Two-Dimensional Passport Option
An Open Problem

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Option on a traded account

Consider an option that confers on its holder the right to

- Begin with initial capital X_0,
- Trade in one or more risky assets and a risk-free asset subject to some constraints,
- Generate an account balance X_t at each time t, $0 \leq t \leq T$,
- At expiration T receive

$$\max\{X_T, \text{Floor}\}.$$

Primary example: Variable annuity.

□

Passport option

A passport option is an option on a traded account in which

- the holder is constrained to have positions between short one share and long one share of the risky asset (or assets),
- At expiration T, the option pays X_T^+.

A short history:

- Developed at Banker's Trust and originally sold on foreign currency; see Hyer, et. al. (1997).
- Long the currency "corresponds" to being in a foreign country. Short "corresponds" to coming home. For this travel, one needs a passport.
- Later extended to other assets, including bonds and interest rate futures. A variety of related options were introduced; see Andersen, et. al. (1998), Penaud, et. al. (1999).

□

The simplest case

- Geometric Brownian motion

$$dS_t = rS_t\, dt + \sigma S_t\, dW_t.$$

- Traded account value

$$dX_t = q_t\, dS_t + r\big(X_t - q_t S_t\big)\, dt.$$

- Constraint on trading

$$-1 \le q_t \le 1, \quad 0 \le t \le T.$$

- At expiration the option pays

$$X_T^+.$$

□

Price the option by solving a stochastic control problem

$$\begin{aligned} &\text{Maximize} \quad \mathbb{E}\big[e^{-rT} X_T^+\big] \\ &\text{Subject to} \quad |q_t| \le 1, \quad 0 \le t \le T. \end{aligned}$$

Value function for $0 \le t \le T, s \ge 0, x \in \mathbb{R}$:

$$V(t,s,x) = \max_{|q| \le 1} \mathbb{E}\left[e^{-r(T-t)} X_T^+ \,\middle|\, S_t = s, X_t = x \right].$$

Hamilton-Jacobi-Bellman (HJB) equation

$$-rV + V_t + rsV_s + rxV_x + \frac{1}{2}\sigma^2 V_{ss} + \sigma^2 s^2 \max_{|q| \le 1} \left[qV_{sx} + \frac{1}{2} q^2 V_{xx} \right] = 0.$$

Terminal condition

$$V(T,s,x) = x^+ \text{ for all } s > 0, x \in \mathbb{R}. \qquad \square$$

Option seller's hedge (given the function $V(t,s,x)$)

Define the "optimal" feedback control $\widehat{q}_t(s,x)$ by

$$\widehat{q}_t(s,x)V_{sx}(t,s,x)+\frac{1}{2}\widehat{q}_t^2(s,x)V_{xx}(t,s,x) = \max_{|q|\le 1}\left[qV_{sx}(t,s,x)+\frac{1}{2}q^2V_{xx}(t,s,x)\right].$$

- Suppose the option holder uses the trading strategy q_t, resulting in account value X_t, $0 \le t \le T$.
- Then the option seller uses the trading strategy

$$\overline{q}_t := V_s\big(t,S_t,X_t\big)+q_tV_x\big(t,S_t,X_t\big).$$

- Also, the option seller consumes at the nonnegative rate

$$C_t = \sigma^2 S_t^2\bigg[\big(\widehat{q}_t(S_t,X_t)-q_t\big)V_{sx}\big(t,S_t,X_t\big) + \frac{1}{2}\big(\widehat{q}_t^2(S_t,X_t)-q_t^2\big)V_{xx}\big(t,S_t,X_t\big)\bigg]. \qquad \square$$

Option seller's hedge (given the function $V(t, s, x)$)

Differential of the discounted option price along the option holder's account value process is

$$
\begin{aligned}
d\big(e^{-rt}V(t, S_t, X_t)\big) \\
&= e^{-rt}\left[-rV + V_t + rSV_s + rXV_x + \sigma^2 S^2\left(\frac{1}{2}V_{ss} + q_t V_{sx} + \frac{1}{2}q_t^2 V_{xx}\right)\right]dt \\
&\qquad + \sigma e^{-rt} S(V_s + q_t V_x)\, dW \\
&= e^{-rt}\left[-rV + V_t + rSV_s + rXV_x + \sigma^2 S^2\left(\frac{1}{2}V_{ss} + \widehat{q}_t V_{sx} + \frac{1}{2}\widehat{q}_t^2 v_{xx}\right)\right]dt \\
&\qquad - e^{-rt} C_t\, dt + \sigma e^{-rt} S(V_s + q_t V_x)\, dW \\
&= -e^{-rt} C_t\, dt + \sigma e^{-rt} S_t \overline{q}_t\, dW_t.
\end{aligned}
$$

Differential of the option seller's discounted portfolio:

$$
d(e^{-rt}\overline{X}_t) = -e^{-rt} C_t\, dt + \sigma e^{-rt} S_t \overline{q}_t\, dW_t = d\big(e^{-rt} V(t, S_t, X_t)\big).
$$

$$
\overline{X}_0 = V(0, S_0, X_0) \Longrightarrow \overline{X}_T = V(T, S_T, X_T) = X_T^+. \qquad \square
$$

Stochastic control problem

Maximize $\mathbb{E}\big[e^{-rT}X_T^+\big]$ Subject to $|q_t| \leq 1, \quad 0 \leq t \leq T.$

Value function $V(t,s,x) = \max_{|q|\leq 1} \mathbb{E}\left[e^{-r(T-t)}X_T^+ \,\middle|\, S_t = s, X_t = x\right].$

THEOREM *$V(t,s,x)$ is convex in x.*
PROOF: Given $x^{(3)} = \lambda x^{(1)} + (1-\lambda)x^{(2)}$ with $0 \leq \lambda \leq 1$.
Let q_θ, $t \leq \theta \leq T$, be a trading strategy.
This generates account value processes $X^{(i)}$ starting from $S_t = s$, $X_t^{(i)} = x^{(i)}$, $i = 1,2,3$. Then

$$X_T^{(3)} = \lambda X_T^{(1)} + (1-\lambda)X_T^{(2)},$$

$$\begin{aligned}\mathbb{E}\big[e^{-rT}(X_T^{(3)})^+\big] &\leq \lambda\mathbb{E}\big[e^{-rT}(X_T^{(1)})^+\big] + (1-\lambda)\mathbb{E}\big[e^{-rT}(X_T^{(2)})^+\big] \\ &\leq \lambda V\big(t,s,x^{(1)}\big) + (1-\lambda)V\big(t,s,x^{(2)}\big).\end{aligned}$$

Now maximize the left-hand side over q_θ, $t \leq \theta \leq T$. □

Bang-bang control

Hamilton-Jacobi-Bellman (HJB) equation

$$-rV+V_t+rsV_s+rxV_x+\frac{1}{2}\sigma^2 s^2 V_{ss}+\sigma^2 s^2 \max_{|q|\leq 1}\left[qV_{sx}+\frac{1}{2}q^2 V_{xx}\right]=0.$$

Because $V_{xx} \geq 0$, the maximum over q is achieved at an extreme point of the convex constraint set $[-1,1]$, i.e., at either

$$q_t = -1 \text{ or } q_t = 1.$$

But which extreme point?

□

Example

We always have

$$\mathbb{E}\left[e^{-rT}X_T\right] = X_0, \qquad \mathbb{E}\left[e^{-rT}X_T^+\right] > X_0.$$

Consider a one-step problem.

- Expiration date: $T = 1$
- Initial account value value: $X_0 = 2$
- Initial risky asset price: $S_0 = 4$
- Parameter values: $r = 2\%$, $\sigma = 20\%$

What is the distribution of X_T

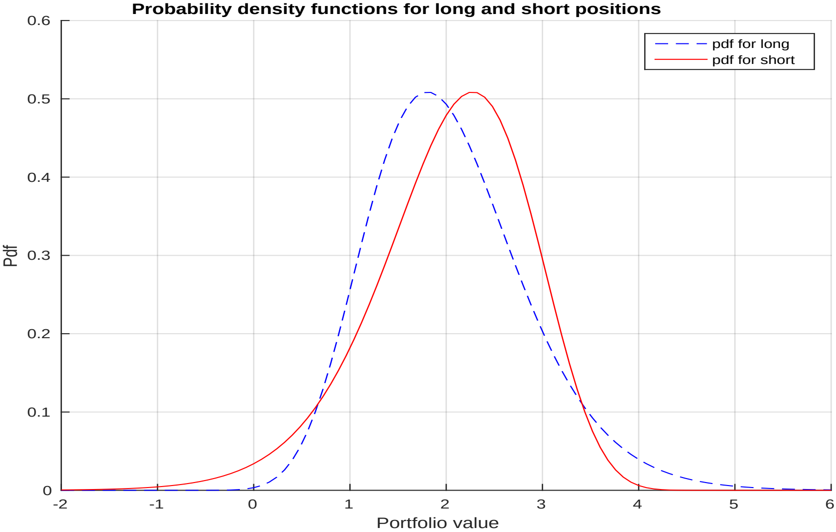
- if the option holder takes a long position?
- if the option holder takes a short position?

Which position maximizes

$$\mathbb{E}\left[e^{-rT}X_T^+\right]?$$

□

Intuition



Conjecture: $\widehat{q}_t = -\text{sign}(X_t)$. □

Change of numéraire

Two martingales:

$$d(e^{-rt}S_t) = \sigma e^{-rt} S_t\, dW_t,$$
$$d(e^{-rt}X_t) = \sigma e^{-rt} S_t q_t\, dW_t.$$

Define

$$Y_t := \frac{X_t}{S_t} = \frac{e^{-rt}X_t}{e^{-rt}S_t}.$$

Then

$$dY_t = \sigma(q_t - Y_t)(dW_t - \sigma\, dt) = \sigma(q_t - Y_t)\, d\widetilde{W}_t,$$

where $\widetilde{W}_t := W_t - \sigma t$ is a Brownian motion under $\widetilde{\mathbb{P}}$ defined by

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \frac{e^{-rT}S_T}{S_0}.$$

$$\mathbb{E}\left[e^{-rT}X_T^+\right] = S_0 \mathbb{E}\left[\frac{e^{-rT}S_T}{S_0}\left(\frac{X_T}{S_T}\right)^+\right] = S_0 \widetilde{\mathbb{E}}\left[Y_T^+\right]. \qquad \square$$

Equivalent stochastic control problem

Maximize $\quad \widetilde{\mathbb{E}}\left[Y_T^+\right]$

Subject to $\quad Y_0 = X_0/S_0,$

$$dY_t = \sigma(q_t - Y_t)\, d\widetilde{W}_t,$$

$$|q_t| \le 1, \quad 0 \le t \le T.$$

Value function

$$u(t, y) = \max_{|q| \le 1} \widetilde{\mathbb{E}}\left[\, Y_T^+ \middle|\, Y_t = y\right].$$

Hamilton-Jacobi-Bellman (HJB) equation

$$u_t(t, y) + \frac{1}{2}\sigma^2 \max_{|q| \le 1} (q - y)^2 u_{yy}(t, y) = 0.$$

Terminal condition

$$u(T, y) = y^+ \text{ for all } y \in \mathbb{R}. \qquad \square$$

Optimal control in feedback form

THEOREM $u(t,y)$ is convex in y.

Recall the Hamilton-Jacobi-Bellman (HJB) equation

$$u_t(t,y) + \frac{1}{2}\sigma^2 \max_{|q|\le 1}(q-y)^2 u_{yy}(t,y) = 0.$$

Therefore,

$$\begin{aligned} \widehat{q}_t &= -\mathrm{sign}(Y_t), \\ dY_t &= \sigma(\widehat{q}_t - Y_t)\, d\widetilde{W}_t \\ &= -\sigma\, \mathrm{sign}(Y_t)(1+|Y_t|)\, d\widetilde{W}_t, \end{aligned}$$

and the HJB equation becomes

$$u_t(t,y) + \frac{1}{2}\sigma^2 (1+|y|)^2 u_{yy}(t,y) = 0.$$

□

Solution of HJB by Skorohod mapping

Delbaen & Yor (2002), Henderson & Hobson (2000), Nagayama (1999).

Let $Y_0 = y$ and

$$dY_t = -\sigma\,\mathrm{sign}(Y_t)(1+|Y_t|)\,d\widetilde{W}_t, \quad 0 \le t \le T.$$

Then

$$\begin{aligned} d(1+|Y_t|) &= \mathrm{sign}(Y_t)\,dY_t + dL_t^Y(0) \\ &= -\sigma(1+|Y_t|)\,d\widetilde{W}_t + dL_t^Y(0), \end{aligned}$$

and

$$\begin{aligned} &d\log(1+|Y_t|) \\ &= \frac{1}{1+|Y_t|}\,d(1+|Y_t|) - \frac{1}{2(1+|Y_t|)^2}\,d\langle 1+|Y|, 1+|Y|\rangle_t \\ &= -\sigma\,d\widetilde{W}_t - \frac{1}{2}\sigma^2\,dt + dL_t^Y(0). \end{aligned}$$

□

Solution of HJB by Skorohod mapping

From the previous page: $Y_0 = y$ and

$$d\log\left(1+|Y_t|\right) = \sigma\left(-d\widetilde{W}_t - \frac{1}{2}\sigma\,dt\right) + dL_t^Y(0), \quad 0 \le t \le T.$$

Define

$$\overline{W}_t = -\widetilde{W}_t - \frac{1}{2}\sigma t,$$

and a measure $\overline{\mathbb{P}}$ under which $\overline{W}$ is a Brownian motion. Then

$$\log(1+|Y_t|) = \log(1+|y|) + \sigma\,\overline{W}_t + L_t^Y(0).$$

The process $\log(1+|Y_t|)$ is nonnegative and $L_t^Y(0)$ grows only when $\log(1+|Y_t|)$ is at zero. Skorohod implies

$$L_t^Y(0) = \max_{0\le u\le t}\left(-\log(1+|y|) - \sigma\overline{W}_u\right)^+.$$

□

Solution of HJB by Skorohod mapping

Conclusion:

$$\log(1+|Y_T|) = \log(1+|y|)+\sigma\overline{W}_T+\max_{0\le u\le T}\left(-\log(1+|y|)-\sigma\overline{W}_u\right)^+,$$

where $\overline{W}$ is a Brownian motion under $\overline{\mathbb{P}}$ and

$$\frac{d\overline{\mathbb{P}}}{d\widetilde{\mathbb{P}}} = \exp\left[-\frac{1}{2}\sigma\widetilde{W}_T - \frac{1}{8}\sigma^2 T\right].$$

Put all the pieces together to compute

$$\widetilde{\mathbb{E}}\left[Y_T^+\,\middle|\,Y_0 = y\right] = \frac{1}{2}y - \frac{1}{2}\log(1+|y|) + \frac{1}{4}\sigma^2 T + \frac{1}{2}\sigma\sqrt{T}\,N'(d_-)$$
$$+\frac{1}{2}(1+|y|)N(d_+) + \frac{1}{2}(\sigma\sqrt{T}\,d_- - 1)N(d_-),$$

where

$$d_\pm = \frac{1}{\sigma\sqrt{T}}\log(1+|y|) \pm \frac{1}{2}\sigma T, \quad N(d_\pm) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_\pm} e^{-x^2/2}\,dx.$$

□

Solution of the control problem by comparison

Hajek (1985), Shreve & Večeř (2000).

$$\begin{aligned} \text{Maximize} \quad & \widetilde{\mathbb{E}}\big[Y_T^+\big] \\ \text{Subject to} \quad & Y_0 = X_0/S_0, \\ & dY_t = \sigma(q_t - Y_t)\, d\widetilde{W}_t, \\ & |q_t| \le 1, \quad 0 \le t \le T. \end{aligned}$$

Let q_θ be any adapted process satisfying $|q_\theta| \le 1$, $0 \le t \le T$, and let

$$Y_t = Y_0 + \sigma \int_0^t (q_\theta - Y_\theta)\, d\widetilde{W}_\theta.$$

Define

$$\varphi(y) = -\text{sign}(y)(1 + |y|), \quad y \in \mathbb{R},$$

so that $\varphi^2(Y_\theta) \ge (q_\theta - Y_\theta)^2$. Define

$$A_t = \int_0^t \frac{(q_\theta - Y_\theta)^2}{\varphi^2(Y_\theta)}\, d\theta \le t, \quad A_s^{-1} = \inf\{t \ge 0 : A_t > s\}.$$

□

Solution of the control problem by comparison

Both

$$Y_t = Y_0 + \sigma \int_0^t (q_\theta - Y_\theta)\, d\widetilde{W}_\theta,$$

and

$$Y_t^2 - \sigma^2 \int_0^t (q_\theta - Y_\theta)^2\, d\theta$$

are continuous martingales.
Optional sampling implies that both

$$Z_s := Y_{A_s^{-1}}$$

and

$$Y_{A_s^{-1}}^2 - \sigma^2 \int_0^{A_s^{-1}} (q_\theta - Y_\theta)^2\, d\theta$$

are continuous martingales. □

Solution of the control problem by comparison

Consider the continuous martingale

$$
\begin{aligned}
& Y^2_{A_s^{-1}} - \sigma^2 \int_0^{A_s^{-1}} (q_\theta - Y_\theta)^2 \, d\theta \\
= \quad & Y^2_{A_s^{-1}} - \sigma^2 \int_0^{A_s^{-1}} \varphi^2(Y_\theta) \frac{(q_\theta - Y_\theta)^2}{\varphi^2(Y_\theta)} \, d\theta \\
= \quad & Z_s^2 - \sigma^2 \int_0^{A_s^{-1}} \varphi(Z_{A_\theta}) dA_\theta \\
= \quad & Z_s^2 - \sigma^2 \int_0^{s} \varphi^2(Z_\nu) \, d\nu.
\end{aligned}
$$

Therefore, both

$$Z_s \text{ and } Z_s^2 - \sigma^2 \int_0^s \varphi^2(Z_\nu) \, d\nu$$

are continuous martingales, so Z is a weak solution of the SDE

$$dZ_s = Y_0 + \sigma \int_0^s \varphi(Z_\nu) \, d\widetilde{W}_\nu. \qquad \square$$

Solution of the control problem by comparison

Conclusion: We have

$$
\begin{aligned}
Y_0 &= Z_0, \\
dY_t &= \sigma(q_t - Y_t)\, d\widetilde{W}_t, \\
dZ_s &= \sigma\varphi(Z_s)\, d\widetilde{W}_s, \\
Y_T &= Z_{A_T}, \\
A_T &\leq T.
\end{aligned}
$$

The submartingale property for Z^+ implies

$$\widetilde{\mathbb{E}}\big[Y_T^+\big] = \widetilde{\mathbb{E}}\big[Z_{A_T}^+\big] \leq \widetilde{\mathbb{E}}\big[Z_T^+\big].$$

□

Two other solution methods

- Solve the HJB equation by analytical methods
  - Andersen, Andreasen & Brotherton-Radcliffe (1998)
  - Hyer, Lipton-Lifschitz & Pugachevsky (1997)
- Solve the discrete-time trading problem and pass to the limit
  - Delbaen & Yor (2002)

□

Open problem: Passport option on multiple risky assets

- Two risky assets

$$
\begin{aligned}
dS_1(t) &= rS_1(t)\,dt + \sigma_1 S_1(t)\,dW_1(t),\\
dS_2(t) &= rS_2(t)\,dt + \sigma_2 S_2(t)\,dW_2(t),\\
dW_1(t)\,dW_2(t) &= \rho\,dt \quad (-1 < \rho < 1).
\end{aligned}
$$

- Traded account value

$$
dX(t) = \sum_{i=1}^{2} q_i(t)\,dS_i(t) + r\left(X(t) - \sum_{i=1}^{2} q_i(t)S_i(t)\right)dt.
$$

- Maximize

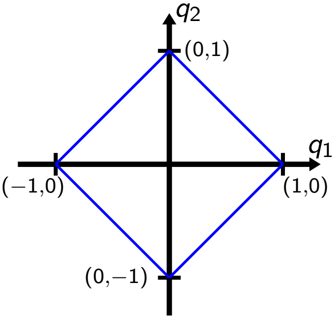
$$
\mathbb{E}\big[X^+(T)\big].
$$

□

Trading constraint and value function

Trading constraint

$$|q_1(t)| + |q_2(t)| \le 1, \quad 0 \le t \le T.$$



Value function

$$V(t, s_1, s_2, x) = \max_{|q_1|+|q_2|\le 1} \mathbb{E}\left[e^{-rT} X^+(T) \middle| S_1(t) = s_1, S_2(t) = s_2, X(t) = x \right].$$

□

Hamilton-Jacobi-Bellman equation

$$V_t + r\sum_{i=1}^{2} s_i V_{s_i} + rxV_x + \frac{1}{2}\text{trace}(C\nabla^2_{ss}V)$$
$$+ \max_{|q_1|+|q_2|\le 1}\left[\langle \nabla^2_{sx}V, Cq\rangle + \frac{1}{2}\langle q, Cq\rangle V_{xx}\right] = 0,$$

where

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad C = \begin{bmatrix} \sigma_1^2 s_1^2 & \rho\sigma_1\sigma_2 s_1 s_2 \\ \rho\sigma_1\sigma_2 s_1 s_2 & \sigma_2^2 s_2^2 \end{bmatrix} \ge 0,$$

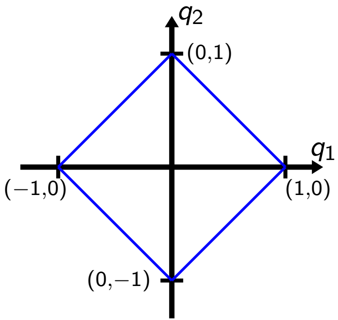
$$\nabla^2_{sx}V = \begin{bmatrix} V_{s_1 x} \\ V_{s_2 x} \end{bmatrix}, \quad \nabla V^2_{ss} = \begin{bmatrix} V_{s_1 s_1} & V_{s_1 s_2} \\ V_{s_1 s_2} & V_{s_2 s_2} \end{bmatrix}.$$

□

Bang-bang control

$$\langle \nabla_{sx}^2 V, Cq \rangle + \frac{1}{2} \langle q, Cq \rangle V_{xx}$$

is convex in (q_1, q_2), so the maximum is always obtained at an extreme points of the constraint set



But which extreme point? □

Observations

- Even if the correlation between the risky assets is zero or one, the solution is not known. These cases might have simple explicit solutions.
- In general, correlation seems to matter. Even when we don't hold an asset, its price changes now affect opportunities it offers later.
- The solution of the HJB equation for multiple risky assets is unknown, although variations of it with two risky assets have been solved numerically (e.g., Penaud (2000)).
- One can choose either S_1 or S_2 as the numéraire and reduce the dimensionality of the problem with two risky-assets. The reduced problem has a two-dimensional state process.
- The Skorohod mapping approach and the comparison argument do not have obvious extensions to the case of a two-dimensional state process. □

And finally....

Don't tell Mete anything!

Sagen Sie Mete nichts!

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