_1}, \cdots, X_{t_N}); \varphi \in C_{l.Lip}((\mathbb{R}^d)^N)$ for some $N \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_N \leq T\}$
- For each $t \in [0, T]$, one can extend the space $L_{ip}(\Omega_t) \subset L_{ip}(\Omega_T), t \leq T$ to a Banach space $L^i_{\mathcal{E}}(\Omega_t)$ under the norm $\| \cdot \|_1 = \mathcal{E}[| \cdot |]$ or $\| \cdot \|_2 = \sqrt{\mathcal{E}[| \cdot |^2]}$. Set $\mathcal{H} = L^i_{\mathcal{E}} := \cup_{T \geq 0} L^i_{\mathcal{E}}(\Omega_T)$

Properties of $\mathcal{E}$

Given such a sublinear expectation space $(\Omega, \mathcal{H}, \mathcal{E})$. For any $\xi, \eta \in \mathcal{H}$

- (Monotonicity)
$$\mathcal{E}_t[\xi] \leq \mathcal{E}_t[\eta] \text{ if } \xi \leq \eta$$
- (Constant preserving) For $c \in \mathbb{R}$ constant,
$$\mathcal{E}[\xi + c] = \mathcal{E}[\xi] + c$$
- (Tower property)
$$\mathcal{E}_s \circ \mathcal{E}_{s+h} = \widetilde{\mathcal{E}}_s, \ h > 0.$$
- (Subadditivity) $\mathcal{E}_t[\xi + \eta] \leq \mathcal{E}_t[\xi] + \mathcal{E}_t[\eta]$.
- (Positive homogeneity)
$$\mathcal{E}_s[\xi\eta] = \xi^+ \mathcal{E}_s[\eta] + \xi^- \mathcal{E}_s[-\eta].$$
In particular,
$$\mathcal{E}_t[\lambda\xi] = \lambda\mathcal{E}_t[\xi]$$
for any constant $\lambda \geq 0$.

For the time dependent G, the tower property becomes the semi-group property

Martingale Problem

Let $\Omega = C_x([0,\infty);\mathbb{R}^d)$ with $\omega_0 = x \in \mathbb{R}^d$, set $X_t(\omega) := \omega_t$. Given the PDE (P) with $\varphi \in C_0^\infty(\mathbb{R})$.

$$\varphi(X_t) - \int_0^t G\left(X_\theta, \varphi_x(X_\theta), \varphi_{xx}(X_\theta)\right) d\theta] = 0,\ 0 \leq s \leq t < \infty.$$

is an $\mathcal{E}-$ martingale. Here $\mathcal{E} = \{\mathcal{E}_t\}$ with $\mathcal{E}_t$ the family of conditional expectations on the sublinear expectation space generated from the PDE (P).

$\mathcal{E}$-Martingale

Given a sublinear expectation space $(\Omega, \mathcal{H}, \mathcal{E})$, a stochastic process $(\xi_t)_{t\geq 0}$ is a collection of random variables on $(\Omega, \mathcal{H})$, i.e., for each $t \geq 0$, $\xi_t \in L^i_{\mathcal{E}}(\Omega_t)$, $i = 1, 2$.

A stochastic process $(M_t)_{t\geq 0}$ is called an *$\mathcal{E}$-martingale* if for each $t \in [0, \infty)$, $M_t \in L^i_{\mathcal{E}}(\Omega_t)$, and for each $s \in [0, t]$,

$$\mathcal{E}_s[M_t] = M_s.$$

Key idea to solving the martingale problem

- The positive homogeneity, monotonicity, and the constant preserving of $\mathcal{E}$ leads to

Given a sublinear expectation space $(\Omega, \mathcal{H}, \mathcal{E})$. Let $X \in \mathcal{H}$ be given. Then for each sequence $\{\varphi_n\}_{n=1}^{\infty} \subset \mathcal{C}(\mathbb{R}^d)$ satisfying $\varphi_n \downarrow 0$, we have $\mathcal{E}[\varphi_n(X)] \downarrow 0$.

- Focus on the simple process and smooth solutions of PDE. Take $\pi = \{\theta_i; \theta_0 = s < \theta_1 < \cdots < \theta_K = t, K \in \mathbb{N}\}, \|\pi\| = \max_{1 \le k \le K} |\Delta\theta_k|, \Delta\theta_k = \theta_k - \theta_{k-1}$
  $g(X_\theta) := G\left(X_\theta, \varphi_x(X_\theta), \varphi_{xx}(X_\theta)\right)$.
  Given a smooth PDE solution in the sense of Evans-Krylov,

$$\mathcal{E}_s[\sum_{k=1}^{K}\{\varphi(X_{\theta_k}) - \varphi(X_{\theta_{k-1}}) - g(X_{\theta_{k-1}})\Delta\theta_k\}] = o(1), \text{ as } \|\pi\| \to 0.$$

The proof needs $\partial_t u$ to be uniformly $\frac{\alpha}{2}$-Hölder continuous in t, because

$$
\begin{aligned}
&\mathcal{E}_{\theta_{k-1}}[\varphi(X_{\theta_k}) - \varphi(X_{\theta_{k-1}}) - g(X_{\theta_{k-1}})\Delta\theta_k] \\
=&\mathcal{E}_{\theta_{k-1}}[\varphi(X_{\theta_k})] - \varphi(X_{\theta_{k-1}}) - g(X_{\theta_{k-1}})\Delta\theta_k \\
=&\ u(\Delta\theta_k, X_{\theta_{k-1}}) - u(0, X_{\theta_{k-1}}) - g(X_{\theta_{k-1}})\Delta\theta_k \\
=&\ \left[\frac{\partial u}{\partial t}(0, X_{\theta_{k-1}}) - g(X_{\theta_{k-1}})\right]\Delta\theta_k + O((\Delta\theta_k)^{1+\frac{\alpha}{2}}) \\
=&\ O((\Delta\theta_k)^{1+\frac{\alpha}{2}})
\end{aligned}
$$

Solution of martingale problem leads to

Key moment estimates

Given $T > 0$. Let $t \in [0, T]$ and $h > 0$ such that $t_i = t + ih \in [0, T], i = 0, 1, 2, 3$. Then we have the following estimates

$$\begin{aligned} |\mathcal{E}_t[\pm(X_t - X_s)]| &\leq L|t - s|, \\ \mathcal{E}_t[(X_{t+h} - X_t)^{2n}] &\leq C(n, L, T)h^n, \\ \mathcal{E}_t[\pm(X_{t+2h} - X_{t+h})(X_{t+h} - X_t)]| &\leq L\sqrt{C(1, L, T)}h^{\frac{3}{2}}, \\ \mathcal{E}_t[|(X_{t_1} - X_{t_0})(X_{t_2} - X_{t_1})(X_{t_3} - X_{t_2})|] &\leq C(1, L, T)^{\frac{3}{2}} h^{\frac{3}{2}}, \end{aligned}$$

where L is the Lipschitz constant of G, $C(n, L, T)$ is a constant depending only on n, L and T.

Remark

- Our estimate is consistent with the G-framework where independent increments assumption is needed.
  Take $X_t = B_t + \langle B \rangle_t$, then under the G-expectation $\hat{\mathbb{E}}$,
  $$\hat{\mathbb{E}}[(X_{2h} - X_h)X_h] = \hat{\mathbb{E}}[(B_{2h} + \langle B \rangle_{2h} - B_h - \langle B \rangle_h)(B_h + \langle B \rangle_h)]$$
  $$\sim O(h^{\frac{3}{2}}).$$
- The moment estimates are crucial for establishing Itô's calculus as the random process in the sublinear expectation space constructed from the PDE in general has no independent increment

Moments estimates leads to

$$\langle X \rangle_t \stackrel{L^2_{\mathcal{E}}}{=} \lim_{n \to \infty} \sum_{\Pi_N} |X_{t_{k+1}} - X_{t_k}|^2,$$

$$\int_0^T \phi(X_t)\, dX_t \stackrel{L^2_{\mathcal{E}}}{=} \lim_{\|\Pi_N\| \to 0} \sum_{t_k \in \Pi_N} \phi(X_{t_k})(X_{t_{k+1}} - X_{t_k}),$$

with $\Pi_N = \{t_k; t_k = kt/N, k = 0, 1, \ldots, N\}$

For general $\phi(X)$ and general partition

To show the limit is independent of the partition, need "sewing lemma" for estimating error term.

To see this, assume $\phi \in C^1(\mathbb{R})$ for each $u \in [s,t]$, and approximate $\phi(X_u)$ by

$$\phi(X_u) \approx \phi(X_s) + \phi'(X_s)X_{s,u}.$$

Then formally the integral

$$\int_0^T \phi(X_\theta)\, dX_\theta \approx \sum_{\Pi_N} \left[\phi(X_{t_k})X_{t_k,t_{k+1}} + \phi'(X_{t_k}) \int_{t_k}^{t_{k+1}} X_{t_k,\theta}\, dX_\theta \right],$$

with the error term

$$R(s,t) := \int_s^t \phi(X_\theta)\, dX_\theta - \left[\phi(X_s)X_{s,t} + \phi'(X_s) \int_s^t X_{s,\theta}\, dX_\theta \right]$$

Establishing "Sewing Lemma" to estimate

$$\|R(s,t)\|_2 := \sqrt{\mathcal{E}[|R(s,t)|^2]} \le C|t-s|^\beta, \quad \text{for some} \quad \beta > 1$$

Generalized Itô's isometry

Suppose $\varphi \in C^1(\mathbb{R})$ and $\varphi' \in \mathcal{P}$. Then

$$\mathcal{E}_s\left[\left(\int_s^{s+h} \varphi(X_\theta)\, dX_\theta\right)^2\right] \leq C_1 h, \tag{1}$$

$$\mathcal{E}_s\left[\left(\int_s^{s+h} \varphi(X_\theta)\, dX_\theta\right)^4\right] \leq C_2 h^2, \tag{2}$$

$$\mathcal{E}_s\left[\pm\left(\int_s^{s+h} \varphi(X_\theta)\, dX_\theta\right)\right] \leq C_3 h, \tag{3}$$

with $h > 0, s, s+h \in [0, T]$, and C_1, C_2, C_3 depending on φ, L, and T,

$$\mathcal{P} = \{\varphi \in C(\mathbb{R}) : \text{ there exist constants } C > 0, p \in \mathbb{N}, \gamma \in (0,1] \text{ s.t. } |\varphi(x) - \varphi(y)| \leq C(1 + |x|^p + |y|^p)|x-y|^\gamma \text{ for any } x, y \in \mathbb{R}\}.$$

Itô's formula

Let

$$\eta_t = \eta_0 + \int_0^t \mu(X_\theta)\, d\theta + \int_0^t \rho(X_\theta)\, d\langle X\rangle_\theta + \int_0^t \varsigma(X_\theta)\, dX_\theta, \quad 0 \le t \le T,$$

where $\eta_0 \in \mathbb{R}$ is a constant. Suppose $\varsigma \in C^1(\mathbb{R})$ and $\varsigma' \in \mathcal{P}$. Then, for any $f \in C^2(\mathbb{R})$ satisfying $f'' \in \mathcal{P}$,

$$f(\eta_t) = f(\eta_0) + \int_0^t f'(\eta_\theta)\, d\eta_\theta + \frac{1}{2}\int_0^t f''(\eta_\theta)\varsigma^2(X_\theta)\, d\langle X\rangle_\theta.$$

Integration by parts

Let $\phi, \psi \in C^1(\mathbb{R})$, and $\phi', \psi' \in \mathcal{P}$, then under the $L^2_{\mathcal{E}}$ norm

$$\phi(X_t)\psi(X_t) - \phi(X_s)\psi(X_s) = \int_s^t \phi(X_\theta)\, d\psi(X_\theta) + \int_s^t \psi(X_\theta)\, d\phi(X_\theta) + \langle \phi(X), \psi(X) \rangle_{s,t}$$

Here the cross variation process $\langle \phi(X), \psi(X) \rangle$ is defined by

$$\langle \phi(X), \psi(X) \rangle = \frac{1}{4} \left[\langle \phi(X) + \psi(X) \rangle - \langle \phi(X) - \psi(X) \rangle \right]$$

and the quadratic variation $\langle \phi(X) \rangle_t$ defined as

$$\langle \phi(X) \rangle_t = \phi(X_t)^2 - \phi(X_0)^2 - 2 \int_0^t \phi(X_\theta)\, d\phi(X_\theta) \quad \text{in } L^2_{\mathcal{E}}.$$

Summary

- A martingale problem under sub-linear expectation is proposed and studied, and associated Ito's calculus is developed
- The analytical properties of PDE, especially the semi-group property and the regularity of the solution, replace the dependence structure of the random process X, and leads to generalized Itô's isometry.
- Extension to path dependent PDEs or jump processes?

More related works

- Construction of stochastic integrals with respect to semi-martingale: Bichleter (1981), Nutz (2012),
- G-SDEs with rough paths: Geng, Qian, and Yang (2014), Peng and Zhang (2015)
- Rough path: Lyons (1998), Friz and Hairer (2014)

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- $u \in C^{\alpha}(Q), \alpha \in (0,1]$, if

$$\|u\|_{C^{\alpha}(Q)} := \sup_{\substack{x \neq y, x, y \in \mathbb{R}^d \\ s \neq t, s, t \in [0,T]}} \frac{|u(t,x) - u(s,y)|}{(|t-s|^{\frac{1}{2}} + |x-y|)^{\alpha}} < \infty.$$

- $u \in C^{2+\alpha}(Q), \alpha \in (0,1]$, if

$$\|u\|_{C(Q)} + \|\partial_t u\|_{C(Q)} + \|Du\|_{C(Q)} + \|D^2 u\|_{C(Q)} + \|\partial_t u\|_{C^{\alpha}(Q)} + \|D^2 u\|_{C^{\alpha}(Q)} < \infty.$$

- $C_0^{\infty}(\mathbb{R}^d)$ is the space of C^{∞} functions having compact support on $\mathbb{R}^d$.

Happy Birthday, METE!