Non local equation in deforming media

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Main topic of the presentation:

The Monge Ampere equation and non local Processes

a) A review of the general regularity

a) Non local versions of Monge Ampere

b) Fractional kernels adapted to the geometry of a solution

The Monge Ampere equation

$$MA(u(x)) = \det D^2 u(x) = \Pi \ \lambda_j$$

is a monotone function of D^2u provided that u in convex (i.e. all eigenvalues of D^2 are positive)

We can also define

$MA(u(x)) = \mathrm{Inf}\{Lu$ / among $Lu = a_{ij\,(x)} D_{ij}\, u$ with the matrix n. $a_{ij\,(x)}$ being an affine transformation of the identity of determinant one$\}$

About right hand sides, there are different important degrees of regularity

$$\mathrm{Det}\, D^2\,(u(x)) = f(x, u, ..) ,$$

(If smooth we expect regularity of u)

(or) $$0 < h < \mathrm{Det}\, D^2\,(u(x)) < H$$

(bounded measurable RHS) important for basic theory, invariant under dilations and deformations)

$$\mathrm{Det}\, D^2\,(u(x)) \text{ a doubling measure}$$

(natural in case of free boundary, when boundary and interior interact, for instance, with harmonic measure: D.Jerison)

About fully non linear equations

We will only consider Concave Fully Non-Linear eqs. ,of the form

$$\mathrm{Inf}_{\alpha} \text{ of } a_{ij\alpha}(x)\, D_{ij}(u(x)),$$

where the index α indicates that $a_{ij\alpha}$ is in some set of positive matrices, for instance between two multiples of the identity (Ex. Pucci extremal operator)

In the non-local case, (second) derivatives are replaced by kernels of the form

$$\int [u(x+y)-u(x)]\, K_x(y)\, dy$$

K is a positive, symmetric kernel that compares the density at y with such at x: For instance in a process with random jumps instead of walks

The final (non linear) equation is invariant under translations since the same family of α's is tested at all x.

→ the derivatives of u satisfy a similar equation, but for "bounded measurable kernels"

Moreover, from the concavity of the operator, second order incremental quotients are subsolutions this same discontinuous equation.

→ That is one of the reasons why it is so important to find regularity properties for equations with discontinuous coefficients

A fundamental development occurred in the early '80s from the work of Krylov-Safanov on the regularity of solutions to fully non linear equations:

their main theorem showed the Harnak Inequality property for solutions of equations with measurable coefficients in non divergence form, and that implied applied to second derivatives (Evans-Krylov) $C^{2,\alpha}$ regularity of u.

Fundamental to this theorem is the ABP theorem, and the role of the Monge Ampere equation being the potential of a volume controlled mapping, with simultaneously non divergence and divergence structure, allows to connect the size of the volume of the map in the interior

(det D^2 w with the slope at the boundary).

Approaches to regularity for fully non linear equations

For equations with a comparison principle there are two ways to study regularity issues:

1- Regularity inherited from the boundary and

2-Interior Regularity

The first one requires regularity of the domain and of the data along the boundary and often *some form of translation invariance of the equation to obtain interior regularity through a maximum or comparison principle.*

Interior regularity, in turn, has deeper result. The Harnack inequality for instance: It establishes that for a non negative solution in say, a ball of radius one, the Sup and Inf in the ball of radius ½ are comparable.

Back to the MA equation partially fits in the discussion above:

It is a (degenerate) fully non linear equation.

Further (det B)$^{1/n}$ is a concave function of matrices B (as long as B stays positive)

In between Laplace and Monge Ampere we have the symmetric functions of the Hessian

$$\Sigma_m = \sum \lambda_{d1}\ d_{l2}\ldots\ \lambda_{dm}$$

Each one has a cone of monotonicity in the first quadrant and $(\Sigma_m)^{1/m}$ is concave.

They are degenerate and harder than both, Monge Ampere and Laplacean.

Likewise, the solution of $\boxed{MA(u(x) = \det D_{ij}\, u = f(x)}$ can also be defined in the first quadrant as

$$\boxed{Mu(x) = \inf_{a_{ij}} a_{ij}(f(x))D_{ij}u(x) = f(x)}$$

where the infimum is taken among all the positive matrices a_{ij} with $\det a_{ij} =$ the appropriate function of $f(x)$.

In other words, we minimize all measure preserving affine transformation of the Laplacean.

For D^2u restricted to the first quadrant $u(x)$ results convex (important for ellipticity) and MA vanishes along the boundary.

- After the work of Krylov and Safanov, the regularity inherited from the boundary for solutions of MA was attained in different ways, but mainly from boundary regularity

- A completely different question emerged with Yann Brenier's "Polar factorization and monotone rearrangement of vector valued functions" Lacking the familiar " boundary conditions" created the need of a different " local" treatment " of the solution

- Also optimal transportation does not see " singular measures"

The Monge Ampere in a local context geometry and regularity

→ **Good** **&** **Bad**:

- **Very degenerated** (bad)

- **Very rich family of invariants** (good):
  The family of "bounded, measurable RHS"

$$\boxed{0 < t < \det D^2 u < T}$$

Is **invariant under** Measure preserving affine transformations (of determinant one) dilations, rigid motions. **(as many invariances as the Laplacean)**

Regularity of weak solutions to Monge Ampere

Dichotomy:

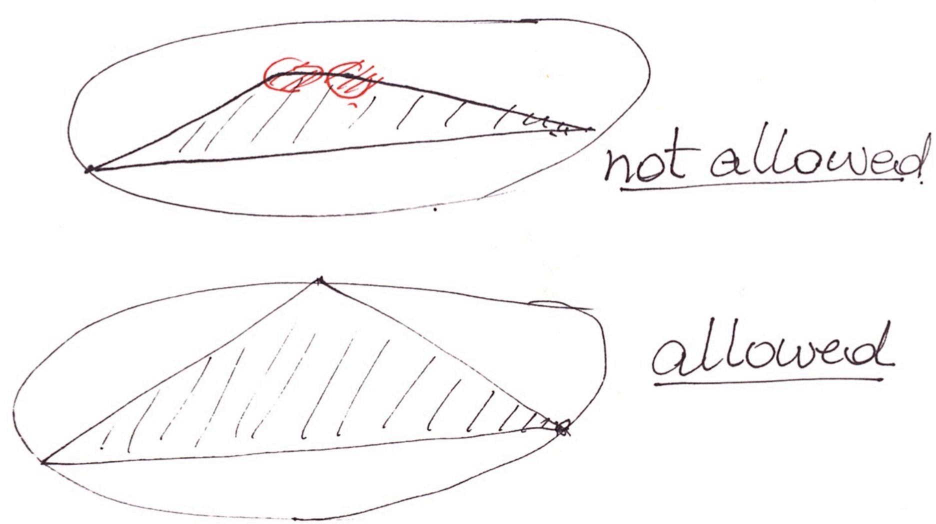
- **It may flat in a set with no extremal points in the interior** (i.e. is generated only by convex combinations of boundary points)

or

- **It is strictly convex** (tangent planes have a single contact point) **and it is $C^{1,\alpha}$**
- **(Flat at most in a set of dimension n-2**

Sketch of the main dicotomy arguments

- Main lemma: The set of **contact points** (where u vanishes) **cannot have interior extremal points.**
  In other words, **the zero set, if not a point, it is a convex combination of points in the boundary of the domain of definition …..**



Some preliminary: John's Lemma and renormalization

John's Lemma: Any convex set S with non empty interior can be trapped between two ellipsoids multiple of each other:

- **E_1 contained in S, contained in t(n) E_1, where t depends only on the dimension n**

Renormalization: Given a solution, **u(x)**, of

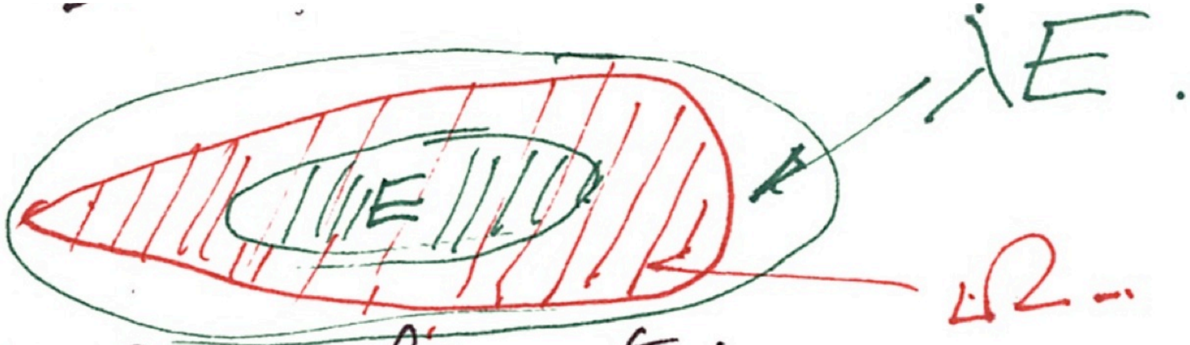
$$\boxed{\mathbf{0 < m < \det D^2(u(x)) < M\,,}}$$

and an affine transformation Ax, then

$$\boxed{\mathbf{u_A = (\det^{-1} A)\, u\,(Ax)\,,}}$$

also satisfies $\mathbf{0 < m < \det D^2(u_A(x)) < M}$

John's Lemma + affine renormalization

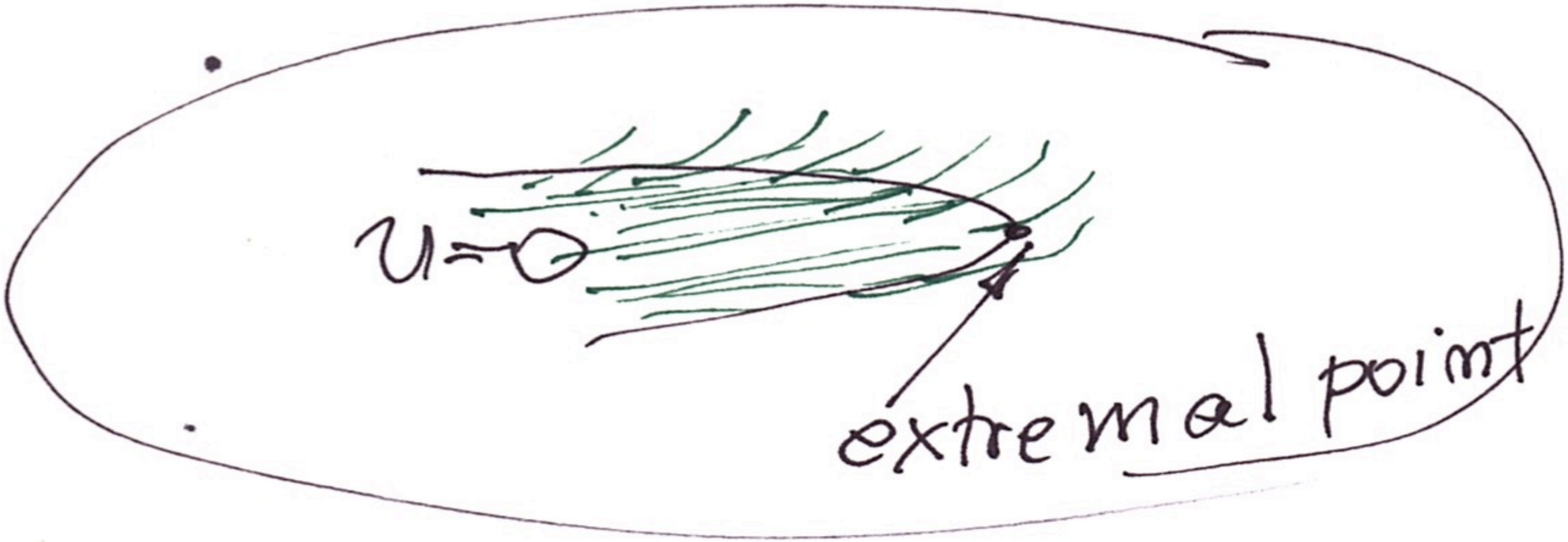


Affine renormalization

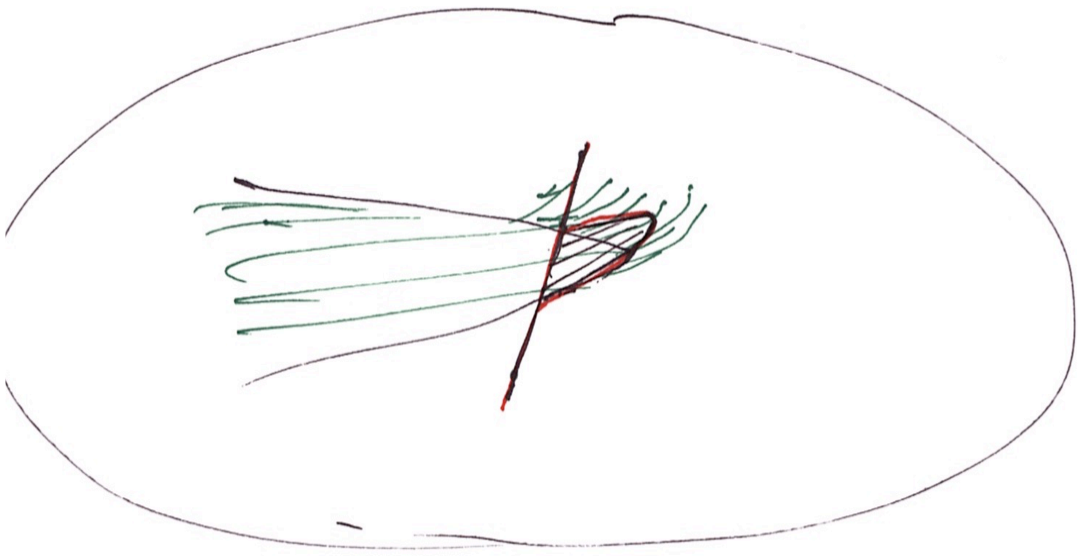
$E \to B_1, \quad \lambda E \to B_\lambda.$

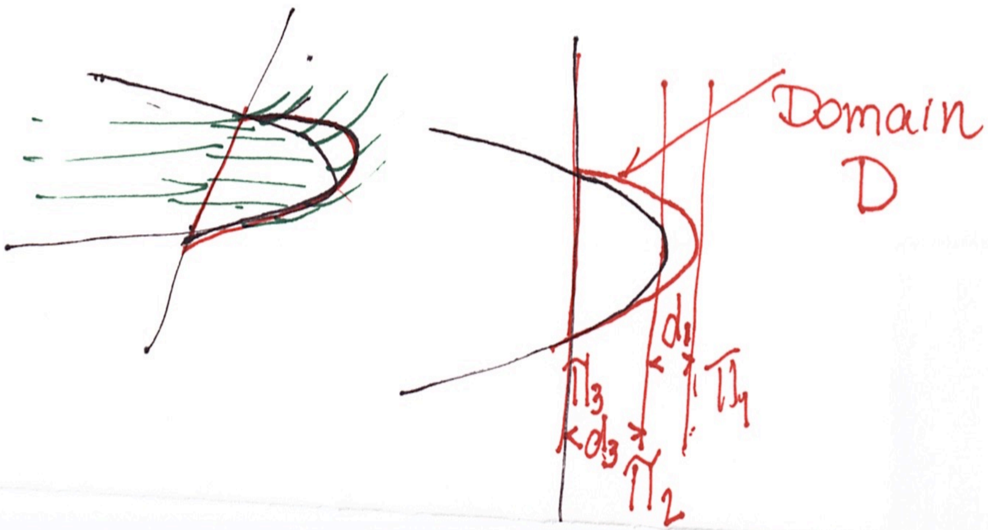
Proof: the **2D picture really corresponds to an nD configuration with n > 2.**

Assume we have the following **geometry:**

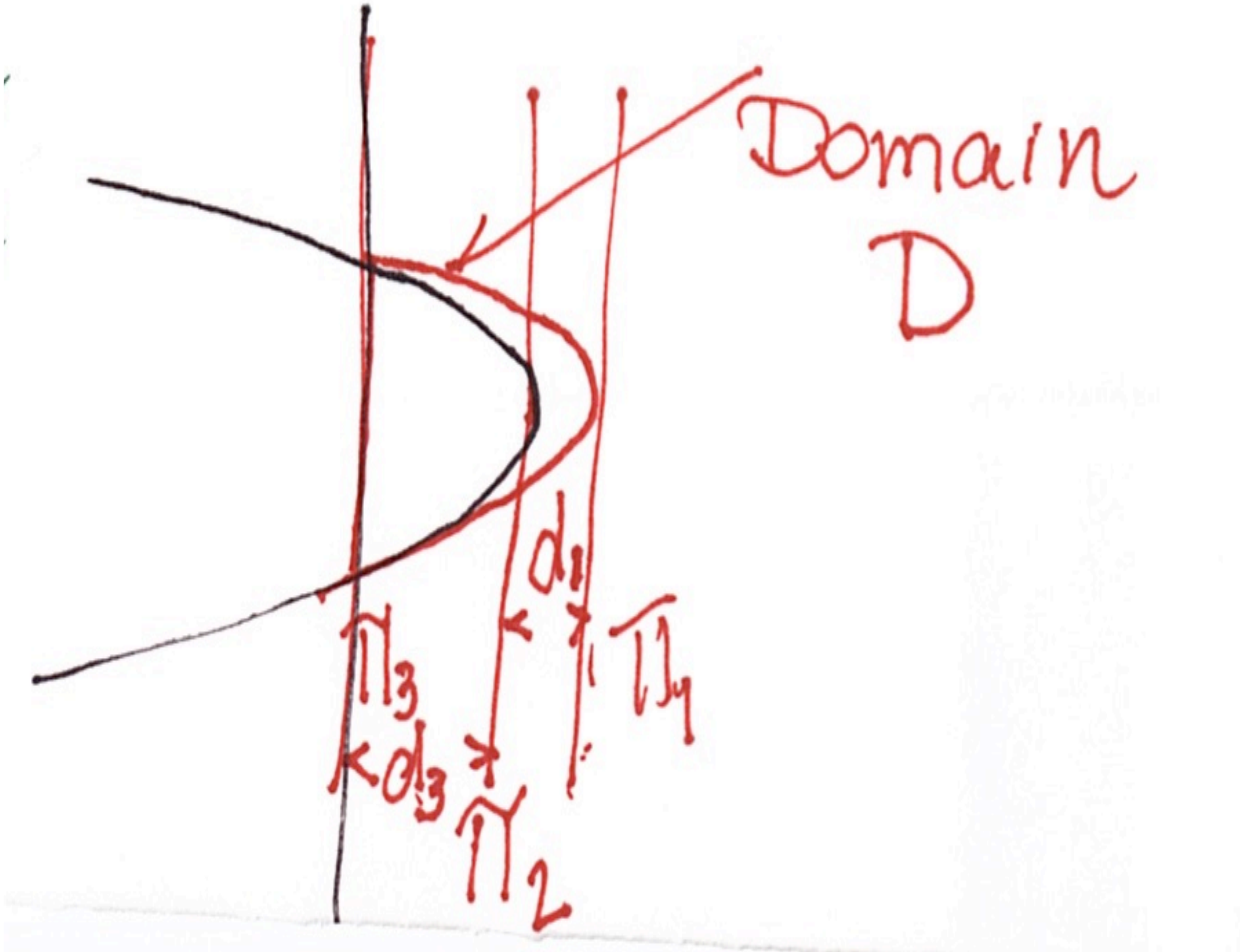


If we have an extremal point , we can slice the solution u with a plane of very small slope

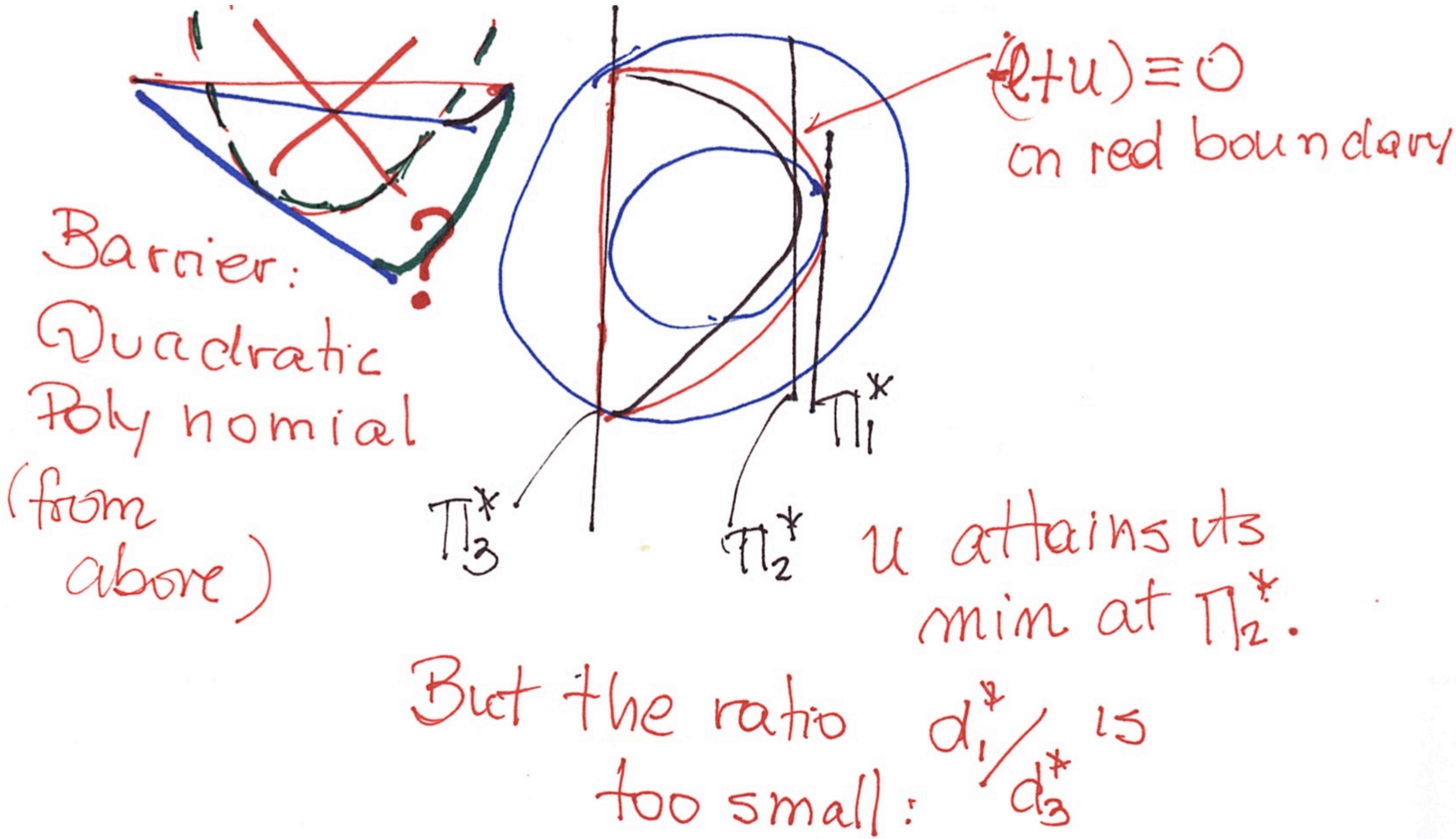




Since the slope is very small the distance, d_1 is much smaller than d_3



Renormalize as discussed above, so **the domain becomes equivalent to a ball.** A quadratic barrier pushes the **min of (-l+u) to be of order -1, at a point in $\prod^*_2$**



On the other hand, **we have a lower barrier** that says that **u–l is bounded below by its distance to the boundary:**

$$\boxed{W = (|x'|^2 - C)\, y^{2/n}}$$

where ***y is in the direction orthogonal to*** $\prod_2$ (horizontal in our picture) and ***x' tangential to*** $\prod_2$,

→ that contradicts the previous configuration

And so this completes the proof.

This observation becomes much stronger if we supplement it with compactness:

Assume that u is very close to zero in a fixed neighborhood of the origin, say **$u < \varepsilon$ in $B_r(0)$.**

- If ε goes to zero within the fixed $B_r(0)$, **u** is forced to converge to zero in a much larger set, generated all the way to the boundary of the domain

- **The counterpart is**: if **u is not flat all the way to the boundary** of the domain, **then u is strictly convex in the interior.**

In other words, **the eccentricity of the level surfaces of u changes in a controlled fashion**

An **important geometric element of solutions of Monge Ampere** eqs. is then its sections:

- Given a point x (**say x = 0 , the origin**), we may **substract a plane l(x), supporting the graph of u at 0** and consider the level surfaces

$$S_t \text{ of } u^* = u\text{-}l, \qquad (S_t = \{u^* < t\})$$

The **surfaces** S_t have **a doubling property and adjacent sections are "comparable"**:

- That is **S_{2t} is trapped between two multiples bigger than one of S_t and also a multiple of S_t engulfs its adjacent sections of comparable height.**

- This sets the geometry in a structure that allows to develop real analysis replacing balls by **sections of u**.

- **It connects,** in some way, **the domain of definition D, with its image by the transport map**

(see the fundamental work of **De Philippis, Figalli and Savin**)

Two Monge Ampere like

fractional equations

A first Monge Ampere fractional equation considers the

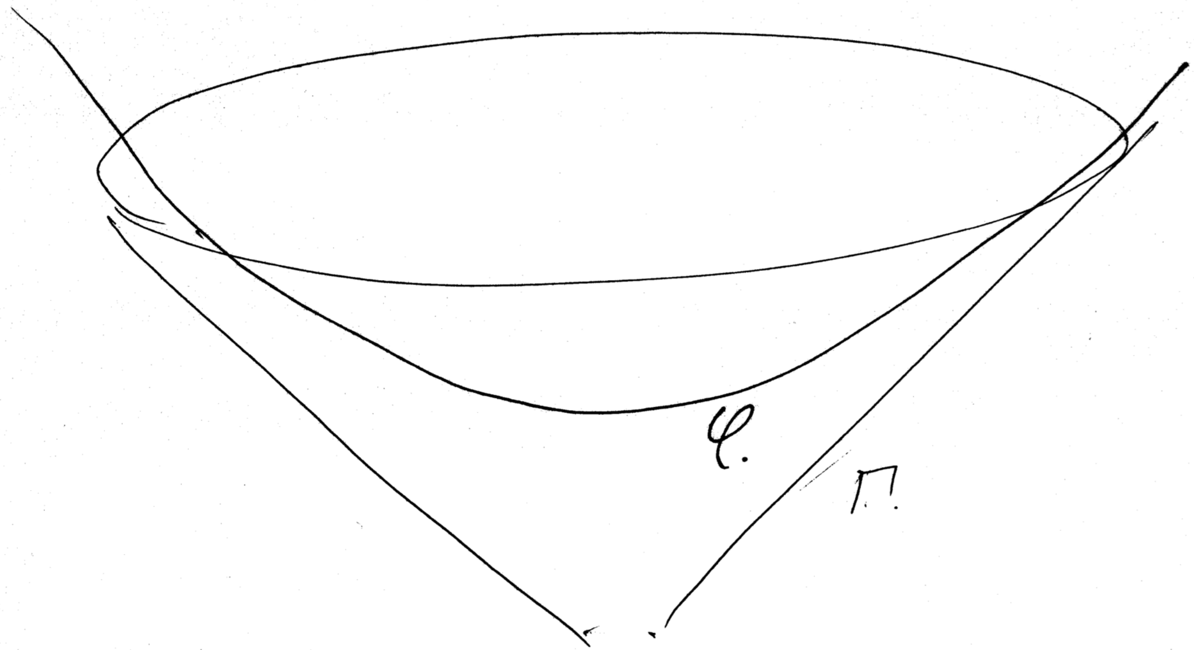
$$\textbf{inf of } (F_k(u(x)), \qquad \text{where}$$

$F_k(x)$ consists of all affine transformations of determinant one, but the equation is now **the fractional s-Laplacean**

Existence and regularity: we prescribe a simple geometry : **a smooth convex, strict subsolution φ, asymptotic to a cone of strictly convex smooth trace,** and solve the equation

$$\boxed{MA(u) = u - \varphi}$$

- Requires the s >1/2 for integrability at infinity,
- May also assume that φ is a smoothing by convolution of Γ



Main steps:

- **Existence of solutions:** Construct appropriate sub and super solution barriers:

  From below, **φ** does, from above is more delicate

- **Regularity:** Solutions are semiconcave, i.e second derivatives are bounded above (controls comes from infinity and concavity of the operator)

- Along each line the fractional Laplacean is bounded above and, on the other hand, **strictly positive (this replaces convexity by a sort of Fractional convexity)**

- The operators that are close to the **infimum remain strictly Elliptic.**

- **A non local version of Krylov Safanov and Evans Krylov applies.**

The first assertion, the fact that the

- **Fractional Laplacean is positive along any line**

is proven by contradiction .

Say e is a "bad line" where the average becomes negative.

By convolving (testing) u with an admissible kernel that is "heavily" charged in the bad direction e .

From the semi-concavity the contribution of the rest **cannot make the whole integral positive**, and so **a contradiction**

To show that the competing kernels are bounded, the crucial estimate is

$$(1-s)\int_{\mathbb{R}^n} \frac{u(y)-u(0)}{\left(\sum_{j=1}^{n} \epsilon_j^2 y_j^2\right)^{\frac{n+2s}{2}}}\,dy \geq \frac{\mu_0\,\omega_n}{2n}\cdot\sum_{j=1}^{n}\frac{1}{\epsilon_j^{2s}},$$

This implies that it is enough to have just one epsilon small to makes the configuration ineligible to be a minimizer, which also implies that all directions are comparable

Comment:
A natural question is what happens with an equivalent of the full family of equations of eigenvalues of the Hessian?

Yijing Wu has existence and regularity results for the second form:

$$\boxed{\Sigma\ \lambda_i \lambda_j}$$

In general, it reduces to an: "inverse problem" from "the corresponding Grassmanian induced by the operator" into the size of the "fractional operator" itself.

In the second instance, instead of considering
measure preserving "affine transformations"
take instead just measure presseving kernels
"each level surface can have a completely different, highly discontinuous image"

The only condition is that the infimum is among all measure preserving transformations of the fractional s-Laplacean kernel (the s-Laplacean kernel is the common rearrangement
of all of these "test" kernels)

We start again with a fixed kernel:

- **Choose again the Fractional Laplacean kernel**

$F_s(x) = (1-s)/|x|^{n+2s}$ **but now look at the equation**

$$Ma_s(u((x)) = \mathrm{Inf}_k \int [u(x+y) - (u(x)+\nabla u(x).y] \, K(y)dy$$

where the infimum runs now among all K that are a measure preserving rearrangement of K_0

The solution of this problem must be a convex function, (the infimum would be $-\infty$ otherwise) and although apparently disorganized, it ends up having a beautiful description:

If K_α is such a kernel,

$$|\{K_\alpha(t)>\}| = |\{K_s(t)>\}| = t^{-n/n+2s}$$

The equation is now

$$Ma_\alpha(u(x)) = \inf_\alpha \int [u(x+y)-(u(x)+\nabla u(x).y)]\, K_\alpha(y)dy$$

We prove existence, $C^{1,1}$ regularity of the solution and convergence to the classic MA when s → 1.

In fact, the cones K_α that realize the minimum have a very strong connection with the geometry of the solution:

This is due to the fact that in order to minimize the result the level surfaces of the kernel $K_x(y)$, i.e. the kernel that minimizes at x

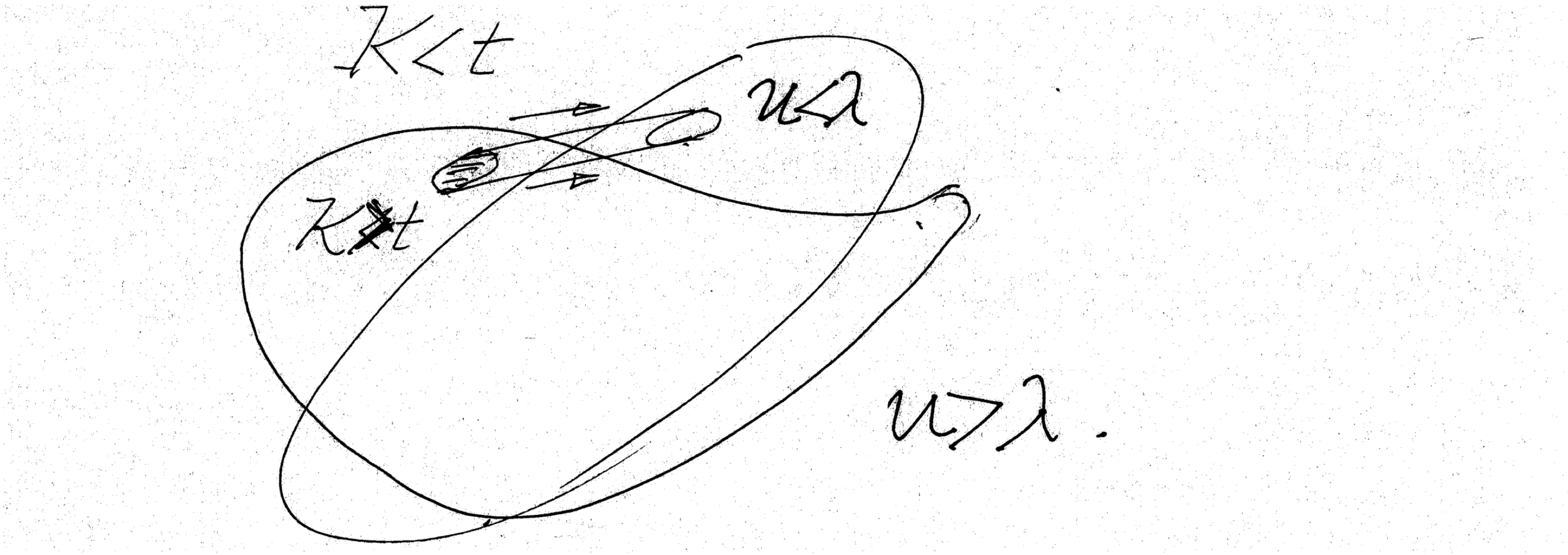
$$\int [u(x+y) - u(x)]\, K(y)\, dy$$

must align its level surfaces to those of

$$[u(x+y) - u(x) - \mathrm{grad}(u(x).y)]$$

For every t we have a λ, such that K> t and u< λ have the same volume, and then they have to match

i.e. we have a λ (t) or a t(λ) that matches the corresponding level surfaces - Assume they do not coincide, then rearranging K,   we would get a smaller value



We can then find a formula for $K_x(y)$ as function of u:

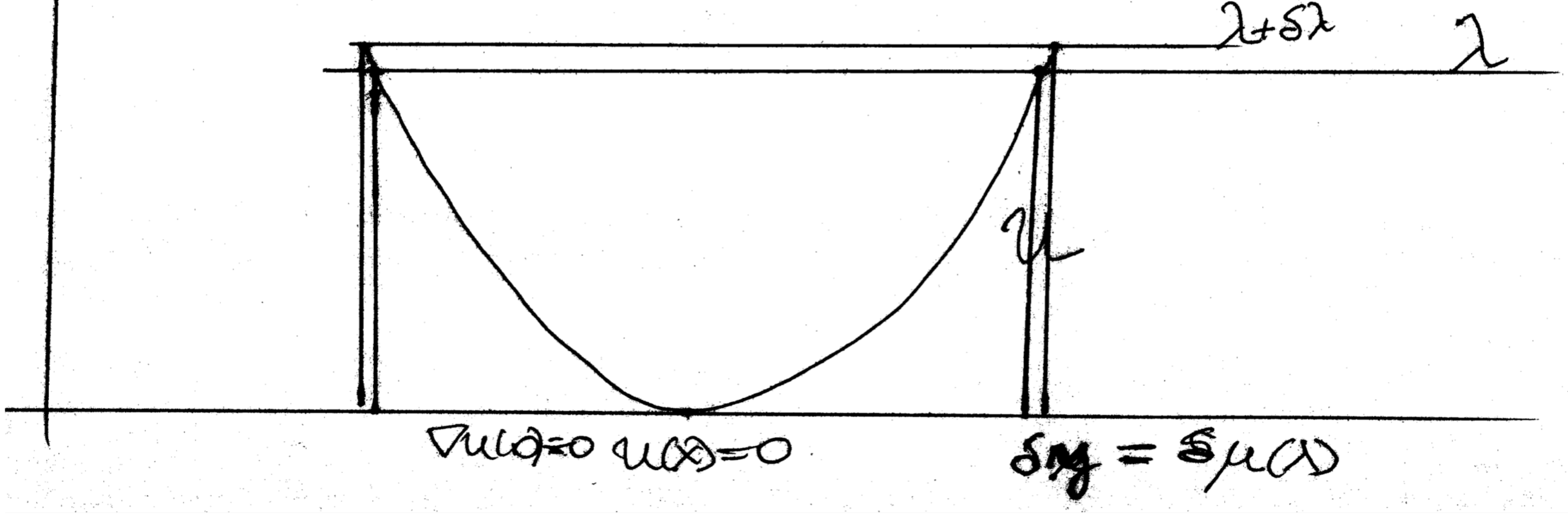
Let us find a formula for K in terms of the level surfaces of u:

- **We integrate in λ, the levels of u.**

- **Then t (λ) is the value that matches |{K>t}| with *μ* (λ), the volume of the level surface of u :**

- **We get:** $\boxed{\mu(\lambda) \text{ must be} = |\{K>t\}| = t^{-n/n+2s} = |\{x< t^{-1/n+2s}\}|}$

$$\int \Big[\overset{\lambda}{\widehat{u(x+y)}} - \overset{0}{\widehat{u(x)}} - \overset{0}{\widehat{\nabla u(x)y}}\Big] \overset{?}{\widehat{K(y)}} \overset{\mu'(\lambda)d\lambda}{\widehat{dy}}.$$



And the equation becomes an equation in the distribution function of the solution itself :

$$\mathbf{(1\text{-}s)\int\lambda\,(\mu)\mu^{-\,(n+2s)/n}d\mu = \text{-}n/2s\,(1\text{-}s)\int\mu^{-\,(2s)/n}(\lambda)d\lambda{=}f}$$

where $\lambda\,(\mu)$ is the high that corresponds to a section of area μ

Note that for s→1, the formula applied to $\mathbf{u(x)=|x|^2}$ near the origin and linear growth near infinity →
converges to standard Monge Ampere

Comment: Instead of prescribing data at infinity, **one may truncate the kernel and in particular make the equation translation invariant**

Is there a Jorgens Calabi Theorem for this equation, i.e global solutions are quadratic polynomials?

A third problem:

Non local kernels adapted to the geometry of a solution to the Monge Ampere equation

This third case is in collaboration with Rafeyel Teymurazyan and Jose Miguel Urbano and has to do with **non local kernels that adapt to the level surfaces a solution to Monge Ampere:**

It is somewhat based in work I did with Gutierrez where we notice that the level surfaces of a solution to the MA (called sections) **have a geometry of "balls" that provides the structure for a Calderon Zygmund type harmonic analysis theory**

In this third issue ,then,we are given roughly a (global) solution φ of

$$0 < p < MA\ \varphi < P$$

and we are interested in solutions of

$$L(u(x)) = \inf_A \int [u(x+y)-u(x)]\, K_{s,x}\,(Ay)\, dy$$

Where now the argument **Ay** adapts the level surfaces of the kernel to those of the “sections” of **φ**

If the solution **φ** to MA were **C |x| ^{2}**, K would be the **standard Fractional Laplacean**, where existence and regularity are well known.

- **If we would have a bounded measurable kernel, or the corresponding fully non linear non local equations the regularity would follow.**

- **The problem is hidden in the proof of the Harnack Inequality for the fractional bounded measurable kernel.**

- **We use there a Calderon Zigmund decomposition lemma argument that involves many scales.**

- **In the Eulidean case they are all comparable, but when we have the level surfaces of a solution to Monge-Ampere the geometry of small sections and large sections start to diverge and they have equal weight.**

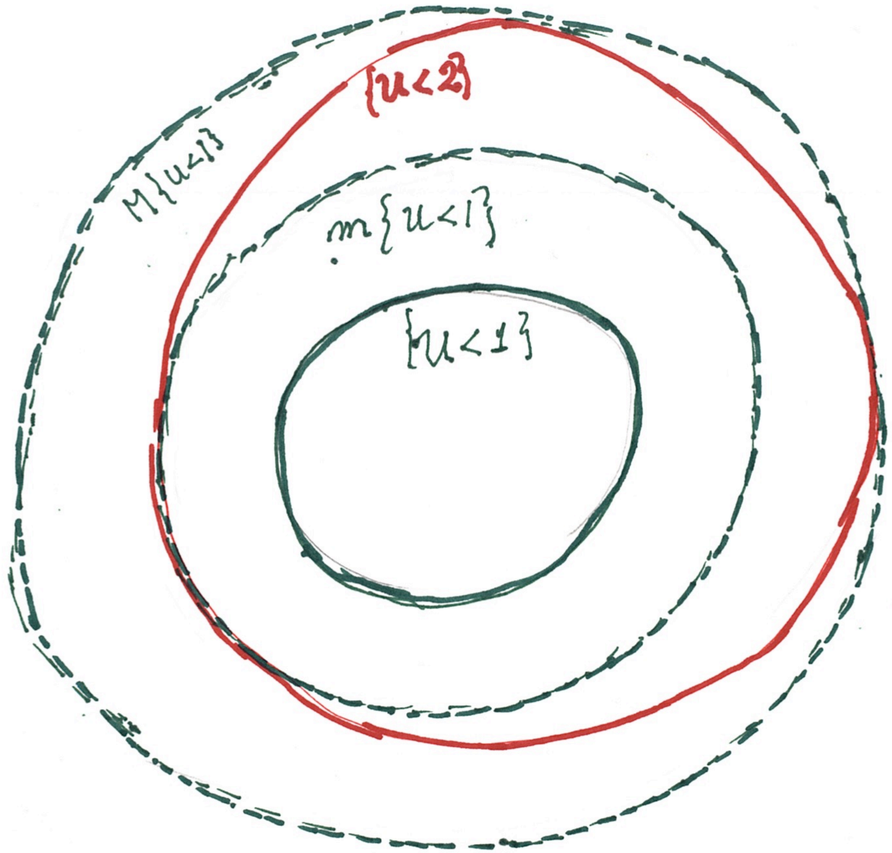
- **The solution was to develop covering lemmas that control interaction among the sections involved.**

- **This is done with a somehow involved diadic decomposition.**

- We prove an **ABP theorem and Harnack Inequalities for such solutions and $C^{1,h}$** for the fully non linear ones.

Comment: This work seems to be related to optimal Transporting Levi processes.

Thank you, very much for your attention



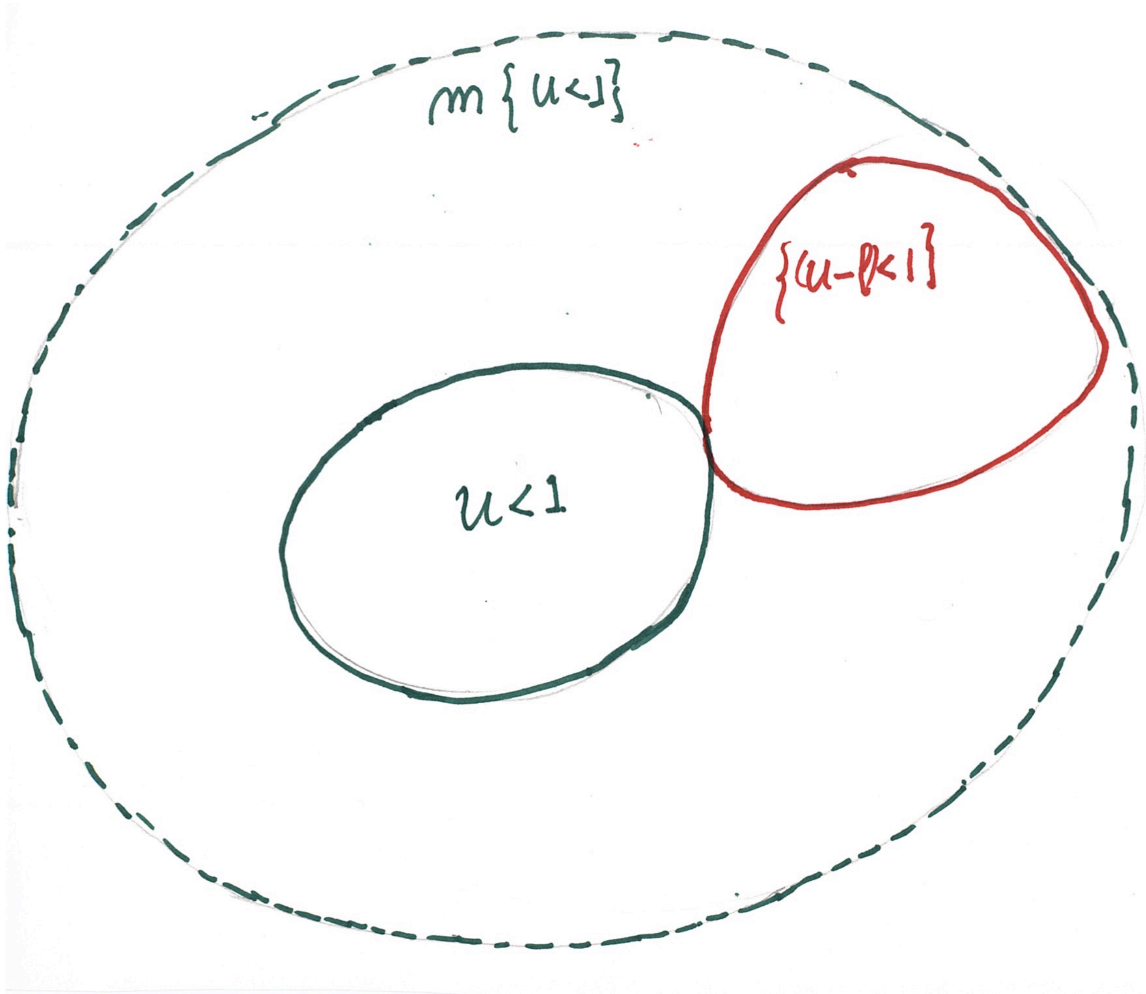
In between Laplace and Monge Ampere we have the symetric functions of the Hessian

$$\Sigma_m = \sum \lambda_{d1} \; d_{l2} \dots \; \lambda_{dm}$$

Each one has a cone of monotonicity containing the first quadrant and $(\Sigma_m)^{1/m}$ is concave

They are degenerate and harder than both
Monge Ampere and Laplacean.

And engulf adjacent slices of the same size



Thank you very much for your attention!

The level surfaces on Monge Ampere as a basis for analysis:

The level surfaces of u provides a structure that allows to develop real analysis, replacing balls by sections of u

This connects in some way the domain of definition D with its image by the transport map (if you have’t yet, see the fundamental work of De Philippis,Figalli and Savin on harmonic analysis of Monge Ampere)

Probably the main difficulty is to adapt to this geometry the non local APP and Harnak inequality of C. Silvestre, valid up to second order, due to the deformation of the kernels that difficult the interactions in the covering lemma.

We use special covering lemmas in deforming media where The space is split in areas of different eccentricity

In this case we non local kernels K_α whose level surfaces reproduce now the sections of a given (convex) solution to the second order Monge Ampere equation

In other words, in a medium where space deforms by the D^2 of a solution to MA, the kernels above are a natural Levy process with good basic regularity

We can think of this geometry as linked to optimal transportation (by grad (φ)) of a standard Levy process

It also suggests connecting Monge Ampere to non local operators, in two different ways:
Providing , with its level surfaces a geometric bases that replaces the standard bases of balls and cubes to do harmonic analysis:
covering lemmas , singular integrals
To give a parallel:

The domains obtained by slicing the parabola $P(x) = x^2$ with planes (the balls with all centers and radia) is substituted by slices of a non degenerate solution $u(x)$ of Monge Ampere

Fully non linear non local equations with kernels decaying along the level surfaces of a solution to Monge Ampere

Given now a solution, w, to the MA equation we consider its level surfaces, sections) provided by $v_x(y) < m$, with v_x the linear function

$v_x(y) = w(y) – w(x) - \nabla(w(x).y-x$

We will consider functions u (x) super and sub solutions of the extremal Pucci operators associated to the function,w:

$P^+ (u) > 0$ and $P^- (u) < 0$

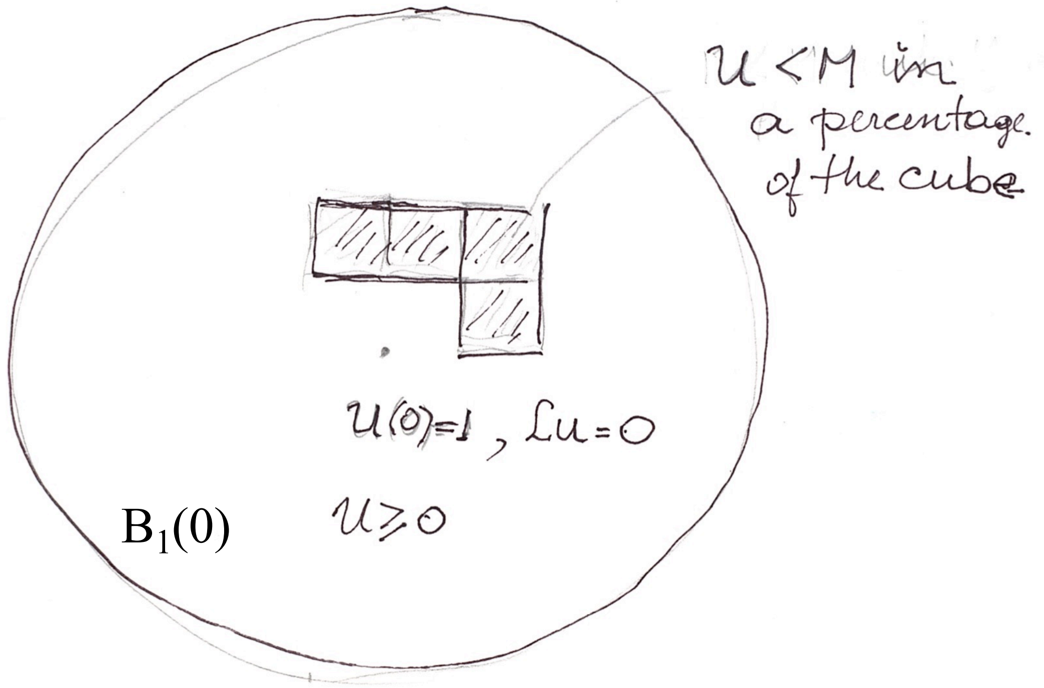
With

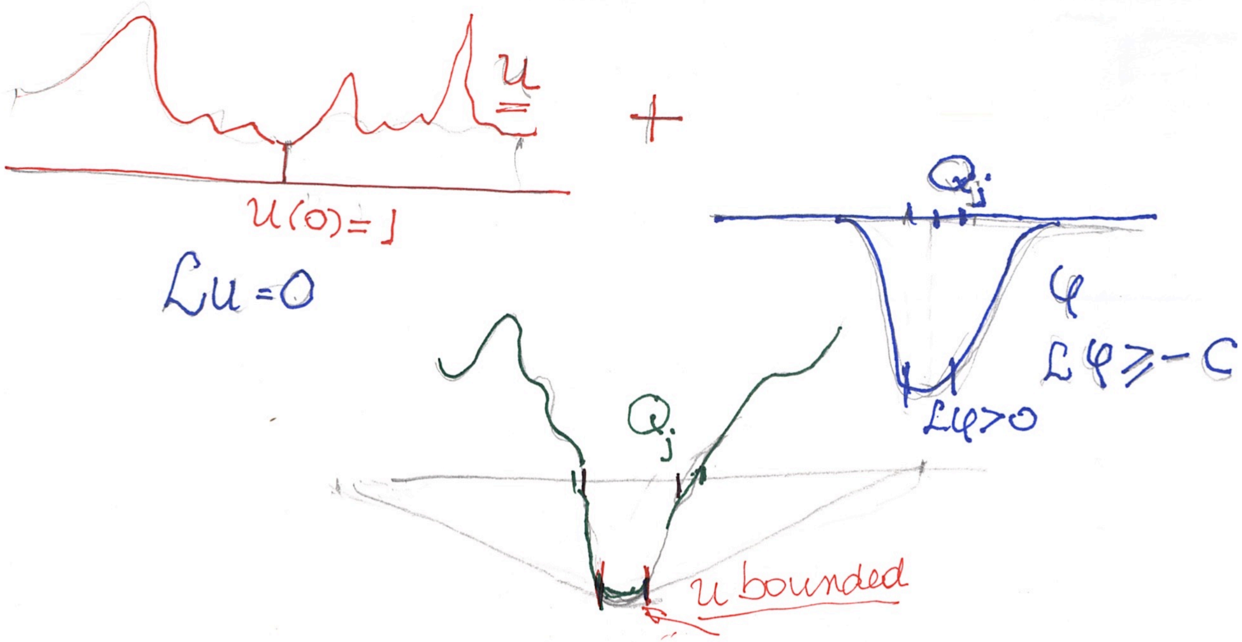
$P^+ (u (x)) = \int (2-\alpha) [\boldsymbol{\Lambda} \boldsymbol{\delta}^+ + \boldsymbol{\lambda} \boldsymbol{\delta}^-] \; v_x(y)^{-(n+\alpha)/2} dy$ (extremal Puci along the level sets of w)

First step:

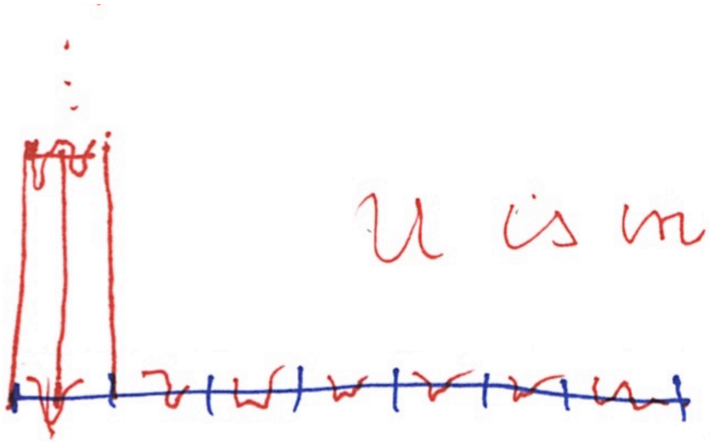
The non negative solution, $u(x)$, is in L^{ε}

Main bound: $u>0$, $u(0)=1$, then u is less than M in a portion of any nearby cube





$\underline{u}$
$+$
$u(0)=1$
$Lu=0$
Q_j
φ
$L\varphi \geq -C$
$L\varphi > 0$
Q_j
u bounded

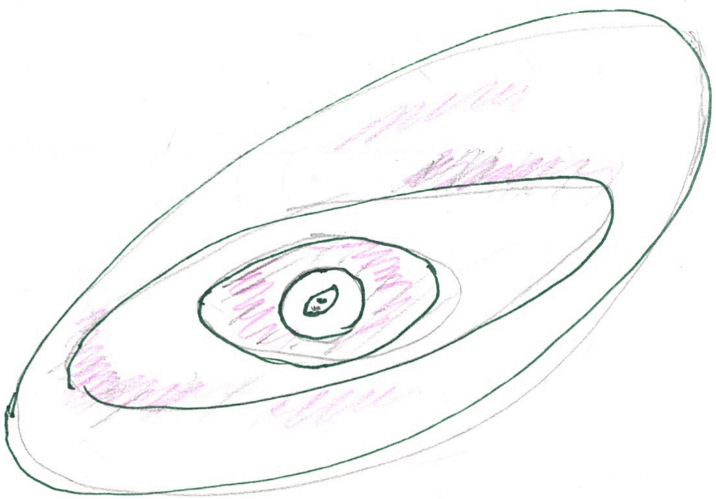


u is in L^{ε} : $|\{u > M^k\}| < \varepsilon^k$.

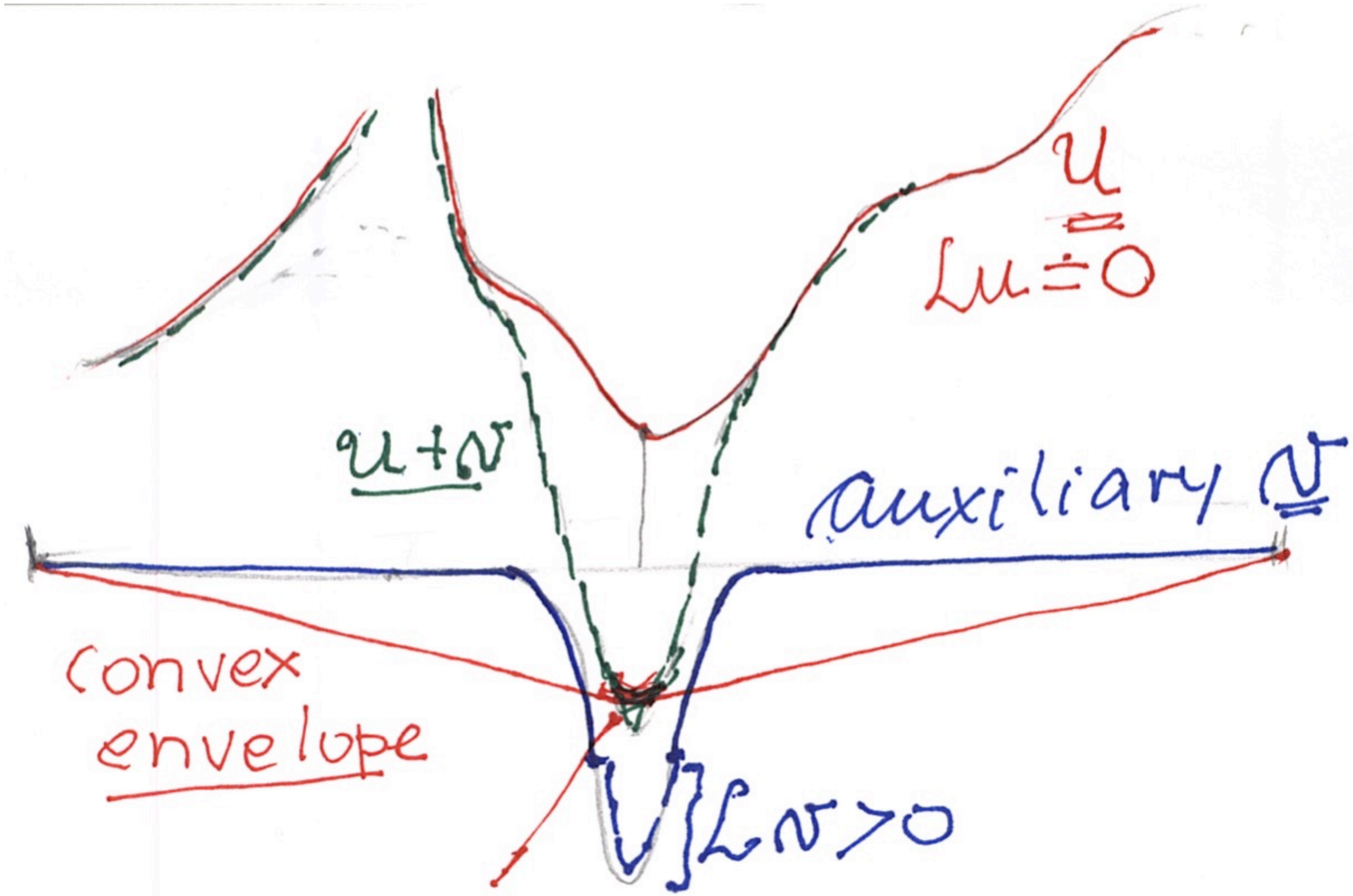
$u = 1$

In the non local case, there is at least one ring where the contact of $u + \boldsymbol{\varphi}$ with the convex envelope, E, covers a good portion of the ring and further forces flatness of the convex envelope in one of the rings

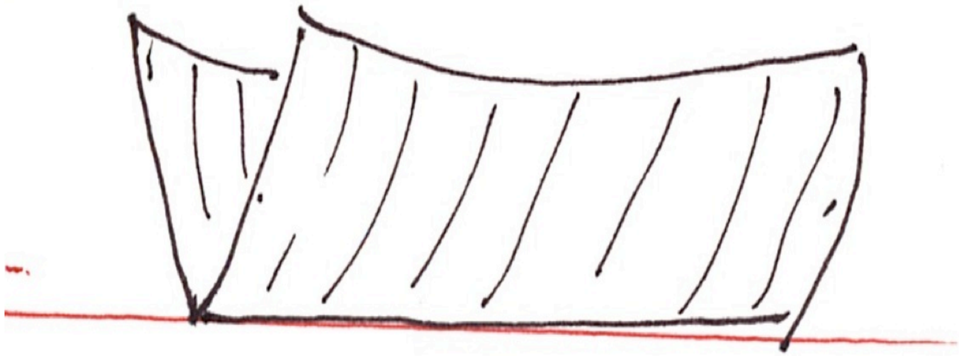
To get u to be L^{ε} we need a covering lemma scale by scale



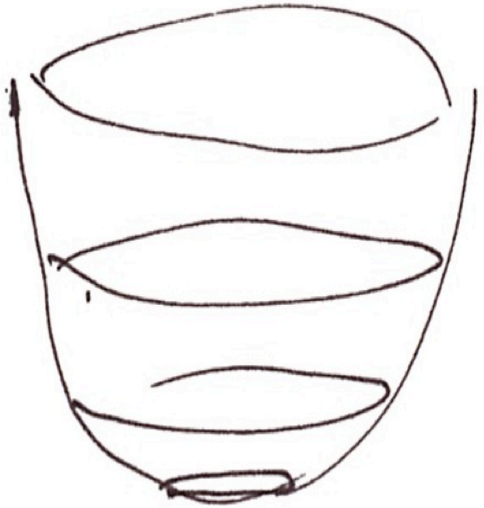

Large contact set of $u+v$ with $\Gamma(u+v)$, the convex envelope.



contact set:
$u+v$ and convex envelope.
(it occus only when.
$Lu+v \geq 0$



a)



b)

(no internal flat parts or "angles"