Arbitrage Theory via Numéraires: A Survey

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OUTLINE

We survey a theory for finance based on the following principle: That *"it should not be possible to fund* **nontrivial** *liability or consumption streams, starting with initial capital which is strictly positive but* **arbitrarily close to zero**".

In other words, by proscribing fairly egregious forms of what is commonly called *arbitrage*.

This requirement

• turns out to be equivalent to the existence of a *growth-optimal portfolio*, a notion at least as old as KELLY (1956);

• can be easily described in terms of the underlying model's characteristics, and is equivalent to the existence of appropriate *deflators*.

Using such tools it is possible to develop an entire mathematical theory for the subject, which deals with

(i) hedging,

(ii) utility maximization,

(iii) portfolio constraints that include “drawdown” restrictions,

(iv) equilibrium, as well as

(v) markets with an arbitrary number of assets.

You will not need to know anything about finance, to be able to follow this talk. You will need, though, some elements of stochastic analysis.

FINANCIAL MARKET

Consider a market with a finite number n assets. The strictly positive prices $S_1, \cdots, S_n$, as well as the returns $R_1, \cdots, R_n$ of these assets, are modeled by continuous **semimartingales** ("signals, in a bath of noise"):

$$\frac{\mathrm{d}S_i(t)}{S_i(t)} = \mathrm{d}R_i(t) = \mathrm{d}A_i(t) + \mathrm{d}M_i(t), \qquad i = 1, \cdots, n.$$

Here

$$R_i = A_i + M_i$$

are the assets' cumulative *return processes:* continuous semimartingales in their own right, with bounded variation ("signal", "mean return", "trend") parts A_i, and local martingale ("noise") parts M_i with covariations

$$C_{ij} := [M_i, M_j] = [R_i, R_j], \qquad 1 \leq i, j \leq n.$$

We recall here the definition

$$[R_i, R_j](t) = \lim_{m\to\infty} \sum_{k=1}^{m} \left(R_i\left(\frac{kt}{m}\right) - R_i\left(\frac{(k-1)t}{m}\right)\right) \cdot \cdot \left(R_j\left(\frac{kt}{m}\right) - R_j\left(\frac{(k-1)t}{m}\right)\right)$$

of the covariation, as a FINITE limit in probability, for every $t \in [0, \infty)$.

(More on this down the road.)

We go from cumulative returns to prices, via the stochastic exponentials

$$\frac{S_i}{S_i(0)} = \mathcal{E}(R_i) = \exp\Big(R_i - \frac{1}{2}C_{ii}\Big) = \exp\Big(\Gamma_i + M_i\Big).$$

Here we recall

$$R_i = A_i + M_i\,,$$

and write

$$\Gamma_i := A_i - \frac{1}{2}C_{ii}$$

for the "cumulative growth" of asset $i = 1, \cdots, n$.

. All processes are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and are adapted to an underlying filtration, or "flow of information",

$$\mathfrak{F} = \{\mathcal{F}(t)\}_{0 \le t < \infty} .$$

This is only assumed to be right continuous – nothing else. For concreteness only, we take $\mathcal{F}(0) = \{\emptyset, \Omega\}$.

CANONICAL EXAMPLE

The "canonical" case concerns cumulative returns

$$R_i = A_i + M_i$$

with signals and noises of the Itô process form

$$A_i = \int_0^{\cdot} \alpha_i(t)\,\mathrm{d}t\,, \qquad M_i = \int_0^{\cdot} \sum_{\nu=1}^{d} \sigma_{i\nu}(t)\,\mathrm{d}W_\nu$$

for suitable "mean-rate-of-return" vector $\alpha = (\alpha_i)_{1 \le i \le n}$ and "volatility" matrix $\sigma = (\sigma_{i\nu})_{\substack{1 \le i \le n \\ 1 \le j \le d}}$, thus "covariation rates"

$$c_{ij}(t) := \sum_{\nu=1}^{d} \sigma_{i\nu}(t)\,\sigma_{j\nu}(t) = \Big(\sigma(t)\,\sigma'(t)\Big)_{ij} = \frac{\mathrm{d}}{\mathrm{d}t}\,C_{ij}(t).$$

Remark: Introducing the "intrinsic clock"

$$\mathcal{O}(t) := \sum_{i=1}^{n} \left(\check{A}_i(t) + \sum_{j=1}^{n} C_{ii}(t) \right) , \qquad 0 \leq t < \infty$$

of the market, we can ALWAYS write

$$A_i = \int_0^{\cdot} \alpha_i(t) \, \mathrm{d}\mathcal{O}(t)$$

for an appropriate, progressively measurable processes $\alpha_i \, , \; i = 1, \cdots , n \, .$

TRADING

Let us place now a small investor in this market:
Give him an initial capital $x \in (0, \infty)$, and let him choose a predicable *investment strategy* $\vartheta = (\vartheta_1, \cdots, \vartheta_n)' \in \mathcal{I}(S)$.
This generates "gains from trade", thus "wealth"

$$X(\cdot\,; x, \vartheta) := x + \int_0^{\cdot} \vartheta'(t)\, \mathrm{d}S(t) \equiv x + \int_0^{\cdot} \sum_{i=1}^{n} \vartheta_i(t)\, \mathrm{d}S_i(t).$$

Interpretation: The quantity $\vartheta_i(t)$ is the number of shares "bought", thus $\vartheta_i(t)\, S_i(t)$ the currency amount invested, at time t, in the i th asset; whereas the difference

$$X(t\,; x, \vartheta) - \sum_{i=1}^{n} \vartheta_i(t)\, S_i(t)$$

is placed under the proverbial mattress (a.k.a "money-market", from which one borrows and lends at interest rate $r \equiv 0$).

ADMISSIBILITY and NUMÉRAIRES

Very outrageous things – gains from trade known as "doubling strategies" – can occur in this setting.

So we need to impose some constraints.

The first thing to do is constrain borrowing, recognizing the hard fact that *"eventually one runs out of other people's money"*.

We do this in a somewhat Draconian manner, prohibiting borrowing altogether.

That is, we say that a wealth process as above is *admissible*, if

$$X(\cdot\,; x, \vartheta) \geq 0.$$

Penury is tolerated; but debt (negative wealth) is not.

Definition

An admissible wealth process $X \equiv X(\cdot\,; 1, \vartheta) > 0$ which is actually strictly positive, is called *numéraire*.

Every such numéraire X is generated equivalently, by a *portfolio process* $\pi = (\pi_1, \cdots, \pi_n)' \in \mathcal{I}(R)$ with weights (proportions)

$$\pi_i(t) = \frac{\vartheta_i(t)\, S_i(t)}{X(t)}\,, \quad \text{via}$$

$$\frac{\mathrm{d}X(t)}{X(t)} = \sum_{i=1}^{n} \pi_i(t)\, \frac{\mathrm{d}S_i(t)}{S_i(t)} = \sum_{i=1}^{n} \pi_i(t)\, \mathrm{d}R_i(t) =: \mathrm{d}R_\pi(t).$$

Remark: We note that $\pi_0(t) := 1 - \sum_{i=1}^{n} \pi_i(t)$ is the proportion of current wealth placed, at any given time t, in the money-market.

Just as before, this leads to the stochastic exponential

$$X \equiv X^{\pi} = \exp\left(R_{\pi} - \frac{1}{2} C_{\pi\pi}\right) = \exp\left(\Gamma_{\pi} + M_{\pi}\right)$$

representations. Here we are setting

$$R_{\pi} = A_{\pi} + M_{\pi} \,,$$

$$A_{\pi} := \int_0^{\cdot} \sum_{i=1}^{n} \pi_i(t)\, \mathrm{d}A_i(t) \,, \qquad M_{\pi} := \int_0^{\cdot} \sum_{i=1}^{n} \pi_i(t)\, \mathrm{d}M_i(t) \,.$$

The scalar processes of finite variation

$$\Gamma_{\pi} := A_{\pi} - \frac{1}{2} C_{\pi\pi} \,, \qquad C_{\pi\pi} := \int_0^{\cdot} \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i(t) \pi_j(t)\, \mathrm{d}C_{ij}(t) < \infty$$

are, respectively, the cumulative growth and the cumulative variation of the portfolio π.

AN IMPORTANT QUANTITY

In the framework of our "canonical" example, we express this cumulative growth in terms of its rate process:

$$\Gamma_\pi = \int_0^{\cdot} \gamma_\pi(t)\, \mathrm{d}\mathcal{O}(t)\,, \qquad \gamma_\pi(t) = \left(p'\alpha(t) - \frac{1}{2}\, p'c(t)p \right)\Bigg|_{p=\pi(t)}\,.$$

- Let us note that, **IF** a portfolio $\nu \in \mathcal{I}(R)$ exists with

$$\alpha(t) = c(t)\,\nu(t)\,, \qquad 0 \le t < \infty\,,$$

then the maximal rate of growth in this market is

$$\mathfrak{g}(t) := \frac{1}{2}\,\nu'(t)\,c(t)\,\nu(t) = \gamma_\nu(t) \ge \gamma_\pi(t)\,, \qquad 0 \le t < \infty\,,$$

with the inequality valid for ANY portfolio $\pi \in \mathcal{I}(R)$.

CUMULATIVE CAPITAL WITHDRAWAL STREAMS

Adapted, increasing, right-continuous processes K with $K(0) = 0$. Models future cumulative *consumption*, or *liabilities.* Class $\mathcal{K}$.

Example

(A) "Smooth" consumption stream

$$K = \int_0^{\cdot} \kappa(t)\,\mathrm{d}t\,.$$

(B) European contingent claim (liability), for some $T \in (0, \infty)$ and $\mathcal{F}(T)$-measurable r.v. $P(T) \geq 0$:

$$K(t) = P(T)\,\mathbf{1}_{[T,\infty)}(t),\;\; 0 \leq t < \infty\,.$$

. We say that a given stream $K \in \mathcal{K}$ is "non-zero" (or "non-trivial"), if

$$\mathbb{P}(K(\infty) > 0) > 0.$$

Example

"European call option"

$$P(T) = \big(S_1(T) - q\big)^+$$

with $\mathbb{P}(S_1(T) > q) > 0$.

FINANCING, OR "HEDGING", C.C.W. STREAMS

Definition

We say that a wealth process $X \equiv X(\cdot\,; x, \vartheta)$ with $x \in [0, \infty)$, $\vartheta \in \mathcal{I}(S)$ **finances** (or "funds", or "hedges") a given c.c.w.s. $K \in \mathcal{K}$, if

$$X(\cdot\,; x, \vartheta) \geq K$$

holds.

. We denote the collection of all c.c.w. streams $K \in \mathcal{K}$ that can be financed starting from a given initial capital $x \in (0, \infty)$, by

$$\mathcal{K}(x) := \left\{ K \in \mathcal{K} \,:\, \exists\, \vartheta \in \mathcal{I}(S)\,, \ \text{ s.t. } \ X(\cdot\,; x, \vartheta) \geq K \right\}.$$

Clearly

$$\mathcal{K}(0) \subseteq \mathcal{K}(y) \subseteq \mathcal{K}(x)$$

if $0 < y < x < \infty$.

As well as

$$\{\mathbf{0}\} \subseteq \mathcal{K}(0) \subseteq \boxed{\mathcal{K}(0+) := \bigcap_{y>0} \mathcal{K}(y)} \subseteq \mathcal{K}(x)$$

for every $x \in (0, \infty)$.

Definition

We say that an adapted, RCLL process $\widehat{X}$ is a **minimal hedge** of the given c.c.w. stream $K \in \mathcal{K}$, if

(i) $\widehat{X} \geq K$ holds,

and if

(ii) $\widehat{X}$ is dominated by every wealth process that finances K.

. That is, if for every $x \geq 0$, $\vartheta \in \mathcal{I}(S)$ with the property $X(\cdot\,; x, \vartheta) \geq K$ as above, we have

$$K \leq \widehat{X} \leq X(\cdot\,; x, \vartheta)\,.$$

A FEW FUNDAMENTAL QUESTIONS

(A) How do we make the market model "VIABLE", that is, ensure it does not contain egregious [1] forms of arbitrage?

(B) Once we have settled on a notion of viability, can we formulate it in terms of market characteristics? If so: How?

[1] Possibilities of financing a lavish lifestyle (consumption pattern) "for free", or "essentially for free".

(C) Given a nontrivial stream $K \in \mathcal{K}$, what is "the smallest" amount of initial capital

$$\mathbf{x}(K) := \inf\left\{x > 0 \,:\, \exists\, \vartheta \in \mathcal{I}(S),\ \text{s.t.}\ \ X(\cdot\,; x, \vartheta) \geq K\,\right\}$$

starting with which the stream K can be financed?
(Viability will ensure, in particular, that $\mathbf{x}(K) > 0\,$.)

(D) Conversely, starting from a given initial capital $x \in (0, \infty)$, how can we describe effectively the collection

$$\mathcal{K}(x) := \left\{K \in \mathcal{K} \,:\, \exists\, \vartheta \in \mathcal{I}(S)\,,\ \ \text{s.t.}\ \ X(\cdot\,; x, \vartheta) \geq K\,\right\}$$

of all streams $K \in \mathcal{K}$, that can be financed starting from it?

We will develop presently some notions, that will help us answer these questions in the most general and efficient manner we are aware of.

Let's get on with it.

VIABILITY

Definition

We shall say that the market under consideration is **viable**, if in the notation of five slides back, we have

$$\mathcal{K}(0+) = \{\mathbf{0}\}.$$

In other words, we call a given market viable, if it is not possible to finance in it any given non-zero stream, starting with initial capital arbitrarily near zero. Formally,

$$K \in \mathcal{K}\,, \quad \mathbf{x}(K) = 0 \quad \Longrightarrow \quad K \equiv \mathbf{0}\,.$$

. Equivalently, viability amounts to the boundedness in probability of wealth levels attainable via portfolios, at any given time T; i.e.,

$$\lim_{m\to\infty} \sup_{\pi \in \mathcal{I}(R)} \mathbb{P}\big[X^{\pi}(T) > m\big] = 0\,, \qquad \forall \quad T \in (0,\infty)$$

LOCAL (and SUPER) MARTINGALE NUMÉRAIRE

A given portfolio $\nu \in \mathcal{I}(R)$ will be said to have the *local (resp., super) martingale numéraire property* if, for every $\pi \in \mathcal{I}(R)$, the ratio

$$\frac{X^{\pi}}{X^{\nu}} \quad \text{is a local (resp., super) martingale.}$$

We call then X^{ν} *the* local (resp., super) martingale numéraire.

The reciprocal $1/X^{\nu}$, of the wealth that the portfolio $\nu \in \mathcal{I}(R)$ generates, is then itself a local (resp., super) martingale.
We shall see that this quantity, when it exists, acts as "natural discount factor" for all economic activity.

Proposition

For any given portfolio $\nu \in \mathcal{I}(R)$, *the following are equivalent:*

(i) ν *has the local martingale numéraire property,*
(ii) ν *has the supermartingale numéraire property,*
(iii) *For every* $i = 1, \cdots, n$ *we have*

$$A_i = C_{i\nu} := \int_0^{\cdot} \sum_{j=1}^{n} \nu_j(t)\, \mathrm{d}C_{ij}(t),$$

or in terms of rates:

$$\alpha(t) = c(t)\, \nu(t).$$

Caution: It must always be kept in mind that the process ν has to be a *portfolio*, that is, belong to $\mathcal{I}(R)$; this means its cumulative variation has to satisfy the $\mathbb{P}$-a.e. local integrability condition

$$C_{\nu\nu}(T) = \int_0^T \sum_{i=1}^n \sum_{j=1}^n \nu_i(t)\,\nu_j(t)\,\mathrm{d}C_{ij}(t) < \infty$$

for every $T \in (0, \infty)$, *in addition to the property*

$$\boxed{\alpha = c\,\nu} \,.$$

Please note that these last properties are expressed ENTIRELY in terms of characteristics: as **descriptive** a theory, as you are likely to get.

Also: we have ALREADY seen this property $\boxed{\alpha = c\,\nu}$ under the guise of local maximization of growth.

DEFLATORS

Definition

An adapted, Right-Continuous process $Y > 0$ with Left-Limits (RCLL) and $Y(0) = 1$ is called **deflator**,
if
YX^{π} is a local martingale (thus also a supermartingale),
for every portfolio $\pi \in \mathcal{I}(R)$.

. In particular, such a Y is itself a local martingale,
as are the products YS_i , $i = 1, \cdots, n$.

And conversely ... .

If the local martingale numéraire X^ν exists, then its reciprocal

$$\boxed{Y_0 = \frac{1}{X^\nu}}$$

is clearly a deflator (the "canonical" one).

- More generally, all products

$$Y = \frac{1}{X^\nu} \cdot \exp\left(L - \frac{1}{2}[L, L] \right)$$

with L a local martingale with RCLL paths, $\Delta L > -1$, and

$$[L, M_i] \equiv 0, \quad i = 1, \cdots, n,$$

are then deflators. Conversely, all deflators are of this form.

We shall denote by $\mathcal{Y}$ the collection of deflators as above.

TOO MANY DEFINITIONS... .
ARE THERE ANY RESULTS TO SPEAK OF ?

Yes, finally, some
(pretty fundamental)
results are coming... .

CORNERSTONE RESULT

Theorem: *The following conditions are equivalent:*

(a) The market is viable.

(b) There exists a deflator: $\mathcal{Y} \neq \emptyset$.

(c) A supermartingale numéraire portfolio $\nu \in \mathcal{I}(R)$ *exists.*

(d) The market has locally finite growth: for every $T \in (0, \infty)$,

$$G(T) = \int_0^T \mathfrak{g}(t)\, \mathrm{d}\mathcal{O}(t) < \infty \,.$$

(e) A growth-optimal portfolio $\nu \in \mathcal{I}(R)$ *exists:*
i.e., for every portfolio π *we have*

$$\gamma_\pi(t) \le \gamma_\nu(t) = \mathfrak{g}(t)\,, \qquad \forall \;\; 0 \le t < \infty.$$

(f) There exists a portfolio $\nu \in \mathcal{I}(R)$ *with the relative logarithmic-optimality property*

$$\mathbb{E}^{\mathbb{P}} \left[\log \left(\frac{X^\pi(T)}{X^\nu(T)} \right) \right] \le 0\,, \qquad \forall \;\; (T, \pi) \in \mathcal{S}_{(0,\infty)} \times \mathcal{I}(R)\,.$$

Please note how this same supermartingale numéraire portfolio $\nu \in \mathcal{I}(R)$ "doubles up" in this result as

- Growth-optimal portfolio, and as
- Log-optimal portfolio.

It was through these properties that this portfolio ν was first studied by KELLY (1956), BREIMAN (1961), and THORP (1969).

It has also information-theoretic properties: ALGOET & COVER (1985).

LONG (1990) introduced formally the "martingale numéraire" property.

INTERESTING STRUCTURE: The supermartingale numéraire is the exponential of **some** Brownian motion W with drift $\mu = 1$ and variance $\sigma^2 = 2$, run according to the clock $G \equiv \Gamma_\nu$ of maximal cumulative growth:

$$X^\nu(t) = \exp\left(\left[u + \sqrt{2}\, W(u) \right] \Big|_{u=G(t)} \right), \qquad 0 \le t < \infty .$$

HEDGING DUALITY

Theorem: *Suppose the market is viable.*
Then, for any given c.c.w. stream $K \in \mathcal{K}$, *the infimum in*

$$\mathbf{x}(K) := \inf \left\{ x > 0 \,:\, \exists\, \vartheta \in \mathcal{I}(S),\ s.t.\ \ X(\cdot\,; x, \vartheta) \geq K \right\} \tag{1}$$

can be expressed as

$$\mathbf{x}(K) = \sup_{Y \in \mathcal{Y}} \mathbb{E}^{\mathbb{P}} \left[\int_0^\infty Y(t)\, \mathrm{d}K(t) \right]. \tag{2}$$

When this quantity is finite, the infimum is ATTAINED; and then there exist $\vartheta_K \in \mathcal{I}(S)$, $F_K \in \mathcal{K}$ *such that the minimal hedge for the stream* K *is of the form*

$$\widehat{X} = X(\cdot\,; \mathbf{x}(K), \vartheta_K) - F_K\,.$$

Remarks: (a) The infimum need not be attained in a non-viable market.

(b) Even with viability, the supremum in (2) can be finite without being attained. It is attained if, and only if, K is *maximal* in $\mathcal{K}(\xi)\big|_{\xi=\mathbf{x}(K)}$. Also if, and only if, $F_K \equiv 0$.

(c) The supremum in (2) can be attained by $Y_0 = 1/X^\nu$.

(d) The supremum in (2) can be attained by some $Y \neq 1/X^\nu$.

Here, **maximality** of K in $\mathcal{K}(\xi)$, means that whenever $H \in \mathcal{K}(\xi)$ dominates K in the sense that the difference $H - K$ is nondecreasing, we have $H \equiv K$.

(The minimal hedge affords then no luxury of 'withdrawals'. Spartan ... thrifty ... squeezes the last ounce ... .)

Corollary: *In a viable market, the collection*

$$\mathcal{K}(x) := \{ K \in \mathcal{K} \, : \, \exists \, \vartheta \in \mathcal{I}(S) \, , \quad \textit{s.t.} \;\; X(\cdot\,; x, \vartheta) \geq K \, \}$$

of c.c.w. streams, that can be financed starting with a given initial capital $x \geq 0$*, is characterized as*

$$\mathcal{K}(x) = \left\{ K \in \mathcal{K} \,\middle|\, \sup_{Y \in \mathcal{Y}} \mathbb{E}^{\mathbb{P}} \left[\int_0^\infty Y(t) \, \mathrm{d}K(t) \right] \leq x \right\} .$$

MODEL-CONSISTENT PROBABILITY MEASURES

Consider now the collection of probability measures

$$\Pi := \{ \mathbb{Q} \sim \mathbb{P} \, : \, M_1, \cdots, M_n \;\; \text{are local martingales under} \;\; \mathbb{Q} \, \}.$$

These are the probability measures $\mathbb{Q} \sim \mathbb{P}$, under which the asset prices $S_1, \cdots, S_n$ (and the returns $R_1, \cdots, R_n$) have the exact SAME dynamics as they do under $\mathbb{P}$.

But these other measures may affect other, "unhedgeable", sources of randomness residing in the filtration, in different ways.

. For every $\mathbb{Q} \in \Pi$, the product process right below is a deflator:

$$Y^{\mathbb{Q}}(t) \; := \; \frac{1}{X^{\nu}(t)} \cdot \left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \bigg|_{\mathcal{F}(t)} \right) , \qquad 0 \leq t < \infty.$$

The inclusion $\{ Y^{\mathbb{Q}} \}_{\mathbb{Q} \in \Pi} \subset \mathcal{Y}$ is typically **strict**.

Theorem: *Under viability, the quantity $x(K)$ of (1) from four slides ago admits, in addition to*

$$\mathbf{x}(K) = \sup_{Y \in \mathcal{Y}} \mathbb{E}^{\mathbb{P}} \left[\int_0^\infty Y(t) \, \mathrm{d}K(t) \right],$$

the "worst-case actuarial representation"

$$\mathbf{x}(K) = \sup_{\mathbb{Q} \in \Pi} \mathbb{E}^{\mathbb{Q}} \left[\int_0^\infty \frac{1}{X^{\nu}(t)} \, \mathrm{d}K(t) \right].$$

(And this, despite the fact that the collection Π in this second representation, is in one-to-one correspondence with a typically *strict* subclass of $\mathcal{Y}$.

- In this scheme of things, the canonical deflator $Y_0 = 1/X^{\nu}$ emerges as a "natural discount" factor that translates future values of c.c.w. streams down to the present $t = 0$.)

MINIMAL HEDGE

Consider now a "European Contingent Claim" (ECC), that is, a c.c.w. stream of the form

$$K(t) = P(T)\,\mathbf{1}_{[T,\infty)}(t)\,, \qquad 0 \le t < \infty$$

for some $T \in (0,\infty)$ and $\mathcal{F}(T)$-measurable r.v. $P(T) \ge 0$.

It can be shown that the minimal hedge $\widehat{X}$ exists for this, and admits the representations

$$\widehat{X}(\sigma) = \operatorname*{ess\,sup}_{Y \in \mathcal{Y}} \mathbb{E}^{\mathbb{P}} \left[\frac{Y(T)}{Y(\sigma)} P(T) \,\middle|\, \mathcal{F}(\sigma) \right]$$

$$= \operatorname*{ess\,sup}_{\mathbb{Q} \in \Pi} \mathbb{E}^{\mathbb{Q}} \left[\frac{X^{\nu}(\sigma)}{X^{\nu}(T)} P(T) \,\middle|\, \mathcal{F}(\sigma) \right],$$

for any given stopping time $\sigma \in \mathcal{S}_{[0,T]}$. In particular,

$$\widehat{X}(T) = P(T). \tag{3}$$

REPLICABILITY AND COMPLETENESS

Definition

A European Contingent Claim

$$K(t) = P(T)\,\mathbf{1}_{[T,\infty)}(t)\,, \qquad 0 \le t < \infty$$

as in the previous slide, with

$$x := \sup_{Y \in \mathcal{Y}} \mathbb{E}^{\mathbb{P}}\big[Y(T)\,P(T)\big] < \infty\,, \tag{4}$$

will be called **replicable**, if the minimal hedge for it is of the form

$$\widehat{X} \equiv X\big(\cdot\,; x, \widehat{\vartheta}\,\big)$$

for some $\widehat{\vartheta} \in \mathcal{I}(S)$.

(No extra withdrawal. Once again, maximality in the class $\mathcal{K}(x)$ ... .)

. The "replicability" terminology comes from the consequence

$$X\big(T\,; x, \widehat{\vartheta}\,\big) = \widehat{X}(T) = P(T)$$

of equation (3) from a couple of slides ago — in contradistinction to "hedging", as in $X(T; x, \vartheta) \geq P(T)$.

Definition

We say that a given viable market is **complete**, if every European Contingent Claim with the property

$$\sup_{Y \in \mathcal{Y}} \mathbb{E}^{\mathbb{P}}\big[Y(T)\,P(T)\big] < \infty$$

in (4) of two slides ago, is replicable.

CHARACTERIZATIONS OF COMPLETENESS

Theorem: *In a viable market, the following conditions are equivalent:*

(a) The market is complete.
(b) There is only one deflator: $\mathcal{Y} = \{1/X^{\nu}\}$.
(c) There is only one model-consistent p.m.: $\Pi = \{\mathbb{P}\}$.

From a purely probabilistic point of view, completeness is equivalent also to the "martingale representation property" of the vector of basic local martingales $M_1, \cdots, M_n$ for the underlying filtration $\mathfrak{F}$:

(d) EVERY LOCAL MARTINGALE OF THE FILTRATION CAN BE REPRESENTED AS A VECTOR STOCHASTIC INTEGRAL W.R.T. THESE BASIC LOCAL MARTINGALES.

UTILITY MAXIMIZATION

These SAME tools can be used to treat very general utility maximization problems, [2] as well as equilibrium problems in complete VIABLE markets, solely in terms of deflators.

Tools from concave FENCHEL duality play an important role here. Particularly when formulated in the context of $\mathbb{L}^0_+$, the nonnegative orthant of the space of measurable functions, equipped with the topology of convergence in probability.

[2] Please note that we have already dealt with log-utility maximization from terminal wealth.

OPTIONAL DECOMPOSITION

Crucial in all these developments, is the following result from Stochastic Analysis.

Theorem

Under viability, the following are equivalent for an adapted process $X \geq 0$ *with RCLL paths:*

(i) The product YX *is a supermartingale, for every deflator* $Y \in \mathcal{Y}$.
(ii) There exist an investment strategy $\vartheta \in \mathcal{I}(S)$ *and an adapted, right-continuous and nondecreasing process* $K \in \mathcal{K}$, *such that*

$$X = x + \int_0^{\cdot} \sum_{i=1}^{n} \vartheta_i(t)\, \mathrm{d}S_i(t) - K\,.$$

With quite a bit of poetic license, this result amounts to a DOOB-MEYER decomposition that holds simultaneously under an entire "family of probability measures", not just one.

There is also a version of this result, with "local martingale" replacing "supermartingale" in statement (i), and with $K \equiv 0$ in statement (ii).

INFINITELY-MANY ASSETS

The theory can be extended to cover markets with an
arbitrary number of assets;
in particular, an infinity of assets — countable or
uncountable, with a topology or without.

This extension uses tools from
reproducing kernel HILBERT space theory.
It is particularly important when one deals with **bond markets**,
which contain a potentially uncountable infinity of assets.

WHY SEMIMARTINGALES?

Let us assume now that the asset price processes $S_1, \cdots, S_n$ are continuous, adapted — and nothing else.

In particular, we will not be assuming *a priori*, as we have done up to this point, that these processes are semimartingales. Integration with respect to them will no longer be possible for us.

But this means, that we cannot define gains-from-trade for an arbitrary strategy, the way we have done up to now.

We shall need to re-think from scratch, the notion of trading strategy.

We consider, then, the simplest possible, a simple predictable strategy:

$$\vartheta(t) = \sum_{\ell=1}^{m} \theta_\ell \, \mathbf{1}_{(\tau_{\ell-1},\tau_\ell]}(t)\,, \qquad 0 \le t < \infty$$

for stoppping times $\tau_0 = 0 < \tau_1 < \tau_2 < \cdots < \tau_m < \infty$
and, for each $\ell = 1, \cdots, m$,
an $\mathcal{F}(\tau_{\ell-1})$-measurable random vector $\theta_\ell = \big(\theta_{\ell i}\big)_{i=1,\cdots,n}$.

A simple predictable process; and a buy-and-hold strategy in each of the intervals between "scheduled trading" times. With initial capital $x \ge 0$, and with such a simple predictable strategy, we define gains-from-trade in the most elementary way imaginable:

$$X(\cdot\,; x, \vartheta) := x + \int_0^{\cdot} \vartheta'(t)\,\mathrm{d}S(t)$$

$$\equiv x + \sum_{\ell=1}^{m} \sum_{i=1}^{n} \theta_{\ell i} \left[S_i(\tau_\ell \wedge \cdot) - S_i(\tau_{\ell-1} \wedge \cdot)\right].$$

We consider from now on *long-only, simple predictable strategies:*

$$\vartheta_1(\cdot) \geq 0\,, \cdots, \vartheta_n(\cdot) \geq 0\,, \quad X(\cdot\,; x, \vartheta) - \sum_{i=1}^{n} \vartheta_i(\cdot)\, S_i(\cdot) \geq 0.$$

These never sell any stock short, and never borrow from the money market. Collection $\mathcal{H}^s_+(S)$.

- For any given c.c.w.s $K \in \mathcal{K}$, we define now, by complete analogy to what we have done already, the quantity

$$\mathbf{x}^s_+(K) := \inf \left\{ x > 0 \;:\; \exists\, \vartheta \in \mathcal{H}^s_+(S),\ \text{s.t.}\ \ X(\cdot\,; x, \vartheta) \geq K \right\} :$$

the smallest initial capital, starting with which the given stream K can be financed using *long-only, simple predictable* strategies.

- We shall say that the market is **weakly viable,** if

$$K \in \mathcal{K}\,, \;\; \mathbf{x}^s_+(K) = 0 \quad \Longrightarrow \quad K \equiv 0\,.$$

A RESULT OF THE BICHTELER-DELLACHERIE TYPE

Theorem: *In the present context of continuous, adapted asset price processes* $S_1, \cdots, S_n$, *the following conditions are equivalent:*

(a) The market is weakly viable.

(b) There exists a strictly positive supermartingale Y with RCLL paths, and the property that each YS_i, $i = 1, \cdots, n$ is a supermartingale.

(c) The processes $S_1, \cdots, S_n$ *are semimartingales.*

In particular, weak viability (condition (a)) leads to asset price processes that have finite quadratic variations and covariations (condition (c)). This phenomenon can be thought of as the *"emergence of volatility"* (i.e., variation of order $p = 2$), in the phraseology of V. VOVK, from purely economic considerations.

Results of the Bichteler-Dellacherie type, single out the class of semimartingales, as consisting of the most general possible "integrators" in a theory of stochastic integration that possesses good continuity properties.

The above result is very similar: It singles out the collection of semimartingales as the most general asset prices, for which the "weak market viability" condition holds.

But in very stark contrast to an earlier, foundational result, no structural conditions are needed for weak viability. This is because we are imposing here restrictions [3] on trading strategies – no short-selling, no borrowing.

[3] And by imposing even harsher ones, such as the imposition of transaction costs, it stands to reason that we can go even beyond semimartingales, as Ch. Czichowsky and W. Schachermayer have demonstrated.

A CONSPICUOUS ABSENCE

We have covered the basic parts of our theory: viability, hedging, portfolio optimization, utility maximization.

We have introduced many notions and results, but not once did we mention that of *Equivalent (Local) Martingale Probability Measure,* or ELMM.

WHAT IS THIS?

This corresponds to the existence of a deflator $Y \in \mathcal{Y}$ which is a **martingale**. Then, for every such deflator and any $T \in (0, \infty)$, the measure $\mathbb{Q}_*^T$ defined via

$$\frac{\mathrm{d}\mathbb{Q}_*^T}{\mathrm{d}\mathbb{P}} = Y(T) \quad \text{on} \quad \mathcal{F}(T)$$

is a probability measure. And under this probability measure, the asset prices $S_1(\cdot \wedge T), \cdots, S_n(\cdot \wedge T)$ are local martingales.

The primary, “operative” reason for this omission is simple:
WE HAD NO NEED TO MAKE THAT ASSUMPTION.

Fortunately. For such a martingale deflator (such an equivalent local martingale measure) may simply **not exist** in a viable market.

There are subtler reasons, though.

• One of them is, that the existence of an ELMM is a **highly normative assumption**: Two market models may very easily have the *exact same characteristics* (local drifts and covariations) — while one of them admits an ELMM, and the other one does not.

• Another important reason is, that the existence of an ELMM is a **pretty strong assumption**: It rules out not just "egregious forms of arbitrage", such as financing something non-trivial for (next to) nothing.

It also tends to rule out "mild forms of arbitrage", such as one portfolio being able to outperform another.

We broach this last issue presently.

OUTPERFORMANCE

Definition

Let us fix a time horizon $T \in (0, \infty)$. We say that a portfolio $\pi \in \mathcal{I}(R)$ **outperforms** [4] another portfolio $\varrho \in \mathcal{I}(R)$ over $[0, T]$, if

$$\mathbb{P}\big(V^\pi(T) \geq V^\varrho(T)\big) = 1\,, \qquad \mathbb{P}\big(V^\pi(T) > V^\varrho(T)\big) > 0\,.$$

. If this last probability is not just positive, but in fact we have

$$\mathbb{P}\big(V^\pi(T) > V^\varrho(T)\big) = 1,$$

then we say that such outperformance (domination; relative arbitrage) is **strict.**

[4] "beats"; "dominates"; "is relative arbitrage with respect to".

When $\varrho \equiv \mathbf{0}$, thus $V^\varrho \equiv 1$, we recover the usual definition of arbitrage w.r.t. cash (the money market).

Remark: A portfolio ϱ which cannot be outperformed ("beaten") by any portfolio π, has the property

$$\mathbb{P}\big(V^\pi(T) \geq V^\varrho(T)\big) = 1 \quad \Longrightarrow \quad \mathbb{P}\big(V^\pi(T) = V^\varrho(T)\big) = 1\,.$$

Maximality — once again ... .

Remark: In a viable market, a portfolio with the local martingale numéraire property *cannot be outperformed by another portfolio,* over ANY horizon of finite length.
But other portfolios CAN be outperformed, in principle.

Proposition: *Suppose that an ELMM exists over a given time interval* $[0, T]$ *of finite length.*
Suppose also that for some portfolio $\varrho \in \mathcal{I}(R)$ *and constant* $K_T \in (0, \infty)$, *the cumulative variation process of* ϱ *satisfies*

$$C_{\varrho\varrho}(T) = \int_0^T \sum_{i=1}^n \sum_{j=1}^n \varrho_i(t)\, \mathrm{d}C_{ij}(t)\, \varrho_j(t) \le K_T\,, \quad \mathbb{P} - a.e.$$

Then the market is viable, and no relative arbitrage can exist with respect to this ϱ *over* $[0, T]$.

Corollary: Suppose that an ELMM exists over a time-interval $[0, T]$ of finite length. Then the market is viable, and no arbitrage relative to cash $\varrho \equiv \mathbf{0}$ can exist over $[0, T]$.

. In fact, the existence of an ELMM is *equivalent* to the conjunction (viability) + (absence of arbitrage relative to cash).

Very deep result of Delbaen & Schachermayer (1994).

Corollary: *Suppose that an ELMM exists over a time-interval* $[0, T]$ *of finite length, for a market with bounded covariation structure:*

$$\sum_{i=1}^{n} C_{ii}(T) \leq K_T \,, \quad \mathbb{P} - a.e.$$

for some $K_T \in (0, \infty)$. *Then the market is viable, and no arbitrage relative to any portfolio with bounded weights can exist over* $[0, T]$.

• In particular, under the assumptions of this last Corollary, no arbitrage can exist relative to the **market portfolio** $\mu = (\mu_1, \cdots, \mu_n)'$ with weights

$$\mu_i(t) = \frac{S_i(t)}{S_1(t) + \cdots + S_n(t)} \,, \qquad i = 1, \cdots, n,$$

for $0 \leq t < \infty$.

- Now, as it turns out, under broad structural conditions involving the market weights and variations, such as

$$\sum_{i=1}^{n} \int_0^T \mu_i(t)\, \mathrm{d}[\log \mu_i, \log \mu_i](t) \ \geq \ \eta\, T\,, \qquad 0 \leq T < \infty$$

for some real constant $\eta > 0$,
it IS possible for the market portfolio to be strictly outperformed *over sufficiently long* (and often over all) time horizons.

And to construct very simple long-only portfolios, functionally generated as in FERNHOLZ (2002), that do this job. These are expressed, at any given time, only in terms of the prevailing market weights.

This means that, under such conditions as in the Proposition from two slides ago, no ELMM can then exist.
Yet such markets can, very easily, be viable.

In such markets, opportunities for (relative) arbitrage may exist. But as far as hedging and/or utility maximization is concerned, they will be irrelevant – the optimal strategies for those problems will choose to ignore them, brush them off.

"YOU CANNOT BEAT THE MARKET"

What does it take, then, for the market portfolio itself to have the numéraire property?

Or a bit more modestly:

Under what conditions will the ratio X^π / X^μ be a local martingale, for every stock-portfolio $\pi \in \mathcal{I}(R)$? [5]

If it has this property, the market portfolio cannot be outperformed ("beaten"), by ANY stock portfolio.

[5] A "stock-portfolio" satisfies $\sum_{i=1}^n \pi_i \equiv 1$, i.e., never holds any cash.

The market portfolio has the numéraire property among stock portfolios

if, and only if,

ALL of its constituent processes $\mu_1, \cdots, \mu_n$ are local – thus also true – martingales: more precisely,

$$\frac{\mathrm{d}\mu_i(t)}{\mu_i(t)} = \sum_{j=1}^{n} \big(\mathbf{1}_{i=j} - \mu_j(t)\big)\, \mathrm{d}M_j(t)\,, \qquad i = 1, \cdots, n\,.$$

Then with “sufficiently distinct” assets (in a sense easily made precise), the entire market capitalization concentrates in just one company in the long run: the limits

$$\mu_i(\infty) := \lim_{t\to\infty} \mu_i(t)\,, \qquad i = 1, \cdots, n$$

exist $\mathbb{P}$-a.e., and their vector $\mu(\infty)$ takes values in the the set of corners of the unit simplex, i.e.,

$$\mathbb{P}\big(\mu(\infty) \in \{\mathfrak{e}_1, \cdots, \mathfrak{e}_n\}\big) \;=\; 1$$

with

$$\mathbb{P}\big(\mu_i(\infty) = 1\big) \;=\; \mu_i(0)\,, \qquad i = 1, \cdots, n\,.$$

(A small modification of these considerations, leads to the so-called “Capital Asset Pricing Model”, or CAPM for short.)

A FEW SOURCES FOR THIS TALK

FERNHOLZ, E.R. & KARATZAS, I. (2009) Stochastic Portfolio Theory: An Overview. In "Handbook of Numerical Analysis", special volume on *Mathematical Modeling and Numerical Methods in Finance* (A. Bensoussan & Q. Zhang, editors), 89-168. Elsevier Publishing Company BV, Amsterdam.

KARATZAS, I. & KARDARAS, C. (2007) The numéraire portfolio in semimartingale financial markets. *Finance & Stochastics* **11**, 447-493.

KARATZAS, I. & KARDARAS, C. (2018+) *Arbitrage Theory via Numéraires.* Working title for book in preparation.

THANK YOU FOR YOUR ATTENTION