Gluing methods for vortex dynamics in Euler flows

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PDEs and Geometric Measure Theory
ETH, Zurich
October 30, 2018

The Euler equation for the velocity of an incompressible fluid confined to $\Omega \subset \mathbb{R}^2$.

$$\begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla p & \text{in } \Omega \times (0, T) \\ \mathbf{u} \cdot \nu = 0 & \text{on } \partial\Omega \times (0, T) \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega \times (0, T) \\ \mathbf{u}(\cdot, 0) = \mathbf{u}_0 & \text{in } \Omega \end{cases} \tag{E}$$

$\mathbf{u}(x, t) : \bar{\Omega} \times [0, T) \to \mathbb{R}^2$, $p(x, t) : \bar{\Omega} \to \mathbb{R}$.
Ω smooth, bounded, simply connected domain in $\mathbb{R}^2$ or entire space.

Vorticity-stream formutation.

Let $\mathbf{u}(x,t)$ solve (E). The conditions

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \quad \mathbf{u}\cdot\nu = 0 \quad \text{on } \partial\Omega$$

imply the existence of a **stream function** $\Psi(x,t)$ with

$$\mathbf{u} = \nabla^{\perp}\Psi, \quad (a,b)^{\perp} = (b,-a)$$

such that $\Psi = \textit{constant}$ on $\partial\Omega$. The *vorticity of* $\mathbf{u}(x,t)$ is the quantity

$$\omega := \nabla\times\mathbf{u} = \partial_{x_2}u_1 - \partial_{x_1}u_2.$$

Then the pair (ω,Ψ) satisfies the vorticity-stream system

$$\omega_t + \nabla^{\perp}\Psi\cdot\nabla\omega = 0, \quad -\Delta\Psi = \omega$$

Reciprocally, if (ω, Ψ) solve the **vorticity-stream system**

$$\begin{cases} \omega_t + \nabla^\perp \Psi \cdot \nabla \omega = 0 & \text{in } \Omega \times (0, T) \\ -\Delta \Psi = \omega & \text{in } \Omega \times (0, T) \\ \Psi = 0 & \text{on } \partial\Omega \times (0, T) \end{cases} \qquad \text{(V)}$$

then $\mathbf{u} = \nabla^\perp \Psi$ satisfies the original Euler equation (E) (The Biot-Savart law).

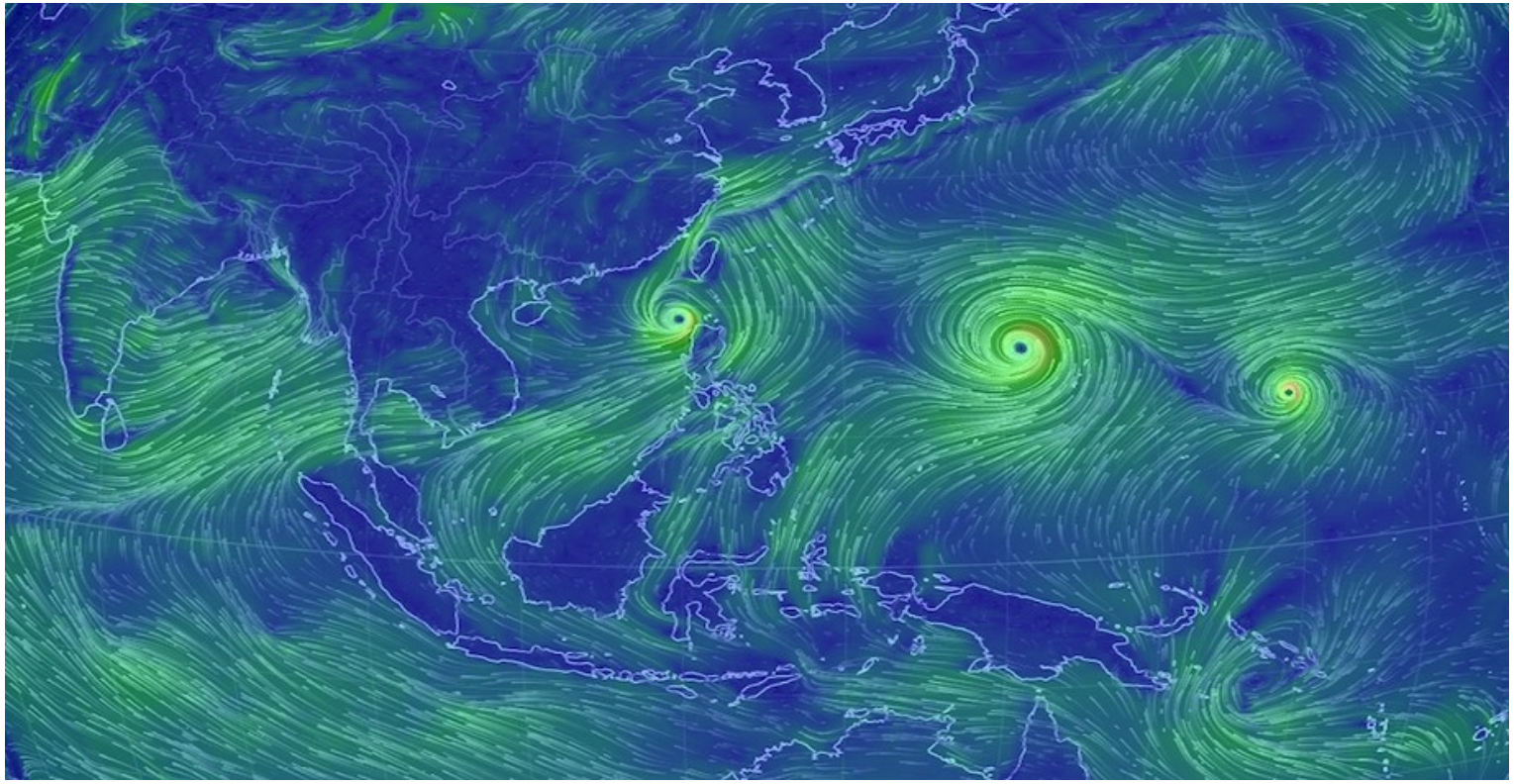
- Steady states: if $\Psi = \Psi(x)$ solves

$$\begin{cases} -\Delta \Psi = f(\Psi) & \text{in } \Omega \\ \Psi = 0 & \text{on } \partial\Omega \end{cases}$$

then the pair (ψ, ω), with $\omega(x) = f(\psi(x))$. is a steady state of (V) since

$$\nabla^\perp \Psi \cdot \nabla \omega = f'(\Psi) \nabla^\perp \Psi \cdot \nabla \Psi = 0.$$

We are interested in describing the evolution of solutions to system (SV) whose vorticities $\omega(x,t)$ are **very concentrated** around a finite number of points.



Velocity $\mathbf{u} = \nabla^{\perp}\Psi$ follows level curves of the stream function Ψ

Analysis of solutions with highly concentrated vorticities:
A mathematical subject with a long history: it traces back to Helmholtz (1858), Kirchoff (1876), Routh (1881).

Formal N-vortex singular solutions (ω^s, Ψ^s) of (V):

$$\omega^s(x,t) = \sum_{j=1}^{N} \kappa_j \, \delta(x - \xi_j(t)),$$

where $\delta(x)$ is the Dirac mass at 0, $\kappa_j \in \mathbb{R}$, $\xi_j : [0, T] \to \Omega$.
Since $-\Delta \Psi^s = \omega^s$, we must have

$$\Psi^s(x,t) = \sum_{j=1}^{N} \kappa_j G(x, \xi_j(t)),$$

with $G(x,\xi)$ Green's function,

$$-\Delta G(x,\xi) = \delta(x-\xi) \quad \text{in } \Omega, \quad G(x,\xi) = 0 \quad \text{on } \partial\Omega.$$

Formally we compute

$$\omega_t^s + \nabla^\perp \psi^s \cdot \nabla \omega^s = -\sum_{j=1}^N \kappa_j \nabla \delta(x-\xi_j)\cdot \dot{\xi}_j, \\ + \sum_{i,j=1}^N \kappa_i \kappa_j \nabla^\perp G(x,\xi_i)\cdot \nabla\delta(x-\xi_j).$$

Let us write

$$G(x,\xi) := \Gamma(x-\xi) - H(x,\xi), \quad \Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$$

$\Gamma(x)$, $\delta(x)$ are "radial", hence $\nabla^\perp \Gamma(x-\xi_j)\cdot\nabla\delta(x-\xi_j) = 0$ and

$$\nabla^\perp G(x,\xi_j)\cdot\nabla\delta(x-\xi_j) = -\nabla_x^\perp H(x,\xi_j)\cdot\nabla\delta(x-\xi_j)$$

$$\omega_t^s + \nabla^\perp \Psi^s \cdot \nabla \omega^s$$

$$= \sum_{j=1}^{N} [-\kappa_j \dot{\xi}_j + \nabla_x^\perp \big(- \kappa_j^2 H(x, \xi_j) + \sum_{i \neq j} \kappa_i \kappa_j G(x, \xi_i) \big)] \cdot \nabla \delta(x - \xi_j).$$

Thus $(\omega^s . \Psi^s)$ is a "solution" of Problem (V) if and only if

$$\dot{\xi}_j(t) = \nabla_x^\perp (-\kappa_j H(x, \xi_j(t)) + \sum_{i \neq j} \kappa_i G(x, \xi_i(t)))\big|_{x = \xi_j(t)}$$

Equivalently, (ω^s, Ψ^s) is a formal N-vortex solution of (SV) if and only if $\xi(t) = (\xi_1(t), \ldots, \xi_N(t))$ solves (without collisions) the Hamiltonian system

$$\kappa_j \dot{\xi}_j = \nabla^{\perp}_{\xi_j} K(\xi), \quad j = 1, \ldots, N \quad \text{in } [0, T] \tag{K}$$

where $K(\xi)$ is the **Kirchoff-Routh function**

$$K(\xi) := -\frac{1}{2} \sum_{i=1}^{N} \kappa_i^2 H(\xi_i, \xi_i) + \frac{1}{2} \sum_{i \neq j} \kappa_i \kappa_j G(\xi_i, \xi_j)$$

Natural question: Do solutions whose initial vorticity is highly concentrated around a finite set of points have "vorticity packets" evolving by a dynamics approximated by (K)?

Problem: (N-vortex desingularization): Can a singular vortex solution be ε-regularized into a "true" smooth solution of (SV)?

- Marchioro and Pulvirenti, 1993:

Given a colissionless solution $\xi(t)$ of System (K) there is a solution $(\omega_\varepsilon, \Psi_\varepsilon)$ of (V) with

$$\omega_\varepsilon(x,t) \rightharpoonup \omega^s(x,t) = \sum_{j=1}^{N} \kappa_j \delta(x - \xi_i(t)),$$

$$\Psi_\varepsilon(x,t) \rightharpoonup \Psi^s(x,t) = \sum_{j=1}^{N} \kappa_j G(x, \xi_j(t)),$$

in the distributional sense.

• **Our purpose:** To solve the N-vortex desingularization problem via gluing methods, providing precise asymptotic behavior of the solution near the vortices.

We find a solution in the form

$$\Psi_\varepsilon = \Psi_{0\varepsilon} + \psi_\varepsilon, \quad \omega_\varepsilon = \omega_{0\varepsilon} + \phi_\varepsilon, \quad -\Delta\Psi_{0\varepsilon} = \omega_{0\varepsilon}$$

where $\omega_{0\varepsilon}$ and $\Psi_{0\varepsilon}$ are explicit ε-regularizations of

$$\omega^s(x,t) = \sum_{j=1}^{N} \kappa_j \delta(x - \xi(t))$$

$$\Psi^s(x,t) = \sum_{j=1}^{N} \kappa_j \Big[\frac{1}{4\pi} \log \frac{1}{|x - \xi_j(t)|^2} - H(x, \xi_j(t))\Big]$$

and we have control on the ε-smallness of ψ_ε and ϕ_ε in stronger norms.

We choose the regularization

$$\boxed{\Psi_{0\varepsilon}(x,t) = \sum_{j=1}^{N} \frac{\kappa_j}{4\pi} \log \frac{1}{|x-\xi_j(t)|^2+\varepsilon^2} - \kappa_j H(x,\xi_j(t)),}$$

$$\boxed{\omega_{0\varepsilon}(x,t) = \sum_{j=1}^{N} \frac{\kappa_j}{\varepsilon^2} U_0\left(\frac{x-\xi_j}{\varepsilon}\right), \quad U_0(y) = \frac{1}{\pi(1+|y|^2)^2}\ .}$$

We have $\omega_{0\varepsilon} = -\Delta\Psi_{0\varepsilon}$ and $\int_{\mathbb{R}^2} U_0 = 1$. We get

$$\omega_{0\varepsilon} \rightharpoonup \sum_{j=1}^{N} \kappa_j \delta(x-\xi_i(t)).$$

For the kinetic energy density we obtain the convergence

$$\frac{1}{|\log\varepsilon|}|\nabla^\perp\Psi_{0\varepsilon}|^2 \rightharpoonup \sum_{j=1}^{N} \kappa_j^2 \delta(x-\xi_j(t)).$$

Theorem (Dávila, del Pino, Musso, Wei, 2018)

Let $\xi(t) = (\xi_1(t), \ldots, \xi_N(t))$ *be a colisionless solution of System* (K). *There exists a solution* $(\omega_\varepsilon, \Psi_\varepsilon)$ *of Problem* (V) *of the form*

$$\boxed{\begin{aligned} \omega_\varepsilon(x,t) =& \omega_{0\varepsilon}(x,t) + \phi_\varepsilon(x,t) \\ \Psi_\varepsilon(x,t) =& \Psi_{0\varepsilon}(x,t) + \psi_\varepsilon(x,t) \end{aligned}}$$

where for some $0 < \sigma < 1$ *and all* $(x,t) \in \Omega \times (0,T)$ *we have*

$$\boxed{\begin{aligned} |\phi_\varepsilon(x,t)| \ &\leq \ \varepsilon^\sigma \sum_{j=1}^{k} \frac{1}{\varepsilon^2} U_0\left(\frac{x-\xi_j}{\varepsilon}\right), \\ |\psi_\varepsilon(x,t)| + \varepsilon |D_x \psi_\varepsilon(x,t)| \ &\leq \ \varepsilon^2. \end{aligned}}$$

In particular:

$$\omega_\varepsilon \rightharpoonup \sum_{j=1}^{k} \kappa_j \delta(x-\xi_i) \quad \frac{1}{|\log \varepsilon|}|\nabla \Psi_\varepsilon|^2 \rightharpoonup \sum_{j=1}^{k} \kappa_j^2 \delta(x-\xi_j).$$

Ingredients in the construction:

• Improvement of the approximation in powers of ε using elliptic and transport equations.

• Setting up the problem as a coupled system of inner problems near the singularities and and an outer problem more regular (the inner-outer gluing scheme)

• A priori estimates to solve by a continuation (degree) argument.

Previous results concern with **steady states**. Liouville's equation:

$$\begin{aligned} -\Delta\Psi_\varepsilon &= \varepsilon^2 e^{\Psi_\varepsilon} \text{ in } \Omega \\ \Psi_\varepsilon &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

If ξ is a "topologically nontrivial" critical point of

$$K(\xi) = \sum_{i<j} G(\xi_i, \xi_j) - \frac{1}{2}\sum_{i=1}^{k} H(\xi_i, \xi_i)$$

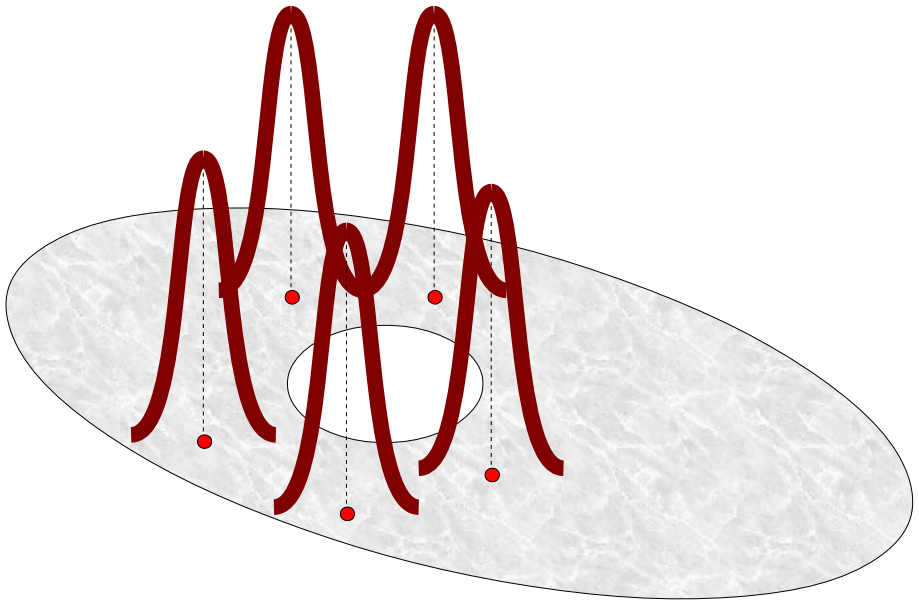
then this problem has a solution such that in distributional sense,

$$\varepsilon^2 e^{\Psi_\varepsilon} \rightharpoonup \sum_{j=1}^{N} 8\pi\delta(x-\xi_i), \quad \frac{1}{|\log\varepsilon|}|\nabla\Psi_\varepsilon(x)|^2 \rightharpoonup \sum_{j=1}^{N} 8\pi\delta(x-\xi_j).$$

Baraket-Pacard 1998, Esposito-Grossi-Pistoia 2005,
del Pino-Kowalczyk-Musso 2005.

$$
\begin{aligned}
-\Delta\Psi_\varepsilon &= \varepsilon^2 e^{\Psi_\varepsilon} \text{ in } \Omega \\
\Psi_\varepsilon &= 0 \qquad \text{on } \partial\Omega
\end{aligned}
$$

In a non-simply connected domain, N-vortex solutions exist for any k (del Pino-Kowalczyk-Musso 2005).



Liouville's equation enters in the proof of our result. The function

$$\Gamma_0(y) := \frac{1}{8\pi} \log \frac{8}{(1+|y|^2)^2}$$

satisfies the Liouville equation

$$-\Delta\Gamma_0(y) = \frac{1}{8\pi} e^{8\pi\Gamma_0(y)} = \frac{1}{\pi} \frac{1}{(1+|y|^2)^2} = U_0(y) \quad \text{in } \mathbb{R}^2$$

and $\int_{\mathbb{R}^2} U_0(y)dy = 1$. We can write

$$\boxed{\Psi_{0\varepsilon}(x,t) = \sum_{j=1}^{k} \kappa_j \Gamma_0 \left(\frac{x - \xi_j(t)}{\varepsilon} \right) - \kappa_j H(x, \xi_j(t)) - \frac{\kappa_j}{8\pi} \log 8\varepsilon^2}$$

Let us show how to improve the approximation for a solution to (V) near the singularity $x = \xi_j$. We compute

$$\Psi_{0\varepsilon}(x,t) = \kappa_j \Gamma_0(y) + \varphi(x) + O(\varepsilon^2) + constant$$

where

$$y = \frac{x - \xi_j}{\varepsilon},$$

$$\varphi(x) = -\kappa_j H(x, \xi_j) + \sum_{i \neq j} \kappa_i G(x, \xi_i).$$

We want to solve the equation

$$E(\omega, \Psi) = 0,$$

$$\boxed{E(\omega, \Psi) := \omega_t + \nabla_x^{\perp}\Psi \cdot \nabla_x \omega, \quad -\Delta_x \Psi = \omega.}$$

We look for a solution of the form

$$\boxed{\Psi = \Psi_{0\varepsilon}(x,t) + \kappa_j \psi(y,t), \quad \omega = \frac{\kappa_j}{\varepsilon^2} U_0(y) + \frac{\kappa_j}{\varepsilon^2}\phi(y,t)}$$

$$\boxed{y = \frac{x - \xi_j(t)}{\varepsilon}.}$$

In terms of the y-variable we get the expression

$$\boxed{\begin{aligned}\varepsilon^4 E(\omega,\Psi) =& \varepsilon^2\phi_t + (-\varepsilon\dot{\xi} + \nabla_y^\perp \Psi_{0\varepsilon} + \kappa_j \nabla_y^\perp \psi)\cdot\nabla_y(U_0+\phi),\\ -\Delta_y\psi =& \phi\end{aligned}}$$

By assumption $\dot{\xi}_j = \nabla_x^\perp \varphi(\xi_j)$. Since we have

$$\Psi_{0\varepsilon}(x,t) = \kappa_j \Gamma_0(y) + \varphi(x) + O(\varepsilon^2) + constant$$

then we get

$$-\varepsilon\dot{\xi}_j + \nabla_y^\perp \Psi_{0\varepsilon}(\xi_j + \varepsilon y) = \kappa_j \nabla^\perp(\Gamma_0 + \mathcal{R})$$

with $\mathcal{R} = O(\varepsilon^2|y|^2)$.

$$\boxed{\begin{aligned}\varepsilon^4 E(\omega,\Psi) =& \varepsilon^2\phi_t + \kappa_j \nabla_y^\perp(\Gamma_0(y)+\mathcal{R}+\psi)\cdot\nabla_y(U_0+\phi),,\\ -\Delta_y\psi =& \phi\\ \mathcal{R} =& O(\varepsilon^2|y|^2)\end{aligned}}$$

Let $f(u) = \frac{1}{8\pi}e^{8\pi u}$. Since $U_0 = f(\Gamma_0)$ we find

$$\boxed{\begin{aligned}\varepsilon^4 E(\omega,\Psi) =& \varepsilon^2\phi_t - \kappa_j \nabla_y^\perp\Gamma_0\cdot\nabla(\Delta\psi + f'(\Gamma_0)\psi)\\ &+ \kappa_j\nabla^\perp\mathcal{R}\cdot\nabla U_0 + \kappa_j\nabla^\perp\mathcal{R}\nabla\phi + \nabla^\perp\psi\nabla\phi.\end{aligned}}$$

The 0-error term:

$$\varepsilon^4 E(\omega_{0\varepsilon},\Psi_{0\varepsilon}) = \nabla^\perp\mathcal{R}\cdot\nabla U_0 = O(\varepsilon^2|y|^{-4}).$$

We obtain a reduction in the error by solving the elliptic equation

$$-\nabla_y^\perp\Gamma_0\cdot\nabla(\Delta\psi + f'(\Gamma_0)\psi) + \nabla^\perp\mathcal{R}\cdot\nabla U_0 = 0$$

In polar coordinates $y = \rho e^{i\theta}$ we see that

$$\nabla_y^{\perp}\Gamma_0 \cdot \nabla(\Delta\psi + f'(\Gamma_0)\psi) = \frac{1}{1+\rho^2}\frac{\partial}{\partial\theta}[\Delta_y\psi + f'(\Gamma_0)\psi]$$

This operator is highly degenerate, since it contains all radial functions in its kernel and the range does not include radial objects.

But our equation is equivalent to

$$\frac{\partial}{\partial\theta}[\Delta_y\psi + f'(\Gamma_0)\psi + \underbrace{f'(\Gamma_0)\mathcal{R}}_{O(\varepsilon^2\rho^{-2})}] = 0.$$

and we check that the larger terms of $\mathcal{R}$ only contain Fourier modes 2 and 3 at which the operator $\Delta_y + f'(\Gamma_0)$ is nicely invertible. We find a solution $\psi = O(\varepsilon^2)$, $\phi = -\Delta\psi = O(\varepsilon^2\rho^{-2})$.

The new error at main order is

$$E(\omega_{0\varepsilon} + \phi, \Psi_{0\varepsilon} + \psi) = O(\varepsilon^4 \rho^{-2}).$$

This is not enough. The equation for a new ψ and ϕ adjusting the previous improvement will come as

$$\varepsilon^2 \phi_t \; - \; \kappa_j \nabla_y^{\perp} \Gamma_0 \cdot \nabla(\Delta\psi + f'(\Gamma_0)\psi) + Q(\phi) + O(\varepsilon^4 \rho^{-2})$$

with Q quadratic and one loses ε^2 in solving the full operator because of the $\varepsilon^2 \phi_t$ term.

To improve the approximation in further powers of ε we solve elliptic equations like

$$\nabla_y^{\perp}\Gamma_0 \cdot \nabla(\Delta\psi + f'(\Gamma_0)\psi) + E(y,t) = 0$$

(new error "wins" two powers of ε of smallness and "loses" two ρ) and transport equations of the form

$$\varepsilon^2\phi_t + \kappa_j\nabla_y^{\perp}\Gamma_0 \cdot \nabla\phi + E(y,t) = 0$$

(new error created wins 4 powers of ρ and loses two ε) using adjustments of ξ to kill the mode 1 terms. We end up with an approximation $(\omega_\varepsilon^*, \Psi_\varepsilon^*)$ with

$$E(y,t) = \varepsilon^4 E(\omega_\varepsilon^*, \Psi_\varepsilon^*) = O\Big(\frac{\varepsilon^5}{\rho^3}\Big)$$

This improvement is sufficient for a priori estimates-continuation method to work.

Let us assume $\kappa_j = 1$. We consider smooth cut-off functions

$$\eta_j(x,t) = \eta_0\left(\frac{|x-\xi_j(t)|}{\delta}\right)$$

where $\eta_0(s) = 1$ if $s < 1$, $\eta_0(s) = 0$ for $s > 2$. We look for a solution the form

$$\boxed{\begin{aligned}\omega(x,t) =& \omega_\varepsilon^*(x,t) + \varepsilon^{-2}\sum_{j=1}^{k}\eta_j\,\phi_j\left(\frac{x-\xi_j(t)}{\varepsilon},t\right) + \phi^{out}(x,t)\\ \Psi(x,t) =& \Psi_\varepsilon^*(x,t) + \sum_{j=1}^{k}\eta_j\,\psi_j\left(\frac{x-\xi_j(t)}{\varepsilon},t\right) + \psi^{out}(x,t)\end{aligned}}$$

We want that (ω, Ψ) given by

$$\omega = \omega_\varepsilon^* + \varepsilon^{-2} \sum_{j=1}^{k} \eta_j \, \phi_j + \phi^{out}, \quad \Psi = \Psi_\varepsilon^* + \sum_{j=1}^{k} \eta_j \, \psi_j + \psi^{out}$$

satisfies

$$\begin{cases} \omega_t + \nabla_x \Psi^\perp \cdot \nabla_x \omega = 0 & \text{in } \Omega \times [0, T] \\ -\Delta_x \Psi = \omega & \text{in } \Omega \times [0, T] \\ \Psi = 0 & \text{on } \partial\Omega \times [0, T] \end{cases}$$

for which it suffices that the functions

$$(\phi_j(y, t), \psi_j(y, t)), \quad (\phi^{out}(x, t), \psi^{out}(x, t))$$

satisfy the **inner-outer gluing system**

$$
(I) \quad \begin{cases} \varepsilon^2 \partial_t \phi_j + \nabla_y^\perp \Gamma_0 \cdot \nabla(\Delta \psi_j + f'(\Gamma_0)\psi_j) \\ + \nabla_y^\perp \psi^{out} \cdot \nabla U_0 \\ + Q_j + E_j = 0 \quad \text{in } \mathbb{R}^2 \times [0, T] \\ - \Delta_y \psi_j = \phi_j, \end{cases}
$$

for $j = 1, \ldots, k$, coupled with $\phi^{out}(\cdot, 0) = 0$ and

$$
(O) \quad \begin{cases} \partial_t \phi^{out} + \nabla_x^\perp [\Psi_* + \eta_j \psi_j + \psi^{out})] \cdot \nabla_x \phi^{out} \\ + \varepsilon^{-2} \phi_j \partial_t \eta_j + \nabla_x^\perp (\eta_j \psi_j + \psi^{out}) \cdot \nabla_x \eta_j \\ + Q_{out} + E_{out} = 0 \quad \text{in } \Omega \times [0, T] \\ \\ \Delta_x \psi^{out} + \phi^{out} + \psi_j \Delta_x \eta_j + 2 \nabla_x \eta_j \cdot \nabla \psi_j = 0, \\ \psi^{out} = 0, \quad \text{on } \partial\Omega \times [0, T]. \end{cases}
$$

We solve System (I)-(O) by a continuation (degree) argument We drop the index j, and (I) becomes approximately

$$\boxed{\begin{aligned} \varepsilon^2\phi_t - \nabla^\perp\Gamma_0\cdot\nabla(\Delta\psi + f'(\Gamma_0)\psi) + E(y,t) = 0 \\ -\Delta\psi = \phi \quad \text{in } \mathbb{R}^2 \times [0,T] \end{aligned}}$$

with $E = O(\varepsilon^5\rho^{-3})$.

We decompose

$$\phi = \sum_{i=0}^{3} \alpha_i(t) W_i(y) + \tilde{\phi}(y,t), \quad \psi = \sum_{i=0}^{3} \alpha_i(t) Z_i(y) + \tilde{\psi}(y,t)$$

$$W_0 = U_0, \quad W_1 = \partial_{y_1} U_0, \quad W_2 = \partial_{y_2} U_0, \quad W_3 = U_0 \frac{1-|y|^2}{1+|y|^2}$$

$$Z_0 = \Gamma_0, \quad Z_1 = \partial_{y_1} \Gamma_0, \quad Z_2 = \partial_{y_2} \Gamma_0, \quad Z_3 = \frac{1-|y|^2}{1+|y|^2}$$

and we impose that the scalar functions $\alpha_i(t)$ satisfy

$$\int_{B_{\delta/\varepsilon}} y\, \tilde{\phi}(y,t)\, dy \;=\; 0, \quad \int_{\mathbb{R}^2} \tilde{\phi}(y,t)\, dy \;=\; 0 = \int_{\mathbb{R}^2} \tilde{\phi}(y,t) \frac{1-|y|^2}{1+|y|^2}\, dy$$

As a conclusion, we have

$$\begin{cases} \varepsilon^2 \tilde{\phi}_t - \nabla^\perp \Gamma_0 \cdot \nabla(\Delta \tilde{\psi} + f'(\Gamma_0)\tilde{\psi}) + \tilde{E}(y,t) = 0 \\ \tilde{\phi}(y,0) = 0 \\ \tilde{\psi}(y,t) = \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}^2} \tilde{\phi}(z,t) \log \frac{1}{|y-z|} dz \quad \text{in } \mathbb{R}^2 \times [0,T] \end{cases}$$

$$\int_{B_{\delta/\varepsilon}} y\, \tilde{\phi}(y,t)\, dy = 0, \quad \int_{\mathbb{R}^2} \tilde{\phi}(y,t)\, dy = 0 = \int_{\mathbb{R}^2} \tilde{\phi}(y,t) \frac{1-|y|^2}{1+|y|^2}\, dy$$

where

$$\tilde{E}(y,t) = E(y,t) + \varepsilon^2 \dot{\alpha}_i(t) Z_i(y),$$

$$\varepsilon^2 \dot{\alpha}_0(t) = \int_{\mathbb{R}^2} E(y,t)\, dy, \quad \varepsilon^2 \dot{\alpha}_i(t) = \int_{\mathbb{R}^2} E(y,t) y_i\, dy, \quad i = 1, 2, ..$$

and we impose for instance $\alpha_i(0) = 0$.

The key step for the degree argument is to obtain a L^2-weighted a priori estimate of the form

$$\|\tilde{\phi}(\cdot,t)U_0^{-\frac{1}{2}}\|_{L^2}^2 \lesssim \varepsilon^{-2}|\log\varepsilon| \sup_{t\in[0,T]} \|\tilde{E}U_0^{-\frac{1}{2}}\|_{L^2}^2 \qquad \text{(L2)}$$

We observe that for $\tilde{E} = O(\varepsilon^5\rho^{-3})$ we would then get

$$\|\tilde{\phi}U_0^{-\frac{1}{2}}\|_{L^2} = O(\varepsilon^{3^-}).$$

We use in the equation the test function

$$g = \frac{\tilde{\phi}}{U_0} - (-\Delta)_{\mathbb{R}^2}^{-1}\tilde{\phi}$$

$$\varepsilon^2\partial_t \int_{\mathbb{R}^2} \phi g = \int_{\mathbb{R}^2} U_0^{-1}\nabla^\perp\Gamma_0\nabla(U_0^2 g^2) + 2\int_{\mathbb{R}^2} \tilde{E}g$$

The second integral is zero for

$$\int_{\mathbb{R}^2} U_0^{-1}\nabla^\perp\Gamma_0\nabla(U_0^2 g^2) = -\int_{\mathbb{R}^2} \underbrace{\nabla\cdot(U_0^{-1}\nabla^\perp\Gamma_0)}_{=0} \nabla(U_0^2 g^2)$$

Thus

$$\varepsilon^2 \partial_t \int_{\mathbb{R}^2} \tilde{\phi} g = 2 \int_{\mathbb{R}^2} \tilde{E} g \leq C \varepsilon^{-2} \| E U_0^{-\frac{1}{2}} \|_{L_2} \| \| g U_0^{\frac{1}{2}} \|_{L^2}$$

and integrating,

$$\varepsilon^2 \int_{\mathbb{R}^2} \tilde{\phi} g(\cdot, t) \leq \max_{\tau \in (0,T)} C \| E(\cdot, \tau) U_0^{-\frac{1}{2}} \|_{L_2} \| \| g(\cdot, \tau) U_0^{\frac{1}{2}} \|_{L^2}.$$

Under the orthogonality conditions assumed on ϕ we can prove (using stereographic projection and a spectral estimate)

$$\frac{\gamma}{|\log \varepsilon|} \int_{\mathbb{R}^2} \tilde{\phi}^2 U_0^{-1} \leq \int_{\mathbb{R}^2} \phi g.$$

while we always have

$$\int_{\mathbb{R}^2} g^2 U_0 \leq C \int_{\mathbb{R}^2} \tilde{\phi}^2 U_0^{-1}.$$

From these inequalities the desired estimate follows.

Nearly singular solutions for Euler in $\mathbb{R}^3$?

Open question: Solutions with concentrated vorticities near curves (filaments): *the Vortex filament conjecture* (Helmholtz, Da Rios, Levi-Civita 1858-1906-1931).

We consider the Euler equation in $\mathbb{R}^3$ including a large time **scaling factor** $|\log \varepsilon|$ with a small $\varepsilon > 0$,

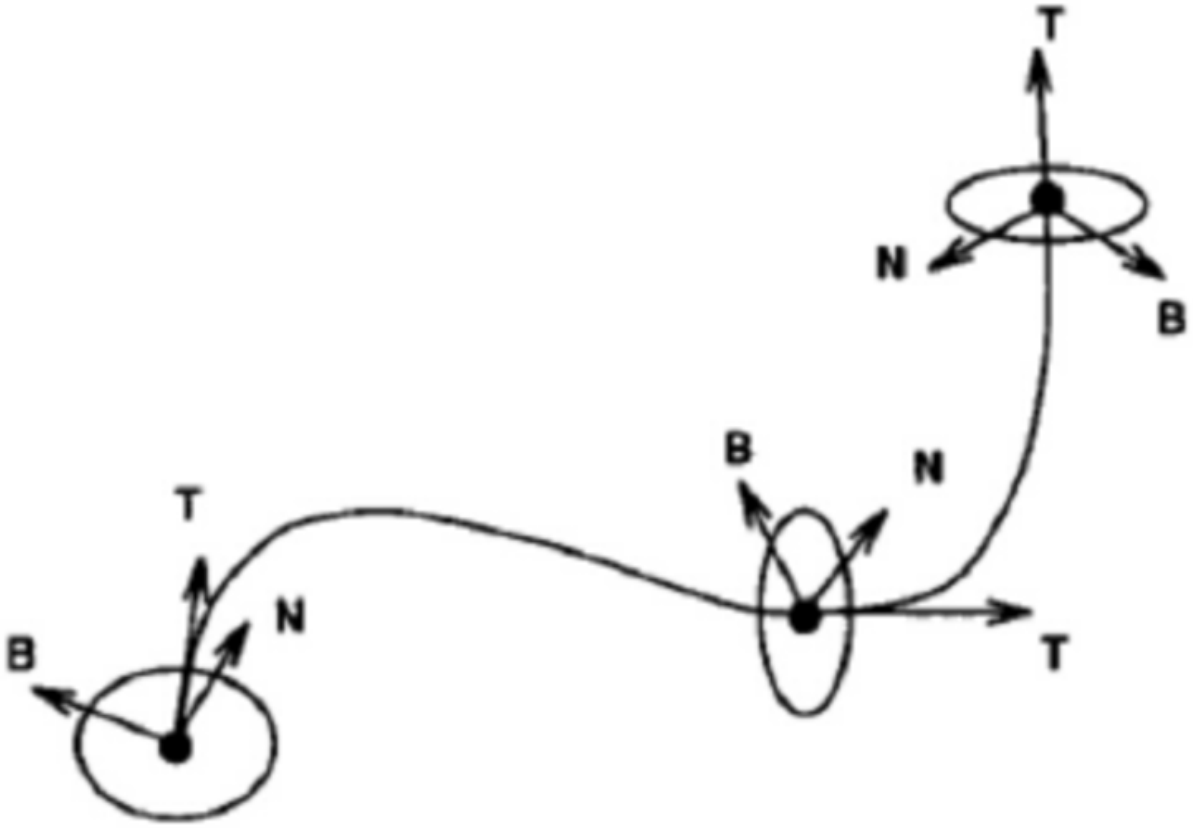
$$\begin{cases} |\log \varepsilon|\, \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla p & \text{in } \mathbb{R}^3 \times (0, T) \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \mathbb{R}^3 \times (0, T) \end{cases} \tag{E}$$

Stream-vorticity formulation ($\omega = \nabla \times \mathbf{u}$ in $\mathbb{R}^3$)

$$\begin{cases} |\log \varepsilon|\, \omega_t + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u = 0 \quad , \\ \mathbf{u} = \nabla \times \psi, \quad -\Delta \psi = \omega. \end{cases} \tag{SV}$$

We want to find solutions with vorticity concentrated on a time evolving curve (filament) $\Gamma(t)$ parametrized by arclength as $\gamma(s, t)$ in $\mathbb{R}^3$.

Let us consider a Frenet frame for $\gamma(\cdot, t)$,
$\mathbf{T} = \gamma_s, \ \mathbf{N} = n_1, \ \mathbf{B} = n_2.$



$$\gamma_{ss} = \kappa\, n_1, \quad n_2 = \gamma_s \times n_1.$$

n_1 normal and n_2 binormal vectors. κ curvature.

In a neighborhood of the curve, we introduce local coordinates

$$X(s, z_1, z_2, t) = \gamma(s, t) + z_i n_i(s, t), \quad |z_i| < \delta$$

and look for an approximate solution of the form

$$\begin{aligned}\Psi_\varepsilon(t, s, z) =& [\psi_{0\varepsilon}(z) + \psi_{01}(s, z)]\, \gamma_s(t, s),\\ \omega_\varepsilon(t, s, z) =& w_\varepsilon(z)\, \gamma_s(t, s)\end{aligned}$$

where

$$\psi_{0\varepsilon}(z) = -2\log(|z|^2 + \varepsilon^2), \quad \psi_{01}(s, z) = -\kappa(s) z_1 \log(|z|^2 + \varepsilon^2)$$

and

$$w_\varepsilon(z) = \frac{8\varepsilon^2}{(|z|^2 + \varepsilon^2)^2} \quad \text{so that} \quad -\Delta_z \psi_{0\varepsilon} = w_\varepsilon$$

We see that in this language we have

$$\boxed{\omega_\varepsilon(x,t) \rightharpoonup 8\pi\delta_{\Gamma(t)}(x)\gamma_s(s,t), \qquad x = \gamma(s,t) + \sum_{i=1}^{2} z_i n_i}$$

where $\delta_{\Gamma(t)}$ denotes uniform Dirac's mass on the curve. We omit the subscript ε.

$$E[\omega,\Psi] := |\log\varepsilon|\,\omega_t + (u\cdot\nabla)\omega - (\omega\cdot\nabla)u,$$
$$u = \nabla\times\Psi, \quad -\Delta\Psi = \omega.$$

The error becomes

$$\varepsilon^4 E[\omega,\Psi] = E_0\gamma_s + e_1 n_1 + e_2 n_2, \quad \rho = \varepsilon^{-1}|z|$$
$$E_0 = O\left(|\log\varepsilon|\frac{\varepsilon}{1+\rho^5}\right), \quad E_1,\, E_2 = O\left(|\log\varepsilon|\frac{\varepsilon^2}{1+\rho^4}\right)$$

We assume now that $\gamma(t,s)$ evolves by the **binormal flow of curves**

$$\gamma_t = 2\,(\gamma_s \times \gamma_{ss}) = 2\,\kappa\, n_2$$

We consider Frenet formulas (τ torsion)

$$\begin{aligned}
\gamma_{ss} &= \kappa n_1 \\
n_{1s} &= -\kappa\gamma_s + \tau n_2 \\
n_{2s} &= -\tau n_2.
\end{aligned}$$

and compute

$$
\begin{aligned}
E[\omega_\varepsilon, \Psi_\varepsilon] = \Big[& |\log\varepsilon| \underbrace{(\gamma_t \cdot n_1)}_{=0} \partial_1 w_\varepsilon \\
& + \Big(\underbrace{|\log\varepsilon| (\gamma_t \cdot n_2)}_{=2|\log\varepsilon|\kappa} \underbrace{-\partial_1\psi_{01} + \kappa\psi_{0\varepsilon} + \kappa\psi_{01}}_{=-2|\log\varepsilon|\kappa + O(1)} \Big) \partial_2 w_\varepsilon \Big] \gamma_s \\
& + \left[-2\kappa\tau |\log\varepsilon| + \underbrace{\tau\kappa\psi_{0\varepsilon} - \tau\partial_1\psi_{01}}_{+2\kappa\tau|\log\varepsilon| + O(1)} \right] w_\varepsilon n_1 \\
& + \left[-2\kappa' |\log\varepsilon| + \underbrace{\partial_s\partial_1\psi_{01} - \kappa'\psi_{0,\varepsilon}}_{=2\kappa'|\log\varepsilon| + O(1)} \right] w_\varepsilon n_2
\end{aligned}
$$

and the error becomes

$$\varepsilon^4 E[\omega, \Psi] = e_0 \gamma_s + E_1 n_1 + E_2 n_2, \quad \rho = |y|$$
$$E_0 = O\left(\frac{\varepsilon}{1+\rho^5}\right), \quad E_1,\, E_2(s,\rho) = O\left(\frac{\varepsilon^2}{1+\rho^4}\right)$$

hence improved in $|\log \varepsilon|^{-1}$ from the initial size. After one adjustment of the curve of this size we use the two-dimensional elliptic improvement in each s-section and improve the whole error to $O\left(\frac{\varepsilon^2}{1+\rho^4}\right)$. This procedure together with the use of the transport equation leads to further improvements in powers of ε.

Jerrard-Seis, 2017: **If** vorticities are concentrated around tubes, tubes they evolve in weak sense by binormal flow.

A result with similar spirit: Wave equation with bi-stable nonlinearity

$$\varepsilon^2 \Box u + u(1-u^2) = 0 \quad \text{in } \mathbb{R}^n \times (0,T) \qquad \text{(W)}$$

$\Box = -\partial_{tt} + \Delta_x$. Let Γ be a compact hypersurface embedded in $\mathbb{R}^n \times (0,T)$ that divides the space into two components O_+ and O_-. We let

$$I(x,t) = \begin{cases} +1 & \text{if } (x,t) \in O_+ \\ -1 & \text{if } (x,t) \in O_- \end{cases}$$

Under what conditions on Γ there is a solution $u_\varepsilon(x,t)$ to (W) with a phase transition from -1 to $+1$?

$$u_\varepsilon(x,t) \to I(x,t) \quad \text{as } \varepsilon \to 0.$$

In the case of the Allen-Cahn equation

$$\varepsilon^2 \Delta u + u(1-u^2) = 0 \quad \text{in } \mathbb{R}^n \qquad (AC)$$

the the answer is: whenever Γ is a minimal hypersurface for which good information is present on its Jacobi operator. The function $w(\zeta) = \tanh\left(\frac{\zeta}{\sqrt{2}}\right)$ solves $w'' + w(1-w^2) = 0$, $w(\pm\infty) = \pm 1$. Then there is a solution u_ε with

$$u_\varepsilon(x) \approx w\left(\varepsilon^{-1} z\right), \quad x = y + z\nu(y), \quad y \in \Gamma, \quad |z| < \delta$$

Basic point: if $|e| = 1$ then $u(x) = w(\varepsilon^{-1}(x-p)\cdot e)$ solves (AC)

Let J denote the $(n+1)\times(n+1)$ diagonal matrix

$$J := \begin{bmatrix} -1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \cdots & \cdots & \ddots & 1 & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

The standard Minkowski inner product

$$\langle a, b\rangle_m = a \cdot Jb, \quad a, b \in \mathbb{R}^{n+1},$$

We assume that Γ is **time-like**, namely $\bar{\nu}\cdot J\bar{\nu} > 0$ for a Euclidean normal vector $\bar{\nu}$ to Γ. We let $\nu := J\bar{\nu}$.
Basic point: if $\langle e, e\rangle_m = 1$ then $u(x,t) = w\left(\varepsilon^{-1}\langle x-p, e\rangle_m\right)$ solves (W)

Let Γ be a time-like Minkowski minimal surface with Minkowski-normal vector ν normalized with $\langle \nu, \nu \rangle_m = 1$ so that

$$(x,t) = y + z\nu(y), \quad y \in \Gamma \quad |z| < \delta$$

define local coordinates near Γ.

Theorem (del Pino, Jerrard, Musso, 2018)

There exists a solution $u_\varepsilon(x,t)$ to the bi-stable wave equation (W) such that $u_\varepsilon(x,t) \to I(x,t)$ as $\varepsilon \to 0$ in the C^k-sense, away from Γ, and

$$u_\varepsilon(x,t) = w(\zeta) + \phi_\varepsilon(y,\zeta), \quad (x,t) = y + z\nu(y), \quad \zeta = \frac{z}{\varepsilon}$$

where for all $j, l \geq 0$ we have

$$|D_y^j D_\zeta^l \phi_\varepsilon(x,t)| \ \leq \ c_{jl}\varepsilon\, e^{-|\zeta|}.$$

(previous results by Jerrard, 2011, 2016).

Thanks for your attention