A critical Hölder exponent for isometric embeddings

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The Onsager Conjecture

Now a Theorem. We consider weak solutions v of the incompressible Euler equations on the periodic 3-dimensional torus $\mathbb{T}^3$

$$\begin{cases} \partial_t v + \mathrm{div}\,(v \otimes v) + \nabla p = 0 \\ \mathrm{div}\, v = 0\,. \end{cases} \tag{1}$$

The kinetic energy is

$$E(t) := \frac{1}{2} \int |v|^2(x,t)\, dx \tag{2}$$

Theorem

(a) *If $v \in C^{0,\alpha}$ with $\alpha > \frac{1}{3}$, then E is constant.*

(b) *For every $\alpha < \frac{1}{3}$ $\exists v \in C^{0,\alpha}$ with E is strictly decreasing.*

The Onsager Conjecture II

(a) proved by Constantin, E and Titi in 1994, after important work of Eyink.

The solutions in (b) are rather counterintuitive!

- Scheffer 1993, $L^2_t(L^2_x)$, nonconservative.
- Shnirelmann 2000, $L^\infty_t(L^2_x)$, dissipative.
- D-Székelhyidi 2008, L^∞, dissipative.
- D-Székelyhidi 2012, C^0, dissipative.
- D-Székelyhidi 2013, $C^{\frac{1}{10}-\varepsilon}$, dissipative.
- Buckmaster-D-Isett-Székelyhidi 2014, $C^{\frac{1}{5}-\varepsilon}$, dissipative.
- Buckmaster-D-Székelyhidi 2015, $L^1_t(C_x^{\frac{1}{3}-\varepsilon})$, nonconservative.
- Daneri-Székelyhidi 2016.
- Isett 2016, $C^{\frac{1}{3}-\varepsilon}$, nonconservative.
- Buckmaster-D-Székelyhidi-Vicol 2017, $C^{\frac{1}{3}-\varepsilon}$, dissipative.

Analogy with isometric embeddings

D-Székelyhidi 2012: C^0 dissipative solutions of Euler through an iteration similar to that used by Nash in 1954 for isometric embeddings.

(Σ, g) smooth (C^2) Riemannian manifold. $u : \Sigma \to \mathbb{R}^N$ is an isometry if it preserves the length of curves:

$$\ell_g(\gamma) = \ell_e(\gamma) \qquad \forall \gamma \subset \Sigma\,.$$

In coordinates, for $u \in C^1$:

$$\int \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}\, dt = \int \underbrace{\left|\frac{d}{dt}(u(\gamma(t))\right|}_{=|Du(\gamma(t))\cdot\dot\gamma(t)|}\, dt$$

$\gamma(t) = p$, $\dot\gamma(t) = v$ arbitrary

Isometric embeddings II

$u \in C^1$ is thus an isometry iff

$$g(v, v) = |Du \cdot v|^2$$

for every tangent vector v.

Note:

$$|Du(p) \cdot v|^2 = v^T \cdot \underbrace{(Du(p))^T \cdot Du(p)}_{=:A} \cdot v$$
$$= A_{ij} v^i v^j \,.$$

$$g(v, v) = g_{ij} v^i v^j \,.$$

Thus $u \in C^1$ is an isometry if and only if $g_{ij} = A_{ij} = (Du^T Du)_{ij}$, namely

$$g_{ij} = (Du^T Du)_{ij} = \partial_i u \cdot \partial_j u \,. \qquad (3)$$

Isometric embeddings III

Remark

If $u \in C^1$ is an isometry, then it is an immersion. If Σ is compact injective isometries are embeddings.

Remark

$u : \Sigma \to \mathbb{R}^N$, $\dim(\Sigma) = n$.
$\frac{n(n+1)}{2}$ equations in N unknowns.

Expect:

- local existence and a certain rigidity if $N = \frac{n(n+1)}{2}$ (Schläfli's conjecture: proved by Janet, Bursztin for real analytic metrics, open for C^∞!);
- no solutions if $N < \frac{n(n+1)}{2}$;
- many solutions if $N > \frac{n(n+1)}{2}$.

The Nash-Kuiper Theorem

And yet...

Corollary (Nash 1954 ($N \geq n+2$), Kuiper 1955 ($N \geq n+1$))

If Σ is compact and there are immersions (resp. embeddings) in $\mathbb{R}^N$, then there are isometric immersions (resp. embeddings) in $\mathbb{R}^N$.

In fact the Nash-Kuiper theorem is even more striking.

Definition

$u : \Sigma \to \mathbb{R}^N$ is short if it decreases the length of curves.

$u \in C^1$ short $\iff$ $Du^T Du \leq g$.

The Nash-Kuiper Theorem II

Theorem (Nash-Kuiper)

Assume (Σ, g) *is compact and* $w : \Sigma \to \mathbb{R}^N$ *a short immersion (resp. embedding) with* $N \geq \dim(\Sigma) + 1$*.*
For every $\varepsilon > 0$ *there is an isometric immersion (resp. embedding)* u *such that* $\|u - w\|_{C^0} < \varepsilon$*.*

Remark

Any $2d$ surface Σ can be isometrically embedded in an arbitrarily small ball $B_\varepsilon(0) \subset \mathbb{R}^3$.

But, Gauss' Theorem: if Σ is a positively curved sphere and $u \in C^2$ is isometric, then $u(\Sigma)$ has positive principal curvatures, i.e. $u(\Sigma)$ is convex.

Conclusion: the Nash-Kuiper theorem cannot produce C^2 isometries.

Critical Hölder exponent

Problem

Is there a Hölder threshold C^{1,α_0} for the Nash-Kuiper theorem?

Conjecture (Gromov)

Yes and moreover $\alpha_0 = \frac{1}{2}$

.

Theorem (D-Inauen 2018)

In an appropriate sense (to be explained later) $\frac{1}{2}$ is a critical exponent for the Nash-Kuiper construction

Strong form of criticality

Conjecture

(a) If (Σ, g) is a positively curved $2d$ surface and $u : \Sigma \to \mathbb{R}^3$ a $C^{1,\frac{1}{2}+\varepsilon}$ isometric immersion, then $u(\Sigma)$ is locally convex.

(b) If $\alpha < \frac{1}{2}$ and (Σ, g) is a compact $2d$ surface, then every short immersion (resp. embedding) $w : \Sigma \to \mathbb{R}^3$ can be uniformly approximated with $C^{1,\alpha}$ isometric immersions (resp. embedding).

From now on:

- "(a) holds for $C^{1,\alpha}$" is a "convexity theorem for $C^{1,\alpha}$";
- (b) is the Nash-Kuiper $C^{1,\alpha}$ approximation property.

Widely open: best exponents are $\frac{1}{5}$ from below and $\frac{2}{3}$ from above.

Borisov's Theorem

Theorem (Borisov 1958)

The convexity theorem holds for $C^{1,\alpha}$ whenever $\alpha > \frac{2}{3}$.

The proof is given in four papers, which reduce the theorem to a known result of Pogorelov.

Conti-D-Székelyhidi 2011: much simpler proof (taking advantage of the Constantin-E-Titi commutator estimate used in the proof of the positive part of the Onsager conjecture!).

Borisov's theorem II

Assume $u : \Sigma \to \mathbb{R}^3$ is a smooth isometric immersion, let N be the unit normal to $u(\Sigma)$.

Gauss' Theorem: Gauss curvature κ equals $\det dN$.

Area formula: for $\Omega \subset \Sigma$ Lipschitz open set and $\varphi \in C_c^\infty(\mathbb{S}^2 \setminus N(\partial\Omega))$,

$$\int_\Omega \varphi(N)\, \kappa\, d\mathrm{vol}_g = \int_{\mathbb{S}^2} \varphi(y) \sum_{z \in N^{-1}(y) \cap \Omega} \mathrm{sign}\,(\det dN)\,. \tag{4}$$

$\sum_{z \in N^{-1}(y) \cap \Omega} \mathrm{sign}\,(\det dN)$ is the Brouwer degree of $N|_\Omega$: $\deg(N, \Omega, y)$.

The Brouwer degree is defined even for N continuous, hence both sides of (4) make sense when $g \in C^2$ and $u \in C^1$.

Weak Gauss' Theorem

Theorem (Conti-D-Székelyhidi 2011)

The "weak Gauss identity" (4) *holds for* $C^{1,\frac{2}{3}+\varepsilon}$ *isometric immersions of* C^2 *Riemannian surfaces in* $\mathbb{R}^3$.

A simple effect

Corollary

If in addition $\kappa \geq 0$*, then* $\deg(N, \Omega, y) \geq 0$ *for every* Ω *and every* $y \in \mathbb{S}^2 \setminus N(\partial\Omega)$ *and thus*

$$\sum_i |N(E_i)| \leq \int_{\bigcup_i E_i} \kappa \tag{5}$$

for every E_i *pairwise disjoint Borel subsets of* Σ.

Pogorelov's theorem

(5) = "$u(\Sigma)$ has bounded extrinsic curvature in the sense of Pogorelov".

Theorem (Pogorelov)

Bounded extrinsic curvature and positive orientation of the Gauss map implies local convexity of $u(\Sigma)$.

Furthermore:

- estimates for Monge-Ampère imply higher regularity of u according to the regularity of g;
- if Σ is topologically a sphere, $u(\Sigma)$ is a closed convex surface and hence uniquely determined up to rigid motions (Cohn-Vossen and Herglotz).

Olbermann and Züst theorem: digression on degree

Theorem (Olbermann 2016, Züst 2016)

Let $U \subset \mathbb{R}^n$ be bounded and Lipschitz and let $z \in C^\alpha(U, \mathbb{R}^n)$ with $\alpha > \frac{n-1}{n}$. Then $\deg(z, U, \cdot) \in L^p$ *for every $p < \frac{n}{n-1}\alpha$.*

D-Inauen 2017, simple (and elementary) alternative proof.

Remark

- $z \in C^1 \Rightarrow \deg \in BV \subset L^{\frac{n}{n-1}}$.
- $z \in C^{\frac{n-1}{n}+} \Rightarrow \deg \in L^{1+}$.

Guess Sobolev regularity behind integrability. Beware:
$z \mapsto \deg(z, U, \cdot)$ is a not a linear map, interpolation is not possible!

Theorem (D-Inauen 2017)

If $z \in C^\alpha$, $\deg \in W^{\beta,p}$ *for every* $0 \le \beta < \frac{n}{p} - \frac{n-1}{\alpha}$ *and $p \ge 1$*

Towards $\frac{1}{2}$ from above

Using Olbermann-Züst estimate + Conti-D-Székelyhidi

Theorem

If (Σ, g) *is a* C^2 *surface and* $u \in C^{1,\frac{1}{2}+\varepsilon}(\Sigma, \mathbb{R}^3)$ *an isometry, then*

$$\int \deg(N, \Omega, y)\, dy = \int_\Omega \kappa\, d\mathrm{vol}_g \quad \forall \Omega \subset\subset U \ \textit{Lipschitz.} \tag{6}$$

(4) allows for two families of tests (Lipschitz open sets Ω and smooth φ), (6) allows just for one!

Towards $\frac{1}{2}$ from above II

Conjecture (De Lellis - Székelyhidi)

Let $z \in C^\alpha(U, \mathbb{R}^2)$ with $U \subset \mathbb{R}^2$ and $\alpha > \frac{1}{2}$. If

$$\int_{\mathbb{R}^2} \deg(z, \Omega, y)\, dy \geq 0$$

for every open set $\Omega \subset\subset U$ with Lipschitz boundary, then $\deg(z, \Omega, y) \geq 0$ $\forall \Omega$ and $\forall z \notin \mathbb{R}^2 \setminus z(\partial\Omega)$.

D-Inauen, forthcoming: the conjecture holds for $\alpha > \frac{2}{3}$ (not obvious from Conti-D-Székelyhidi, because the argument there uses more structure).

To be checked (but should be OK): the full conjecture implies a convexity theorem for $C^{1,\frac{1}{2}+\varepsilon}$ isometric immersions of positively curved surfaces.

Flexibility: low exponents

Theorem (Borisov 1963, announcement)

(Σ, g) *Riemannian manifold with:*

- Σ *diffeomorphic to the n-dimensional ball;*
- *g real analytic.*

Then the Nash-Kuiper $C^{1,\alpha}$ *approximation property holds for* $\alpha < \frac{1}{1+n(n+1)}$.

Note, $n = 2$, $\alpha < \frac{1}{7}$.

Borisov 2004, proof with $n = 2$ and $\alpha < \frac{1}{13}$.

Flexibility: low exponents II

Theorem (Conti-D-Székelyhidi 2011)

(Σ, g) *Riemannian manifold with* $g \in C^2$ *and dimension n. The Nash-Kuiper* $C^{1,\alpha}$ *approximation property holds*

- *for* $\alpha < \frac{1}{1+n(n+1)}$ *if* Σ *is diffeomorphic to a ball;*
- *for* $\alpha < \frac{1}{1+n(n+1)^2}$ *for general compact* Σ*.*

The annoying discrepancy between the two cases has been recently removed.
[Cao-Székelyhidi 2018] + [Cao-Székelyhidi, forthcoming]: $\alpha < \frac{1}{1+n(n+1)}$ for general n-dimensional compact Σ.

Flexibility: low exponents

Finally, combining conformal geometry and Nash-Kuiper:

Theorem (D-Inauen-Székelyhidi 2015)

Let (Σ, g) be a C^2 Riemannian manifold diffeomorphic to a 2-dimensional disk. Then the Nash-Kuiper $C^{1,\alpha}$ approximation property holds for $\alpha < \frac{1}{5}$.

Summarizing, on isometric embedding of positively curved surfaces in $\mathbb{R}^3$:

- $0 \leq \alpha < \frac{1}{5}$ Nash-Kuiper approximation property;
- $\frac{1}{5} \leq \alpha \leq \frac{2}{3}$ unknown;
- $\frac{2}{3} \leq \alpha$ convexity theorem.
- Clues pointing at the criticality of $\frac{1}{2}$.

A philosophical slide

Onsager conjecture (theorem):

- The Euler equations come with an additional conservation law (the energy identity), valid only for solutions above a certain threshold regularity.
- Rigidity in isometric embeddings is a much stronger property. In Euler a close analog would be uniqueness of solutions (which is known to hold for C^1 solutions and known to fail below $C^{\frac{1}{3}}$ as a byproduct of the convex integration methods proving the Onsager conjecture).
- Rigidity is ad-hoc for some *special* geometries.
- It uses the Gauss identity, where the second derivatives of u are involved, while the original equations involve only first derivatives.
- The energy identity in Euler involves as many derivatives as there are in the equations.

Parallel transport

Consider a smooth isometric embedding $u : \Sigma \to \mathbb{R}^N$ of a smooth Riemannian manifold.
Classical result in differential geometry:

Theorem

The Levi-Civita connection ∇^g of (Σ, g) is induced by the Euclidean connection ∇^e.

Fix vector fields X and Y on Σ and identify them with $du(X)$ and $du(Y)$ ("$X = du(X)$"...), namely identify the tangent $T\Sigma$ and the tangent to $u(\Sigma)$.

$$\nabla^\Sigma_X Y = \pi_{T\Sigma}(\nabla^e_X Y) .$$

Using $u^* X$ for $du(X)$, the correct version is

$$u^*(\nabla^\Sigma_X Y) = \pi_{Tu(\Sigma)}(\nabla^e_{u^* X} u^* Y) .$$

Extrinsic and intrinsic parallel transport

Fix an arc $\gamma \subset \Sigma$ with endpoints x and y and a $C^{1,\alpha}$ isometric immersion $u : \Sigma \to \mathbb{R}^N$.

The intrinsic parallel transport is given by a vector field X such that $\nabla^g_{\dot\gamma} X = 0$.

The "extrinsic parallel" transport can be defined via a discretization procedure:

- Take a mesh $x = x_0, x_1, \ldots, x_N$ on γ;
- Set $\tilde{X}_0 := du_{x_0}(X(0))$;
- Define inductively $\tilde{X}_n = \pi_{T_{u(x_n)} u(\Sigma)}(\tilde{X}_{n-1})$.
- Interpolate to get a map $\tilde{X} : \gamma \to \mathbb{R}^N$.
- Refine the mesh and take limits.

Extrinsic and intrinsic parallel transport II

Problem

How much regularity is needed to ensure that the discretization above converges to u^*X?

We expect $C^{1,\frac{1}{2}+\varepsilon}$.

Theorem (D-Inauen)

If $u \in C^{1,\alpha}$ for $\alpha > \frac{\sqrt{5}-1}{2}$, then the discretization converges to u^*X.

$$\frac{\sqrt{5}-1}{2} < \frac{2}{3}$$

$$\iff \frac{\sqrt{5}}{2} < \frac{7}{6} \iff 3\sqrt{5} < 7 \iff 45 < 49\,.$$

Back to Levi-Civita

Fix (Σ, g) smooth and $\gamma \subset \Sigma$ smooth curve. Consider X smooth vector field on Σ and $u : \Sigma \to \mathbb{R}^N$ $C^{1,\frac{1}{2}+\delta}$ isometric immersion.

$$t \mapsto \tilde{X}(t) = u^*(X(\gamma(t))) \qquad \text{is in } C^{\frac{1}{2}+\delta}$$

$$t \mapsto \frac{d}{dt}\tilde{X}(t) \qquad \text{is in } C^{-\frac{1}{2}+\delta}$$

$$t \mapsto L(t) := T_{u(t)}u(\Sigma) \qquad \text{is in } C^{\frac{1}{2}+\delta}$$

Conclusion:

$$t \mapsto Z(t) = \pi_{L(t)}\left(\frac{d}{dt}\tilde{X}(t)\right)$$

is well-defined as distribution.

Back to Levi-Civita II

The following is a weak version of "Levi Civita of the Euclidean ambient induces Levi Civita of the Riemannian manifold".

Theorem (D-Inauen 2018)

Since $u \in C^{1,\frac{1}{2}+\delta}$ is an isometry, the following identity holds:

$$Z(t) = u^* \left(\nabla_{\dot\gamma} X(t)\right) .$$

Polar caps

Consider the standard $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ and denote by σ the standard metric (induced by the euclidean one).

For $a \in]0,1[$ set

$$\Sigma_a := \Sigma \cap \{x_3 \geq a\}\,.$$

The Riemannian manifold (Σ_a, σ) is a polar cap.

We set $\gamma := \partial \Sigma_a$ and from now on we consider isometric embeddings $u : \Sigma_a \to \mathbb{R}^N$ with the additional constraint

$$u(x_1, x_2, a) = (x_1, x_2, a, \ldots, 0) \qquad \forall (x_1, x_2, a) \in \gamma\,.$$

Infinitesimal rigidity

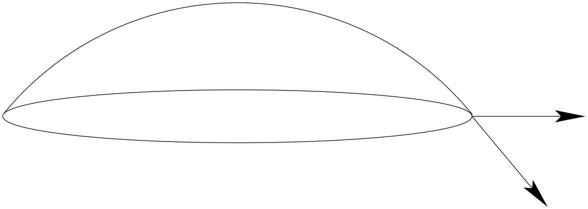
Consider now the unit tangent vector $\tau(t)$ to $\gamma(t)$.

$\nabla^\Sigma_\tau \tau$ is orthogonal to τ in Σ_a and if Y is the unit normal to γ in Σ,

$$g(\nabla^\Sigma_\tau \tau, Y) = k_g\,,$$

the geodesic curvature of $\gamma = \partial \Sigma_a$ in Σ_a.
Observe next that $\frac{d}{dt} u^*(\tau(t))$ is a vector in the plane $\{x_3 = x_4 = \ldots = 0\} \subset \mathbb{R}^N$ normal to γ and with length $\frac{1}{\sqrt{1-a^2}}$.
If u is smooth,

$$\left\langle \frac{d}{dt} u^*(\tau(t)), u^* Y \right\rangle = k_g\,.$$



Infinitesimal rigidity II

If $u \in C^{1,\frac{1}{2}+\delta}$ the previous computations are still valid!

Corollary

Let $u \in C^{1,\frac{1}{2}+\delta}(\Sigma_a, \mathbb{R}^N)$ with $\delta > 0$ be an isometric embedding of the polar cap which maps $\partial\Sigma_a$ onto $\{(x_1, x_2, a, 0, \dots, 0) : x_1^2 + x_2^2 = (1 - a^2)\}$. If Y is the exterior unit normal to γ_a in $\partial\Sigma_a$ and Z is the exterior unit normal to $\partial\Sigma_a$ in the plane $\{(x_1, x_2, a, 0, \dots, 0)\}$, then

$$\langle u^* Y, Z \rangle = a\,.$$

For $N = 3$, this means that the tangents to $u(\Sigma_a)$ at γ_a are determined and coincide (up to symmetry) with the ones of the standard embedding.

Flexibility

The first to prove that a suitable version of the Nash-Kuiper Theorem holds in the *constrained case* are Hungerbühler and Wasem in 2017 (C^1 version).

Gaining a Hölder exponent is more subtle. Using some ideas of [Källen 1978] we can prove:

Theorem

For any $\delta > 0$ there is $u \in C^{1,\frac{1}{2}-\delta}(\Sigma_a, \mathbb{R}^{14})$ isometric embedding of the polar cap with the following properties:

- *u maps $\partial\Sigma_a$ onto $\{(x_1, x_2, a, 0, \ldots, 0) : x_1^2 + x_2^2 = (1 - a^2)\}$.*
- *If Y is the exterior unit normal to γ_a in $\partial\Sigma_a$ and Z is the exterior unit normal to $\partial\Sigma_a$ in the plane $\{(x_1, x_2, a, 0, \ldots, 0)\}$, then*

$$\langle u^* Y, Z \rangle = 1 > a\,.$$

Thank you

for your attention!