The Amazing Power of Dimensional Analysis in Finance: Market Impact and the Intraday Trading Invariance Hypothesis

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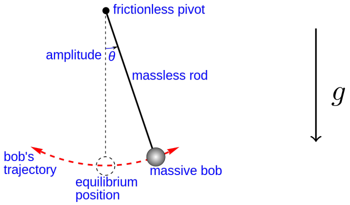
Dimensional Analysis

The period of the pendulum

Functional relation:

period $= f(l, m, g)$.



Dimensions:

- the length l of the pendulum, measured in meters: dimension $\mathbb{L}$.
- the mass m of the bob, measured in grams: dimension $\mathbb{M}$.
- the acceleration g caused by gravity, measured in meters per second squared: dimension $\mathbb{L}/\mathbb{T}^2$.

Basic assumption: The three variables l, m, g **fully explain** the period of the pendulum.

Dimensional Analysis: The period of the pendulum

Ansatz:

$$\text{period} = p = f(l, m, g) = \text{const} \cdot l^{y_1} m^{y_2} g^{y_3}.$$

	l	m	g	p
$\mathbb{L}$ length	1	0	1	0
$\mathbb{M}$ mass	0	1	0	0
$\mathbb{T}$ time	0	0	-2	1

$$\begin{aligned} y_1 \quad + y_3 &= 0 \\ y_2 \quad &= 0 \\ -2y_3 &= 1 \end{aligned}$$

Unique solution: $y_1 = \frac{1}{2}$, $y_2 = 0$, $y_3 = \frac{1}{2}$,

$$\text{period} = \text{const} \cdot \sqrt{\frac{l}{g}}.$$

Remark: The Ansatz

$$\text{period} = \text{const} \cdot l^{y_1} m^{y_2} g^{y_3},$$

does **not restrict the generality** of the relation

$$\text{period} = f(l, m, g),$$

or, equivalently

$$\log(\text{period}) = F(\log(l), \log(m), \log(g)).$$

Indeed, the first row of the matrix translates into the requirement

$$\log(\text{period}) = F(\log(l) + \log(x), \log(m), \log(g) + \log(x)), \forall x > 0.$$

Hence $F(\cdot, \cdot, \cdot)$ must be constant along any line in $\mathbb{R}^3$ parallel to the vector $(1, 0, 1)$.
As the row vectors of the above matrix span $\mathbb{R}^3$, this fully determines the function F (up to a constant).

Market Impact

Consider an agent who intends to buy/sell a large amount (*"meta-order"* or *"bet"*) of some fixed stock. This bet will – ceteris paribus – move the market price to the disadvantage of the agent. The agents are trying hard to minimize the *market impact*, but are unable to avoid it completely.

Definition

The market impact G is the size of price change caused by a bet (in percentage of the price).

Dimensional Analysis: The size of the market impact

Functional relation:

$$\text{market impact } G = g(Q, P, V, \sigma^2).$$

- Q the size of the meta-order, measured in units of shares $\mathbb{S}$,
- P the price of the stock, measured in units of money per share $\mathbb{U}/\mathbb{S}$,
- V the traded volume of the stock, measured in units of shares per time $\mathbb{S}/\mathbb{T}$,
- σ^2 the squared volatility of the stock, measured in percentage of the stock price per unit of time $\mathbb{T}^{-1}$,
- G the market impact is a dimensionless quantity.

Basic assumption: the four variables Q, P, V, σ^2 **fully explain** the size of the market impact G.

Dimensional Analysis: The size of the market impact

Ansatz:

$$G = g(Q, P, V, \sigma^2) = \text{const} \cdot Q^{y_1} P^{y_2} V^{y_3} \sigma^{2y_4},$$

Q= size of bet, P=price of share, V=traded daily volume, σ=volatility.

	Q	P	V	σ^2	G
shares $\mathbb{S}$	1	-1	1	0	0
money $\mathbb{U}$	0	1	0	0	0
time $\mathbb{T}$	0	0	-1	-1	0

Leads to **three** linear equations in **four** unknowns y_1, y_2, y_3, y_4.
The solution has one degree of freedom

$$G = \text{const} \cdot \left(\frac{Q\sigma^2}{V} \right)^y,$$

where $y \in \mathbb{R}$ and const > 0 are still free.

This time the ansatz **does** restrict the generality! The general solution for G, respecting the dimensional restrictions is

$$G = g\Big(\frac{Q\sigma^2}{V}\Big),$$

where $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is an *arbitrary* function.

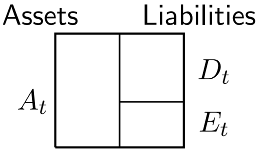
Can we find one more equation which will allow us to get a unique solution?

Kyle, Obizhaeva (2016): YES

Leverage neutrality

Theorem of Modigliani-Miller (1958):

A_t assets
D_t debt
E_t equity



Basic assumption: $(A_t)_{t\geq 0}$ follows a stochastic process, e.g. Samuelson (1965):

$$\frac{dA_t}{A_t} = (\sigma dW_t + \mu dt).$$

Keeping the debt D_t constant, we therefore get $dA_t = dE_t$ so that

$$\frac{dE_t}{E_t} = \frac{A_t}{E_t}(\sigma dW_t + \mu dt).$$

Conclusion: Denoting by $L_t = \frac{A_t}{E_t}$ the *leverage* of the company, the relative dynamics of $(E_t)_{t\geq 0}$ are simply proportional to the the leverage L_t.

What happens to the stock price $(P_t)_{t \geq 0}$, if you change the leverage? Say, the leverage L is doubled by paying out half of the equity by dividends

- P is multiplied by $\frac{1}{2}$.
- σ is multiplied by 2.
- G is multiplied by 2.
- Q, V remain unchanged.

Leverage neutrality: (Kyle, Obizhaeva, 2017)

The value of a firm does not depend on its capital structure (Modigliani, Miller, 1958).

This no-arbitrage-type condition yields one more equation involving the *Modigliani-Miller* dimension $\mathbb{M}$ measuring the leverage of a company.

Mathematically speaking, the variation of the leverage (dimension $\mathbb{M}$) is analogous to the scalings of the dimensions $\mathbb{S}, \mathbb{U}$, and $\mathbb{T}$.

	Q	P	V	σ^2	G
$\mathbb{S}$	1	-1	1	0	0
$\mathbb{U}$	0	1	0	0	0
$\mathbb{T}$	0	0	-1	-1	0
$\mathbb{M}$	0	-1	0	2	1

Theorem (Pohl, Ristig, S., Tangpi, '17 based on Kyle, Obizhaeva, '16):

Assume $G = g(Q, P, V, \sigma^2)$ is such that

- the variables Q, P, V, and σ^2 fully explain G,
- the function g is invariant under scalings of the dimensions $\mathbb{S}, \mathbb{U}, \mathbb{T}$, and leverage neutrality holds true.

Then there is a const > 0 such that

$$G = \text{const} \cdot \sigma \sqrt{\frac{Q}{V}}.$$

In particular, the market impact G is proportional to the *square root of the size* Q of the meta-order.

Empirics

- Does this relation hold true in the real world?
- Do we have to introduce more explanatory random variables (as analyzed by Kyle and Obizhaeva)?

Unfortunately it is hard (if not impossible) to analyze empirically the "true" market immpact G of an order size Q.

We can hardly observe the **meta-orders**, however we can observe the **actual orders**.

The Intraday Trading Invariance Hypothesis

We now apply the method of dimensional analysis to a different issue which has the advantage of being empirically observable:

N : the number of trades (actual orders), measured per unit of time

$$[N] = \mathbb{T}^{-1}.$$

What are the variables which might explain the quantity N? What are their dimensions?

Following Kyle and Obizhaeva (2017) and Bouchaud et al. (2016) the following quantities come into one's mind.

- V traded volume (per day), $[V] = \mathbb{S}\mathbb{T}^{-1}$
- P price of a share, $[P] = \mathbb{U}\mathbb{S}^{-1}$
- σ^2 squared volatility, $[\sigma^2] = \mathbb{T}^{-1}$
- C cost per trade, $[C] = \mathbb{U}$.

First attempt of explanatory variables: P, V, σ^2

Proposition:
Assume that the number of trades N depends *only* on the 3 quantities σ^2, P and V, i.e.,

$$N = g(\sigma^2, P, V),$$

where the function $g : \mathbb{R}_+^3 \to \mathbb{R}_+$ is *dimensionally invariant*.
Then, there is a constant $c > 0$ such that the number of trades N obeys the relation

$$N = c \cdot \sigma^2.$$

This relation was investigated e.g. in Jones et al. (1994).

Too simplistic!

Second attempt of explanatory variables: P, V, σ^2, C

C: cost per trade

$C = \langle Q \rangle \cdot S =$ average order size $\cdot$ bid-ask spread

$[C] = \mathbb{U} =$ money

Second attempt of explanatory variables: P, V, σ^2, C

Theorem [$(3/2)$-law] (Benzaquen, Bouchaud, Donier, 2016):

Suppose that the number of trades N depends *only* on the four quantities σ^2, P, V, C and N, i.e.,

$$N = g(\sigma^2, P, V, C),$$

where the function $g : \mathbb{R}_+^4 \to \mathbb{R}_+$ is *dimensionally invariant* and *leverage neutral*.

Then, there is a constant $c > 0$ such that the number of trades N obeys the relation

$$N^{3/2} = c \cdot \frac{\sigma P V}{C}.$$

	P	V	σ^2	C	N
$\mathbb{S}$	-1	1	0	0	0
$\mathbb{U}$	1	0	0	1	0
$\mathbb{T}$	0	-1	-1	0	-1
$\mathbb{M}$	-1	0	2	0	0

Table: A labelled overview of the dimensions of P, V, σ^2, C and N.

Empirical Results

Our empirical analysis is based on limit order book data provided by the LOBSTER database (`https://lobsterdata.com`). The considered sampling period begins on January 2, 2015 and ends on August 31, 2015, leaving 167 trading days. Among all NASDAQ stocks, $d = 128$ sufficiently liquid stocks with high market capitalizations are chosen.

Let us fix an interval length $T \in \{30, 60, 120, 180, 360\}$ min for which a developed hypothesis is tested.

N_j denotes the number of trades in the interval j,

Q_j $= N_j^{-1} \sum_{k=1}^{N_j} Q_{t_k}$ denotes the average size of the trades in the interval j, where Q_{t_k} denotes the number of shares traded at time t_k,

V_j $= N_j \times Q_j$ is the traded volume in the interval j,

P_j $= N_j^{-1} \sum_{k=1}^{N_j} P_{t_k}$ denotes the average midquote price in the interval j, where $P_{t_k} = (A_{t_k} + B_{t_k})/2$ and A_{t_k} (resp. B_{t_k}) denotes the best ask (resp. bid) price after the transaction at time t_k,

Empirical Results contd.

$\hat{\sigma}_j^2$ denotes the estimated squared volatility in the interval j,

S_j $= N_j^{-1} \sum_{k=1}^{N_j} S_{t_k}$ denotes the average bid-ask spread in the interval j, where $S_{t_k} = A_{t_k} - B_{t_k}$ is the bid-ask spread after the transaction at time t_k, and

C_j $= Q_j \times S_j$ is the spread cost per trade in the interval j.

Statistical analysis of the hypotheses

$$N \sim \sigma^2 \quad \text{versus} \quad N \sim \left(\frac{\sigma PV}{C}\right)^{2/3}.$$

General form:

$$N \sim (\sigma^2)^\beta \left(\frac{PV}{C}\right)^\gamma$$

$N \sim \sigma^2$ corresponds to $\beta = 1, \gamma = 0$,

$N \sim \left(\frac{\sigma PV}{C}\right)^{2/3}$ corresponds to $\beta = 1/3, \gamma = 2/3$,

Linear constraint: $\beta + \gamma = 1$.

Multiplicative model:

$$N_{ij} \sim \left(\hat{\sigma}_{ij}^2\right)^{\beta_i} \left(\frac{P_{ij}V_{ij}}{C_{ij}}\right)^{\gamma_i} \exp(\epsilon_{ij}),$$

Linear model:

$$\log(N_{ij}) \sim \beta_i \log(\hat{\sigma}_{ij}^2) + \gamma_i \log\left(\frac{P_{ij}V_{ij}}{C_{ij}}\right) + \epsilon_{ij}.$$

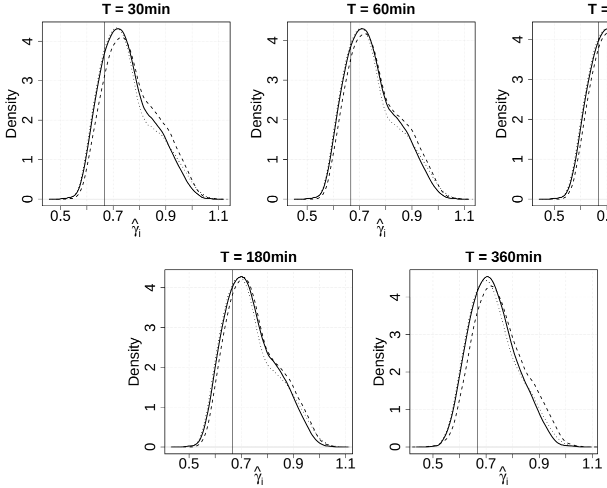


Figure: The panels show kernel density estimates across the estimated parameters $\hat{\gamma}_i$ for different interval lengths $T \in \{30, 60, 120, 180, 360\}$ min.

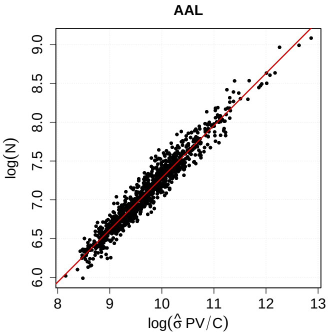


Figure: The dependent variable $\log N$ is plotted versus the explanatory variable $\log(\hat{\sigma} PV/C)$ for the fixed interval length $T = 60$min and the two stocks AAL and AAPL. The lines indicate the estimated linear relations between the considered quantities.

R^2 equals 0.936 and 0.978 respectively.

An Afterthought: the dimension of volatility

Definition of volatility per time T:

$\sigma^2 = \mathrm{Var}(\log(P_{t+T}) - \log(P_t))$.

Example of price process (Black-Scholes):

$dP_u = P_u(\sigma dW_u + \mu du)$.

Obvious consequence: $[\sigma^2] = \mathbb{T}^{-1}$.

But what we really plug into a formula like

$$N^{3/2} = c \cdot \frac{\sigma PV}{C}$$

is an **estimate** $\hat{\sigma}^2$ of the "true" volatility σ^2 (whatever this is).

What does the empirical data tell us on this issue?

Typical estimator for σ^2:

$$\hat{\sigma}^2 := \sum_{k=1}^{n} \Big(\log(P_{t_k}) - \log(P_{t_{k-1}}) \Big)^2.$$

where $t = t_0 < t_1 < \dots t_n = t + T$ are the points in $[t, t+T]$ where P_t jumps.

But we could also consider, for $H \in]0, 1[$,

$$\hat{\sigma}^2(H) := \Big(\sum_{k=1}^{N} \Big| \log(P_{t_k}) - \log(P_{t_{k-1}}) \Big|^{1/H} \Big)^{2H}$$

Possible reasons for $H \neq \frac{1}{2}$:

- fractional Brownian motion W^H instead of W (Mandelbrot, 1961,...)
- rough volatility (Bayer, Gatheral, Rosenbaum, ...)
- market micro structure effects (Bouchaud, Rosenbaum, ...)

$$\mathbb{E}[(W_{t+T} - W_t)^2]^{1/2} = T^{1/2},$$

but $\mathbb{E}[(\lceil W_{t+T} \rceil - \lceil W_t \rceil)^2]^{1/2} \sim T^{1/4}$, for $T \mapsto 0$.

Theorem [$(1+H)$-law] (Pohl, Ristig, S., Tangpi, 2018) : Suppose that $[\hat{\sigma}^2(H)] = \mathbb{T}^{-2H}$ and suppose again that the number of trades N depends *only* on the four quantities $\hat{\sigma}^2(H), P, V$ and C, i.e.,

$$N = g(\hat{\sigma}^2(H), P, V, C),$$

where the function $g : \mathbb{R}_+^4 \to \mathbb{R}_+$ is *dimensionally invariant* and *leverage neutral*.

Then, there is a constant $c > 0$ such that the number of trades N obeys the relation

$$N^{1+H} = c \cdot \frac{\hat{\sigma}(H) P V}{C}.$$

	P	V	$\hat{\sigma}^2(H)$	C	N
$\mathbb{S}$	-1	1	0	0	0
$\mathbb{U}$	1	0	0	1	0
$\mathbb{T}$	0	-1	-2H	0	-1
$\mathbb{M}$	-1	0	2	0	0

Table: An overview of the dimensions of quantities $P, V, \hat{\sigma}^2(H), C$ and N.

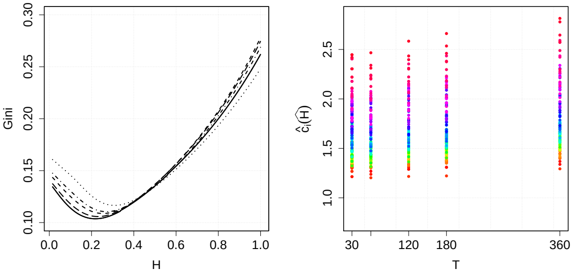
Empirical Analysis

Which estimator $\hat{\sigma}^2(H)$ gives the best fit to the empirical data?

<u>Optimality Criterion:</u> The constant $c = c(H)$ in front of the relation

$$N_{ij}^{1+H} = c_{ij}(H)\frac{\hat{\sigma}_{ij}(H)P_{ij}V_{ij}}{C_{ij}}$$

should vary as little as possible.



The left panel illustrates the Gini-coefficient in dependence of H for $T = 30$min (solid), $T = 60$min (long-dashed), $T = 120$min (dashed), $T = 180$min (dashed-dotted) and $T = 360$min (dotted).

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Happy birthday, Mete!