The evolution of some geometric structures under the Euler and Navier-Stokes equations

Alberto Enciso
ICMAT

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Plan of the talk

Central idea:

Ideas and techniques developed for the analysis of stationary problems in fluid mechanics hold the key to unlock certain evolution problems.

- I Vortex structures in **smooth** fluid flows
  - I.s Knotted vortex structures in stationary 3D Euler [E., Peralta-Salas]
  - I.d Vortex reconnection for 3D Navier–Stokes: creation and destruction of vortex structures [E., Lucà, Peralta-Salas]
- II Fluid-squeezing singularities in **free-boundary** Euler
  - II.s Stationary fluid-squeezing interfaces [Córdoba, E., Grubic]
  - II.d Formation of cusps (and drops!) [Córdoba, E., Fefferman, Grubic]

I.s. Knotted vortex structures in stationary 3D Euler

The stationary Euler equation in 3D

Stationary Euler equations on $\mathbb{R}^3$:

$$(u \cdot \nabla) u = -\nabla p \,, \qquad \operatorname{div} u = 0 \,.$$

What does one mean by "vortex structures"? (Vorticity: $\omega := \operatorname{curl} u$)

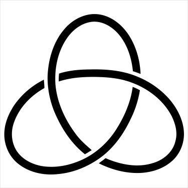
- Vortex lines (VL): periodic integral curves $\frac{dx}{ds} = \omega(x)$.
- Vortex tubes (VT): a surface $\mathcal{T} \subset \mathbb{R}^3$ homeomorphic to a 2-torus consisting of vortex lines (equivalently, an invariant torus of ω).

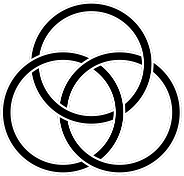
N.B.: Vorticity is smooth, and in principle not supported on these sets!

Setting up the problem

Question

Are there stationary solutions to the Euler equation with knotted/linked vortex lines and tubes?





Motivation

- Lord Kelvin's conjecture (1875): knotted and linked thin vortex tubes can arise in stationary solutions to the Euler equation.
- Arnold's conjecture (1965): vortex lines of any knotted topology and "Hamiltonian-type chaos" can appear in a specific kind of stationary Euler flows called Beltrami fields.
- Lagrangian theory of turbulence and experimental evidence.

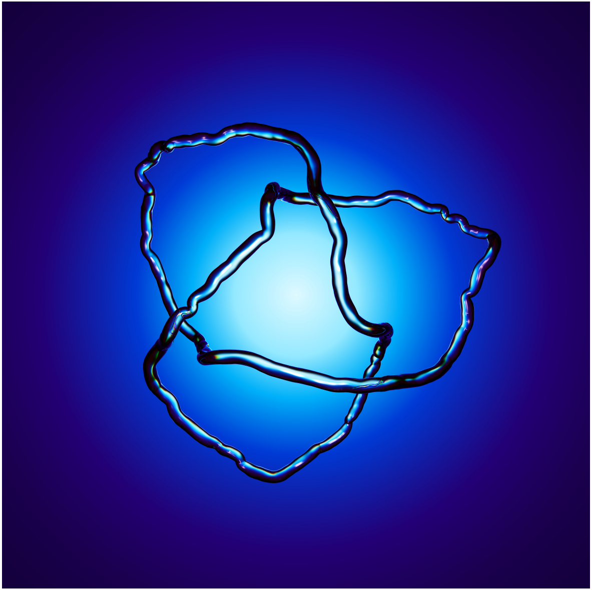


Figure: One of the vortex tubes experimentally constructed by Irvine–Kleckner (2013).

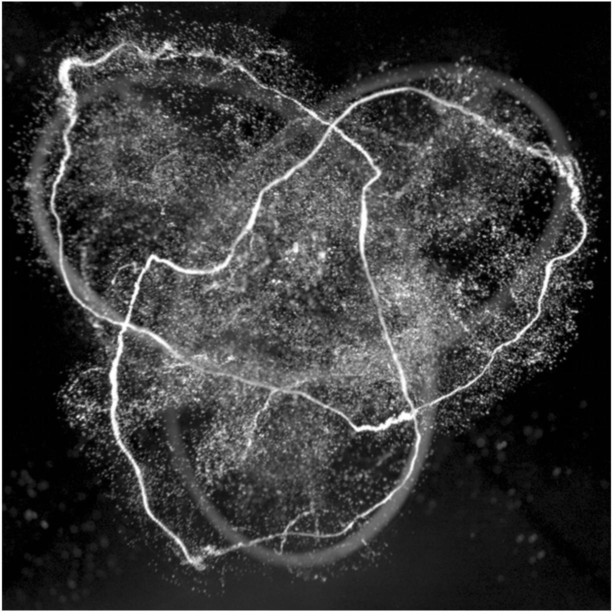


Figure: One of the vortex tubes experimentally constructed by Irvine–Kleckner (2013).

Beltrami fields

We will resort to a particular class of stationary solutions to the Euler equation known as Beltrami fields:

$$\operatorname{curl} u = \lambda u\,, \qquad \lambda \text{ nonzero real constant.}$$

One can check that any such $u(x)$ solves Euler with pressure $p = -\frac{1}{2}|u|^2$, and that Beltrami fields also satisfy the Helmholtz equation

$$\Delta u + \lambda^2 u = 0\,.$$

Why Beltrami?

- The Beltrami equation is linear (although not quite elliptic).
- Arnold's structure theorem (1965).

The realization theorem

$$\mathcal{T}_\epsilon(\gamma) := \left\{x \in \mathbb{R}^3 : \mathrm{dist}(x,\gamma) < \epsilon\right\}$$

Realization theorem (E. & Peralta-Salas, 2015)

Let $\{\gamma_i\}_{i=1}^N$ be (nonintersecting, possibly knotted and linked) closed curves in $\mathbb{R}^3$. For all small enough ϵ and some nonzero λ, the collection of disjoint tubes of width ϵ $\{\mathcal{T}_\epsilon(\gamma_i)\}_{i=1}^N$ can be transformed using a diffeomorphism Φ of $\mathbb{R}^3$, arbitrarily close to the identity in the C^m norm, so that $\{\Phi[\mathcal{T}_\epsilon(\gamma_i)]\}_{i=1}^N$ are vortex tubes and $\{\Phi(\gamma_i)\}_{i=1}^N$ are vortex lines of a Beltrami field u, which satisfies curl $u = \lambda u$ in $\mathbb{R}^3$ for some $\lambda > 0$ with the sharp decay rate $|u(x)| < C/|x|$.

Strategy of proof

The construction of stationary solution to Euler with invariant tori is a problem with both PDE and topological aspects. Topological techniques are too "soft" to capture what happens inside a PDE.

Strategy

1. Construct a local Beltrami field v (that is, in a neighborhood of the tube) with a prescribed set of vortex tubes (or invariant tori).
2. Check that these invariant tori are robust (that is, they are preserved under small perturbations of the field v).
3. Approximate the local solution v by a global Beltrami field u (defined in all $\mathbb{R}^3$ and with the sharp fall off at infinity).

The proof involves fine PDE estimates to apply a KAM theory with small twist (almost KAM-degenerate situation $\implies$ hard PDE analysis), and a global approximation theorem with decay. The result for VL is considerably simpler and we proved it in 2012.

Major open question: A conjecture of Arnold (1965)

Is there chaos inside the tubes??

I.d. Vortex reconnection for 3D Navier–Stokes: creation and destruction of vortex structures

Vortex reconnection in Navier–Stokes

Setting: Take the Navier–Stokes equations on $\mathbb{T}^3$ with a smooth initial datum:

$$\partial_t u + (u \cdot \nabla) u - \nu \Delta u = -\nabla p\,, \qquad \operatorname{div} u = 0\,, \qquad u(x,0) = u_0(x)\,.$$

Here $\nu > 0$ is a possibly small but fixed positive constant. We'll assume that $\operatorname{div} u_0 = 0$ and $\int_{\mathbb{T}^3} u_0\, dx = 0$.

Fact 1: In Euler ($\nu = 0$), vorticity is transported

If the solution to the Euler equations remains smooth, the topology of the (frozen-time) vortex structures of the fluid does not change in time.

Fact 2: there is reconnection (:= change of topology) of vortex structures

In a smooth solution to the Navier–Stokes equations, its vortex structures (say VL and VT) can break down or be created without a loss of regularity.

How do we know that there is vortex reconnection?

- Overwhelming experimental evidence.
- Overwhelming numerical evidence.

Vortex reconnection in Navier–Stokes

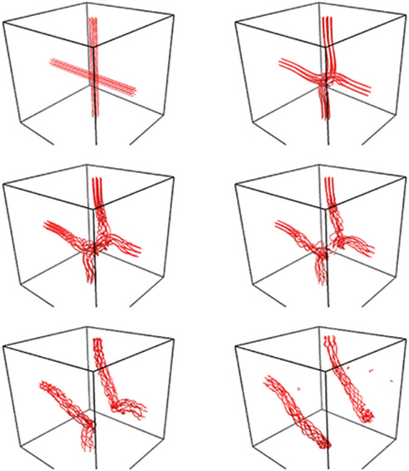


Figure: A possible scenario of vortex reconnection.

Vortex reconnection in Navier–Stokes

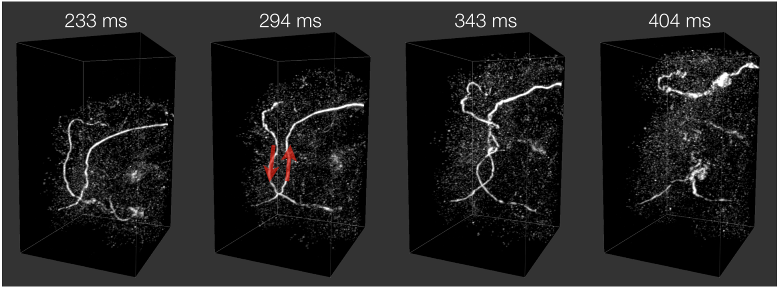


Figure: An actual experiment of Kleckner–Irvine (Nature Phys., 2013).

The enemy:

To come up with a rigorous mechanism of VR, the enemy is that *arbitrary diffeomorphisms are tricky to control:* how can one make sure that an a VL is gone, and not just still in the picture but with a diameter of 10^{-80}?

A cascade of vortex reconnections

Theorem (E., Lucà & Peralta-Salas, 2017)

Given any constants $0 =: T_0 < T_1 < \cdots < T_n$ and $M > 0$, for each odd integer k in $[1, n]$ let us denote by $\mathcal{S}_k$ any finite collection of closed curves and toroidal domains contained in the unit ball of $\mathbb{T}^3$.
Then there is a global C^∞ solution, with a high-frequency initial datum of norm $\|u_0\|_{L^2} = M$, which, for each odd integer $k \in [1, n]$, exhibits at time T_k a set of VT/VL diffeomorphic to $\mathcal{S}_k$ that is not homeomorphic to any of the VL/VT at time $T_{k\pm1}$. This scenario of vortex reconnection is structurally stable.

More visually: *For a suitably chosen smooth but highly oscillatory u_0, at time T_k with k even all the VL/VT wind around a direction of the torus, while at time T_k with k odd the solution has VL/VT of any prescribed knot types and contained in a small ball. Hence the VL/VT must have been created at some time between T_{k-1} and T_k and are destroyed between T_k and T_{k+1}. This phenomenon is physically observable.*

- The construction is a high-frequency argument but it does not involve a transition to higher frequencies. Hence it is smooth, "non-turbulent" VR and will not survive in the inviscid limit $\nu \to 0$.

Heuristics: A dead wrong proof of vortex reconnection (I)

- VR occurs in NS (even for small data) but not in Euler. Hence it should be driven by the linear diffusion. Hence we'll first prove it for the (vector-valued) heat equation on $\mathbb{T}^3$:

$$\partial_t u = \nu \Delta u \,, \qquad u|_{t=0} = u_0 \,.$$

- With $j = 0, 1$, let $\mathcal{W}_j(x)$ be vector fields of frequency N_j (i.e., $\Delta \mathcal{W}_j = -N_j^2 \mathcal{W}_j$) with $\|\mathcal{W}_j\|_{L^2} = 1$. If

$$u_0(x) := M\, \mathcal{W}_0(x) + \delta\, \mathcal{W}_1(x) \,,$$

then

$$u(x,t) = M\, e^{-\nu N_0^2 t} \mathcal{W}_0(x) + \delta e^{-\nu N_1^2 t}\, \mathcal{W}_1(x) \,.$$

- If $\delta \ll M$, at time 0 we have $u(x,0) \approx M\, \mathcal{W}_0(x)$, so the flow at time 0 should look like $\mathcal{W}_0$.
- However, if $N_0 \gg N_1$ (depending on ν, T, M and δ), $u(x,T) \approx c_T \mathcal{W}_1(x)$, so the flow at time T should look like $\mathcal{W}_1$.
- *Therefore, if the flows of $\mathcal{W}_0$ and $\mathcal{W}_1$ are not conjugate, there must have been a change of topology in the time interval $[0, T]$: that's VR.*

Heuristics: A dead wrong proof of vortex reconnection (II)

- The proof, of course, is wrong. We have taken an easier PDE than NS, but we saw that the enemy here is not the strong nonlinearity of the equation but the fact that arbitrary diffeomorphisms are very hard to control. This is not easier for the heat equation.
- What's wrong is that small perturbations of $\mathcal{W}_j$ are *not*, in general, conjugate to $\mathcal{W}_j$, so the whole argument fails. This is not a just a sophisticated technicality, but a very real problem.

But just look on the bright side of life:

The idea of "transition to lower frequencies" will, however, turn out to be useful to construct VR in an easy, general way. But for that we will to fight the real enemy first.

Steps of the proof of vortex reconnection

1. A useful observation is that if $\mathcal{W}$ is a Beltrami field,

$$\operatorname{curl} \mathcal{W} = N\, \mathcal{W}\,,$$

then the solution w to NS with $w|_{t=0} = M\, \mathcal{W}$ is global, explicit and exponentially decaying in time:

$$w(x,t) = M\, e^{-\nu N^2 t}\, \mathcal{W}(x)\,.$$

2. We'll use a suitable stability theorem to control the behavior of solutions to NS whose initial datum is a small perturbation of a Beltrami field:

$$u_0 = M\, \mathcal{W}_0 + \delta_1\, \mathcal{W}_1 + \cdots + \delta_n\, \mathcal{W}_n\,,$$

where $\operatorname{curl} \mathcal{W}_j = N_j\, \mathcal{W}_j$. We'll choose δ_j, N_j such that $u(x, T_j) \approx c_j\, \mathcal{W}_j(x)$ for each $1 \leqslant j \leqslant n$ via "transition to lower frequencies" as before.

3. Key step: by ensuring that the Beltrami field $\mathcal{W}_j$ is "stably non-equivalent" to $\mathcal{W}_{j\pm 1}$ we will then prove that VR occurs. We'll even create and destroy VT/VL of prescribed topology.

The first family of Beltramis

In our argument we will play with two families Beltrami fields of high prescribed frequency, which we will call B and W. Recall that a general Beltrami field of frequency $N \neq 0$ is a nontrivial solution to

$$\operatorname{curl} V = NV \,,$$

which means that $N = \pm |k|$ with $k \in \mathbb{Z}^3$, and that these are stationary solutions to the Euler equations on the torus. The most general Beltrami field of frequency N is a vector-valued trigonometric polynomial of the form

$$W = \sum_{|k|=|N|} \left(b_k \cos(k \cdot x) + \frac{b_k \times k}{N} \sin(k \cdot x) \right) ,$$

where $b_k \in \mathbb{R}^3$ are vectors orthogonal to k: $k \cdot b_k = 0$. R

The first family of Beltramis

With N a large integer, consider the Beltrami of frequency N given by

$$B_N := (\sin Nx_3, \cos Nx_3, 0) \,.$$

The second family of Beltramis

Lemma (The second family of Beltramis)

For every odd $1 \leqslant k \leqslant n$, let $\mathcal{S}_k \subset \mathbb{B}_1$ be any finite collection of closed curves and tubes. Then for any large enough odd integer N_k there are W_k on $\mathbb{T}^3$ such that:

1. $\operatorname{curl} W_k = N_k \, W_k$.
2. W_k has a collection of VL and VT that is diffeomorphic to $\mathcal{S}_k$, *uniformly* structurally stable and contained in the ball $\mathbb{B}_{1/N_k}$ of radius $1/N_k$.
3. $\dfrac{1}{CN_k} < \|W_k\|_{L^2} < \dfrac{C}{\sqrt{N_k}}$.

Lemma (Robust non-equivalence)

There is ϵ_0 independent of N and N_k such that, If W' and B' are any divergence-free vector fields on $\mathbb{T}^3$ with

$$\|W_k - W'\|_{H^7} + \|B_N - B'\|_{H^7} < \epsilon_0 \,,$$

then W' has a collection of VL/VT diffeomorphic to $\mathcal{S}_k$ and B' does not have any contractible VL/VT.

Proof of the Robust Non-equivalence Lemma (I)

1. Starting point: result on $\mathbb{R}^3$ (E., Peralta-Salas, 2015): There is a Beltrami field w on $\mathbb{R}^3$ satisfying

$$\operatorname{curl} w = w$$

   which falls off at infinity as $|w(x)| \leqslant C/|x|$ and has a structurally stable set $\mathcal{S}' \subset \mathbb{B}_R$ of VL/VT diffeomorphic to $\mathcal{S}$.

2. Herglotz theorem: By the sharp decay, there exists $f \in L^2(\mathbb{S}^2)$ such that

$$w(x) = \int_{\mathbb{S}^2} f(\xi)\, e^{ix\cdot\xi}\, d\sigma(\xi)\,.$$

   Necessarily, $f(\xi) = \overline{f(-\xi)}$ and $i\xi \times f(\xi) = f(\xi)$.

3. Approximate by a smooth density: Take a function $g \in C^\infty(\mathbb{S}^2)$ with $\|f - g\|_{L^2(\mathbb{S}^2)} < \epsilon$ (small), so

$$w_1(x) := \int_{\mathbb{S}^2} g(\xi)\, e^{ix\cdot\xi}\, d\sigma(\xi)$$

   satisfies $w \approx w_1$ in C^0. We can assume that $g(\xi) = \overline{g(-\xi)}$ (w_1 real).

Proof of the Robust Non-equivalence Lemma (II)

4. Uniformly distributed rationals (Duke, 2003): The subset of rationals

$$\mathcal{X}_N := \{\xi \in \mathbb{S}^2 \cap \mathbb{Q}^3 : \text{height}(\xi) = N\} \subset \{\xi \in \mathbb{S}^2 \cap \mathbb{Q}^3 : N\xi \in \mathbb{Z}^3\}$$

becomes uniformly distributed as $N \to \infty$ through the odd integers and

$$\frac{N}{C} < |\mathcal{X}_N| < CN^2 \,.$$

5. Discrete sum: Then for a large enough odd integer N, in $C^0(\mathbb{B}_{2R})$

$$w_1(x) \approx w_2(x) := \frac{1}{|\mathcal{X}_N|} \sum_{\xi \in \mathcal{X}_N} g(\xi)\, e^{ix\cdot\xi} \,. \tag{1}$$

Since w, w_1 and w_2 satisfy the Helmholtz equation $\Delta w + w = 0$, in fact

$$\|w - w_2\|_{C^{6,\alpha}(\mathbb{B}_R)} < C\epsilon \,. \tag{2}$$

6. Field on $\mathbb{T}^3$: The field $\widetilde{W}(x) := \frac{1}{|\mathcal{X}_N|} \sum_{\xi \in \mathcal{X}_N} g(\xi)\, e^{ix\cdot(N\xi)}$ is bounded as

$$\frac{1}{CN} < \|\widetilde{W}\|_{L^2} < \frac{C}{\sqrt{N}} \,, \quad \text{and} \quad \left\|\widetilde{W}\left(\frac{x}{N}\right) - w(x)\right\|_{C^{6,\alpha}(\mathbb{B}_R)} < C\epsilon \,.$$

Proof of the Lemma

7. Algebraic fine-tuning of the field: Since $\widetilde{W}$ is not Beltrami but almost, one can now set

$$W := \frac{\operatorname{curl}(\operatorname{curl} \widetilde{W} + N\widetilde{W})}{2N^2}.$$

(The idea of converting solutions of Helmholtz into Beltramis goes back to Chandrasekhar.)

Comments and remarks

- As the non-equivalence follows from a non-contractibility argument, it is key to consider Navier–Stokes on $\mathbb{T}^3$ instead of $\mathbb{R}^3$.
- The proof is essentially linear (i.e., driven by $e^{\nu t\Delta} u_0$), which is not surprising given that it is a viscosity effect, and works with minor modifications for different diffusive terms (e.g., $(-\Delta)^s$ instead of $-\Delta$)..
- If we drop the condition that the phenomenon be structurally stable ("physically observable"), VR can happen instantaneously (and the proof also holds on $\mathbb{R}^3$):

Theorem (Instantaneous VR: E., Lucà & Peralta-Salas, 2017)

Given any $M > 0$, there is a global C^∞ solution of the Navier–Stokes equations u, with initial datum of norm $\|u_0\|_{L^2} = M$ and of zero mean, having a vortex tube at time 0 that breaks instantaneously.

Did we get lucky with the Beltramis on the torus?

A quantum mechanical detour

Since we employ the nontrivial topology of $\mathbb{T}^3$ to prove VR, to some extent one can argue that key in the proof of VR is the fact that we managed to transplant what we know about Beltramis on $\mathbb{R}^3$ (and their knotted VL/VT) to $\mathbb{T}^3$.

Philosophy: Inverse localization

Beltramis: "The behavior of a BF on $\mathbb{R}^3$ in a ball B can be reproduced, modulo a rescaling and up to controllable errors, by a BF on $\mathbb{T}^3$ of arbitrarily large frequency."

Scalar analog: "The behavior of a monochromatic wave on $\mathbb{R}^n$ $(\Delta\varphi + \varphi = 0)$ in a ball B can be reproduced, modulo a rescaling and up to controllable errors, by a Laplace eigenfunction on $\mathbb{T}^n$ of arbitrarily large frequency."

Rule of thumb: Still true if we replace "BFs" or "Laplace eigenfunctions" on the torus by eigenfunctions of any other "superintegrable" problem.

Evidence: BFs on $\mathbb{S}^3$ work even better than on $\mathbb{T}^3$. Of course, trivial if we replace "eigenfunctions" by "quasimodes", without "superintegrability".

A conjecture of Berry

Setting: Eigenvalue problem for a Schrödinger operator on $\mathbb{R}^3$:

$$H_V\psi = E\psi\,,\quad H_V := -\Delta + V(x)\,,\quad x\in\mathbb{R}^3\,,\qquad \psi\in H^2(\mathbb{R}^3)\,.$$

Conjecture (Berry, 2001)

Given any knot γ, there is a potential V on $\mathbb{R}^3$ and an eigenfunction ψ of H_V such that a connected component of its nodal set $\psi^{-1}(0)$ is diffeomorphic to γ.

1. Berry found a complex-valued eigenfunction of the Coulomb potential ($V = -2/|x|$) with nodal set given by a trefoil knot or a Hopf link.
2. *Physical motivation:* the nodal set determines the locus of phase dislocations (singularities of the phase $\mathrm{Im}(\overline{\psi}\nabla\psi)$.)
3. *Why hard?* It is essentially a question about degenerate eigenvalues, since otherwise the eigenfunctions are all real and generically their nodal set is a surface, not a curve. But degenerate eigenvalues are non-generic too! (Uhlenbeck, 1976).

A realization theorem

Let us denote the eigenvalues of the operator H_V by E_N, $N \geqslant 0$.

Theorem (E., Hartley & Peralta-Salas, 2018)

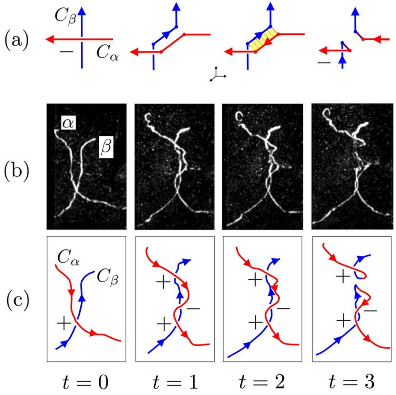
Let the potential be either the harmonic oscillator $V(x) = |x|^2$ or the Coulomb potential $V(x) = -2/|x|$, and let $\mathcal{S}$ be a finite link (collection of closed curves). Then for any large enough N there is an eigenfunction satisfying $H_V\psi = E_N\psi$ such that $\Phi(\mathcal{S})$ is the union of connected components of the nodal set $\psi^{-1}(0)$, where Φ is a diffeomorphism (almost a rescaling).

And now we close the quantum mechanical detour and go back to vortex reconnection . . .

A look ahead: observing vortex reconnections

Major open problem: How does vortex reconnection occur?

In our theorems, we did not say a word about the actual way vortex structures break down. Can one describe how VR happens? There are physically relevant open questions in 2D and 3D.



Remember who the enemy is!

As a first step in our program, we have recently developed a powerful approximation theory for linear parabolic equations:

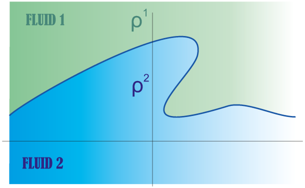
Theorem (E., García-Ferrero, Peralta-Salas, 2018)

1. If $\partial_t v - \Delta v = 0$ on a compact spacetime set $K \subset \mathbb{R}^{n+1}_+$ whose intersection with any time slice is connected, then $\exists\, f \in C_0^\infty(\mathbb{R}^n)$ such that $u := e^{t\Delta} f$ satisfies $\|u - v\|_{C^r(K)} < \delta$.
2. If $\gamma : \mathbb{R} \to \mathbb{R}^n$ is any continuous curve and $\delta : \mathbb{R} \to (0, \infty)$ is continuous, $\exists$ a solution of $\partial_t u - \Delta u = 0$ on $\mathbb{R}^{n+1}$ that has at all times $t \in \mathbb{R}$ a *local hot spot* X_t (:= local maximum of $u(\cdot, t)$) with $|X_t - \gamma(t)| < \delta(t)$.

However, prescribing local extrema (or even level sets) is comparatively easy: even in the case of the (vector-valued) heat equation, prescribing a VR scenario is a very hard open problem!

II. Fluid-squeezing singularities in the free-boundary Euler equation

Setting: the free-boundary Euler equation with two fluids



In each domain $\Omega^j(t)$ $(j = 1, 2)$, the fluid flow is governed by the incompressible, irrotational Euler equations; that is, the respective velocities $u^j(x, t)$ and the corresponding pressures $p^j(x, t)$ satisfy

$$\rho_j(\partial_t u^j + u^j \cdot \nabla)u^j = -\nabla p^j - g\rho_j\, e_2 \quad \text{in} \quad \Omega_j(t), \tag{3a}$$

$$\nabla \cdot u^j = 0 \quad \text{and} \quad \nabla^\perp u^j = 0 \quad \text{in} \quad \Omega_j(t), \tag{3b}$$

$$p^1 - p^2 = -\sigma K \quad \text{on} \quad \partial\Omega(t) \tag{3c}$$

$$(\partial_t z - u^j) \cdot (\partial_\alpha z)^\perp = 0 \quad \text{on} \quad \partial\Omega(t), \tag{3d}$$

where $\partial\Omega(t) = \{z(\alpha, t) = (z_1(\alpha, t), z_2(\alpha, t)) \,:\, \alpha \in \mathbb{R}\}$.

Which kind of evolution problem do we have?

Without getting into technicalities, let us recall that if one linearizes the problem about the trivial equilibrium $z = 0$, $u = 0$ and writes the perturbed interface as a graph

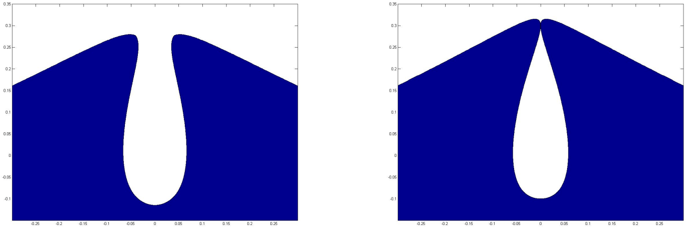
$$z(\alpha, t) = (\alpha, h(\alpha, t))\,,$$

the linearized problem can be rewritten as the nonlocal dispersive equation

$$\partial_{tt} h + g|\partial_\alpha| h - \sigma \partial_{\alpha\alpha} |\partial_\alpha| h = 0\,,$$

where $|\partial_\alpha| := (-\partial_{\alpha\alpha})^{1/2} = H\partial_\alpha$, with H the Hilbert transform. (Here $g > 0$ and $\sigma \geqslant 0$.)

Formation of splash singularities

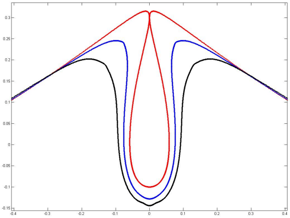


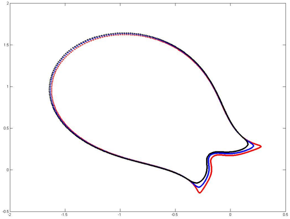
Theorem (Castro, Córdoba, Fefferman, Gancedo, Gómez-Serrano, 2011)

The free-boundary Euler equation with one fluid develops singularities. More precisely, there are smooth initial data for which the interface self-intersects in finite time T^*. The velocity and the interface remain smooth up to T^*.

How can one prove that splash singularities do form?

- The water wave equations are invariant under time reversal, so one can take the splash configuration as the initial datum and (try to) solve the equations backwards in time. This is a local existence problem.
- If one can choose the "splash-type" initial datum so that the normal component of the velocity on the interface at $t = 0$ is positive, the splash opens up and we win.
- One can prove local existence from a splash-type initial datum, morally, because the region where one needs to estimate things is Ω^2, where the fluid lives, which is "inner regular". An efficient way of seeing this is by means of the conformal map $P(w) := \sqrt{\tan \frac{w}{2}}$, which opens up the splash domain.





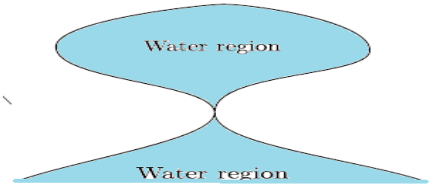
Squeezing a fluid

The above argument fails if there is a fluid in the region Ω^1, which is not "inner-regular". In particular, conformal maps cannot save the day. And it is not hard to see why, physically: this situation would correspond to squeezing an incompressible fluid.

Question:

How can one squeeze an incompressible fluid?

(It should be feasible: that is the way a water drop forms.)



Squeezing singularities involve a loss of regularity

Theorem (No splash singularities with two fluids; Fefferman, Ionescu & Lie 2016)

If $\sup_{t\in[0,T]}(\|z(\cdot,t)\|_{C^4} + \|u^j(\cdot,t)\|_{C^3}) < \infty$ and the interface does not self-intersect at $t = 0$, it does not self-intersect at $t = T$ either.

Proof: Consider the vorticity curl $u(x,t) =: \omega(\alpha,t)\delta(x - z(\alpha,t))$:

1. The regularity assumption implies that $\sup_{t\in[0,T]} \|\omega(\cdot,t)\|_{L^\infty} < C$ because ω satisfies a variant of the Burgers equation.
2. As the interface moves with the fluid, the boundedness of ω implies that the inverse of the chord-arc distance, $F(t) := 1/\mathrm{CA}(t)$ can be controlled as

$$\left|\frac{dF}{dt}\right| \leqslant C|F|\log(|F|+2)\,,$$

remaining therefore bounded at $t = T$.

II. Fluid-squeezing singularities in the free-boundary Euler equation

II.s Stationary fluid-squeezing interfaces

Back to the stationary case

In each domain, the fluid flow is governed by the stationary, incompressible, irrotational Euler equations; that is, the respective velocities v^j and the corresponding pressures p^j satisfy

$$\rho_j(v^j \cdot \nabla)v^j = -\nabla p^j - g\rho_j\, e_2 \quad \text{in} \quad \Omega_j, \tag{4a}$$

$$\nabla \cdot v^j = 0 \quad \text{and} \quad \nabla^\perp v^j = 0 \quad \text{in} \quad \Omega_j, \tag{4b}$$

$$v^j \cdot (\partial_\alpha z)^\perp = 0 \quad \text{on} \quad \partial\Omega, \tag{4c}$$

$$p^1 - p^2 = -\sigma K \quad \text{on} \quad \partial\Omega. \tag{4d}$$

We assume that the interface satisfies periodicity conditions

$$z_1(\alpha + 2\pi) = z_1(\alpha) + 2\pi, \quad z_2(\alpha + 2\pi) = z_2(\alpha)$$

and is symmetric with respect to the x_2-axis:

$$z_1(-\alpha) = -z_1(\alpha), \quad z_2(-\alpha) = z_2(\alpha).$$

Rewriting the stationary equations

The unknowns are $z(\alpha)$ (the parametrized interface) and $\omega(\alpha)$ (the vorticity on the interface). To fix the parametrization, we use the hodograph transform with respect to the lower fluid. Then, as long as there is no self-intersections, finding a stationary solution of the two-fluid system amounts to finding 2π-periodic functions $\omega(\alpha)$ and $z(\alpha) - (\alpha, 0)$ satisfying

$$\begin{aligned} 2|\partial_\alpha z|^2 M(z) + \epsilon\,\omega(\omega - 2) &= 2, \\ 2\,\mathrm{BR}(z,\omega)\cdot\partial_\alpha z + \omega &= 2, \\ \mathrm{BR}(z,\omega)\cdot\partial_\alpha^\perp z &= 0, \end{aligned}$$

where $\mathrm{BR}(z,\omega)$ and $M(z)$ are given by

$$\mathrm{BR}(z,\omega) := \frac{1}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}} \frac{(z(\alpha,t)-z(\beta,t))^\perp}{|z(\alpha,t)-z(\beta,t)|^2}\omega(\alpha,t)d\alpha$$

$$M(z) := -\epsilon q K(z) - 2gz_2 + 1,$$

$q := \frac{\sigma}{\rho_2}$ and $\epsilon := \frac{2\rho_1}{\rho_2-\rho_1}$ are essentially the surface tension and the density of the second fluid, and $K(z)$ is the curvature of the interface.

The capillary case: $\epsilon = g = 0$

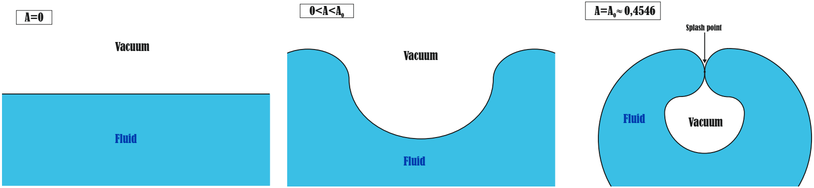
- The system decouples and one recovers pure capillary waves. Family of exact solutions depending on the parameter q (Crapper, 1958):

$$z_A(\alpha) = \alpha + \frac{4i}{1 + Ae^{-i\alpha}} - 4i\,, \quad q = \frac{1 + A^2}{1 - A^2}.$$

To find ω, one inverts the equation

$$2BR(z_A, \omega) \cdot \partial_\alpha z_A + \omega = 2$$

- *For $A = A_0 \simeq 0.45$, the curve $z_A(\alpha)$ exhibits a splash, while for $A < A_0$ the curve does not self-intersect and for A slightly larger than A_0 the curve intersects at exactly two points, and the intersection is transverse.*



Stationary splash and almost-splash solutions

One can readily believe that, with some work, one can perturb the Crapper waves using the implicit function theorem to get solutions with small positive values of ϵ and g:

Theorem (Córdoba, E. & Grubic, 2016)

For any small enough g there is some σ for which there exists a stationary solution such that the interface $\partial\Omega$ has a splash point.

Likewise, for any small enough $\rho_1 > 0$ and g there is some σ for which there exists a stationary solution such that the interface $\partial\Omega$ has is an almost-splash.

In a way, the key step is to control the inverse of the operator

$$A\omega := \omega + 2BR(z,\omega) \cdot \partial_\alpha z \,.$$

This was first done by Baker, Meiron and Orszag (1982) for H^3 curves without self-intersections, which one can apply after opening up the splash domain if necessary using a conformal transformation.

Stationary fluid-squeezing singularities

Theorem (Córdoba, E. & Grubic, preprint)

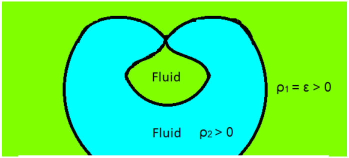
For any small enough $\rho_1 > 0$, g there is some σ for which there exists a stationary solution such that the interface $\partial\Omega$ has a splash point. The interface $\partial\Omega$ is of class $C^{2,\alpha}$, $0 < \alpha < \frac{1}{2}$.

The proof hinges on new weighted estimates for the inverse of the operator A when the curve z has a cusp, which owe much to the work of Maz'ya. E.g., with suitably chose parameters, we show that

$$A : W^1_{p,\beta} \to X_{\beta,\mu}$$

is invertible for a certain $X_{\beta,\mu} \subset W^1_{p,\beta}$, where

$$f \in W^1_{p,\beta} \iff \| |x|^\beta f \|_{L^p} + \| |x|^{\beta+1} \partial_x f \|_{L^p} < \infty.$$



II. Fluid-squeezing singularities in the free-boundary Euler equation

II.d Formation of cusps (and drops!)

Formation of fluid-squeezing singularities

"Theorem" (Córdoba, E., Fefferman & Grubic, in progress)

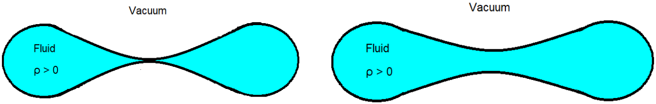


Figure: At times $t = 0$ and $t = T$.

Sketch of proof:

- Goal: to prove local existence starting from a fluid-squeezing (cusped) splash.
- Key point: to obtain a priori estimates for a carefully chosen energy functional within suitably weighted Sobolev spaces.
- Show that one can choose an initial data that opens the splash.

Bonus!

Convexity of Whitham's highest cusped wave

Non-smooth traveling waves for Whitham's equation

Whitham's equation is the very weakly dispersive 1D shallow water wave model

$$\partial_t v + \partial_x(Lv + v^2) = 0\,, \tag{5}$$

where L is the Fourier multiplier $\widehat{Lf}(\xi) := (\tanh \xi/\xi)^{1/2}\, \widehat{f}(\xi)$. This reproduces the full dispersion relation of water waves.

The traveling wave ansatz $v(x,t) := \varphi(x - \mu t)$, where μ is the velocity, leads to the equation

$$L\varphi - \mu\varphi + \varphi^2 = 0\,.$$

There is a whole family of solutions $\varphi \in C^\infty(\mathbb{T})$, $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ (or any other period), but the highest in the family shouldn't be smooth:

Non-smooth traveling waves

Conjecture (Whitham, 1967): There is a sharp crest: $\varphi \in C^{1/2}(\mathbb{T})$ but not better.

Theorem (Ehrnström, Wahlén 2015): Indeed …
Conjecture (Ehrnström, Wahlén 2015): …and φ can be taken *convex*.

Why convex?

In the case of the Euler equations, it is a landmark result that the corresponding non-smooth traveling (or Stokes) waves are convex (Plotnikov, Toland 2004). But the proof hinges on complex analysis, using the connection between free-boundary Euler and harmonic functions on the plane, while here one must exclusively deal with real-variable methods for nonlocal equations:

Theorem (E., Gómez-Serrano, Vergara 2018)

There is a *convex* highest cusped wave $\varphi \in C^{1/2}(\mathbb{T})$.

Key ingredient: (different) weighted estimates and a very carefully constructed approximate solution to the equation.

Thank you for your attention!