Magnetic interpolation inequalities in dimensions 2 and 3

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Outline

- General link between interpolation inequalities and spectral estimates.

- Magnetic interpolation inequalities and spectral consequences.

- Magnetic rings and the case of Aharonov-Bohm magnetic fields.

General link between interpolation inequalities and spectral estimates

For a second order elliptic operator L, consider the minimization problem

$$(1) \qquad \mu_D(\alpha) := \min_{u \in H^1(D)} \frac{(Lu, u) + \alpha\,(u, u)}{\left(\int_D |u|^p\right)^{2/p}}\,, \quad 2 < p < 2^* := \frac{2d}{d-2}\,,$$

equivalent to the inequality $(Lu, u) + \alpha\,(u, u) \geq \mu_D(\alpha)\,\left(\int_D |u|^p\right)^{2/p}$

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Consider the function $\mu \mapsto \alpha_D(\mu)$, the inverse of $\alpha \mapsto \mu_D(\alpha)$.

Then, for $\mu = \|V_+\|_{L^q(D)}, \quad \frac{1}{q} + \frac{2}{p} = 1$,

$$\frac{((L-V)u, u)}{(u, u)} \geq \frac{(Lu, u) - (V_+ u, u)}{(u, u)} \geq \frac{(Lu, u) - \mu\,\left(\int_D |u|^p\right)^{2/p}}{(u, u)} \geq -\alpha_D(\mu)$$

$$(2) \qquad \lambda_1(L - V) \geq -\alpha_D\left(\|V_+\|_{L^q(D)}\right) \quad \text{(Keller-Lieb-Thirring inequality)}$$

$$V_{opt} = \mu\,|u_{min}|^{p-2}$$

Three interpolation inequalities and their dual forms: Magnetic Laplacian and spectral gap

In dimensions $d = 2$ and $d = 3$: the *magnetic Laplacian* is

$$-\Delta_{\mathbf{A}}\,\psi = -\Delta\,\psi - 2\,i\,\mathbf{A}\cdot\nabla\psi + |\mathbf{A}|^2\psi - i\,(\operatorname{div}\mathbf{A})\,\psi$$

where the magnetic potential (resp. field) is $\mathbf{A}$ (resp. $\mathbf{B} = \operatorname{curl}\mathbf{A}$) and

$$\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d) := \left\{\psi \in \mathrm{L}^2(\mathbb{R}^d)\,:\,\nabla_{\mathbf{A}}\psi \in \mathrm{L}^2(\mathbb{R}^d)\right\}\,,\quad \nabla_{\mathbf{A}} := \nabla + i\,\mathbf{A}$$

Spectral gap inequality:

$$\|\nabla_{\mathbf{A}}\psi\|_2^2 \ge \Lambda[\mathbf{B}]\,\|\psi\|_2^2 \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d) \tag{1}$$

− Λ depends only on $\mathbf{B} = \operatorname{curl}\mathbf{A}$
− Assumption: *equality holds for some* $\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d)$
− If $\mathbf{B}$ is a constant magnetic field, $\Lambda[\mathbf{B}] = |\mathbf{B}|$

Magnetic interpolation inequalities

THEOREM.- Under some technical conditions on $\mathbf{A}$, there exist constants $\mu_{\mathbf{B}}(\alpha)$, $\nu_{\mathbf{B}}(\beta)$ and $\xi_{\mathbf{B}}(\gamma)$ such that:

$$\|\nabla_{\mathbf{A}}\psi\|_2^2 + \alpha\,\|\psi\|_2^2 \ge \mu_{\mathbf{B}}(\alpha)\,\|\psi\|_p^2\,, \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d) \tag{2}$$

for any $\alpha \in (-\Lambda[\mathbf{B}], +\infty)$ and any $p \in (2, 2^*)$,

$$\|\nabla_{\mathbf{A}}\psi\|_2^2 + \beta\,\|\psi\|_p^2 \ge \nu_{\mathbf{B}}(\beta)\,\|\psi\|_2^2\,, \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d) \tag{3}$$

for any $\beta \in (0, +\infty)$ and any $p \in (1, 2)$

$$\|\nabla_{\mathbf{A}}\psi\|_2^2 \ge \gamma \int_{\mathbb{R}^d} |\psi|^2 \log\left(\frac{|\psi|^2}{\|\psi\|_2^2}\right) dx + \xi_{\mathbf{B}}(\gamma)\,\|\psi\|_2^2\,, \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d) \tag{4}$$

(limit case corresponding to $p = 2$) for any $\gamma \in (0, +\infty)$

Technical assumptions (E. - Lions, 1989)

$\mathbf{A} \in \mathrm{L}^{\alpha}_{\mathrm{loc}}(\mathbb{R}^d)$, $\alpha > 2$ if $d = 2$ or $\alpha = 3$ if $d = 3$ and

$$\lim_{\sigma \to +\infty} \sigma^{d-2} \int_{\mathbb{R}^d} |\mathbf{A}(x)|^2 \, e^{-\sigma \, |x|} \, dx = 0 \quad \text{if} \quad p \in (2, 2^*)$$

$$\lim_{\sigma \to +\infty} \frac{\sigma^{\frac{d}{2}-1}}{\log \sigma} \int_{\mathbb{R}^d} |\mathbf{A}(x)|^2 \, e^{-\sigma \, |x|^2} \, dx = 0 \quad \text{if} \quad p = 2$$

$$\lim_{\sigma \to +\infty} \sigma^{d-2} \int_{|x| < 1/\sigma} |\mathbf{A}(x)|^2 \, dx \quad \text{if} \quad p \in (1, 2)$$

Magnetic Keller-Lieb-Thirring inequalities

$\lambda_{\mathbf{A},V}$ is the principal eigenvalue of $-\Delta_{\mathbf{A}} - V$,
$\alpha_{\mathbf{B}} : (0,+\infty) \to (-\Lambda,+\infty)$ is the inverse function of $\alpha \mapsto \mu_{\mathbf{B}}(\alpha)$

Corollary

(i) *For any* $q = p/(p-2) \in (d/2,+\infty)$ *and any potential* $0 \geq V \in \mathrm{L}^q(\mathbb{R}^d)$

$$\lambda_{\mathbf{A},V} \geq -\,\alpha_{\mathbf{B}}(\|V\|_q)$$

$$\lim_{\mu\to 0_+} \alpha_{\mathbf{B}}(\mu) = \Lambda \quad \textit{and} \quad \lim_{\mu\to+\infty} \alpha_{\mathbf{B}}(\mu)\,\mu^{\frac{2\,(q+1)}{d-2-2\,q}} = -\,\mathsf{C}_p^{\frac{2\,(q+1)}{d-2-2\,q}}$$

(ii) *For any* $q = p/(2-p) \in (1,+\infty)$ *and any* $0 < W^{-1} \in \mathrm{L}^q(\mathbb{R}^d)$

$$\lambda_{\mathbf{A},W} \geq \nu_{\mathbf{B}}\left(\|W^{-1}\|_q^{-1}\right)$$

(iii) *For any* $\gamma > 0$ *and any* $W \geq 0$ *s.t.* $e^{-W/\gamma} \in \mathrm{L}^1(\mathbb{R}^d)$

$$\lambda_{\mathbf{A},W} \geq \xi_{\mathbf{B}}(\gamma) - \gamma\,\log\left(\int_{\mathbb{R}^d} e^{-W/\gamma}\,dx\right)$$

Limit behaviour

Theorem

$p \in (2, 2^*)$: $\mu_{\mathbf{B}}$ is monotone increasing on $(-\Lambda[\mathbf{B}], +\infty)$, concave and

$$\lim_{\alpha \to (-\Lambda)_+} \mu_{\mathbf{B}}(\alpha) = 0 \quad \textit{and} \quad \lim_{\alpha \to +\infty} \mu_{\mathbf{B}}(\alpha)\, \alpha^{\frac{d-2}{2} - \frac{d}{p}} = \mathsf{C}_p$$

$p \in (1, 2)$: $\nu_{\mathbf{B}}$ is monotone increasing on $(0, +\infty)$, concave and

$$\lim_{\beta \to 0_+} \nu_{\mathbf{B}}(\beta) = \Lambda \quad \textit{and} \quad \lim_{\beta \to +\infty} \nu_{\mathbf{B}}(\beta)\, \beta^{-\frac{2\,p}{2\,p + d\,(2-p)}} = \mathsf{C}_p$$

$\xi_{\mathbf{B}}$ is continuous on $(0, +\infty)$, concave, $\xi_{\mathbf{B}}(0) = \Lambda[\mathbf{B}]$ and

$$\xi_{\mathbf{B}}(\gamma) = \frac{d}{2}\, \gamma\, \log\left(\frac{\pi\, e^2}{\gamma}\right)(1 + o(1)) \quad \textit{as} \quad \gamma \to +\infty$$

$$\mathsf{C}_p := \begin{cases} \min_{u \in \mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}} \frac{\|\nabla u\|_2^2 + \|u\|_2^2}{\|u\|_p^2} & \textit{if} \quad p \in (2, 2^*) \\ \min_{u \in \mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}} \frac{\|\nabla u\|_2^2 + \|u\|_p^2}{\|u\|_2^2} & \textit{if} \quad p \in (1, 2) \end{cases}$$

Constant magnetic fields: equality is achieved
Nonconstant magnetic fields: only partial answers are known

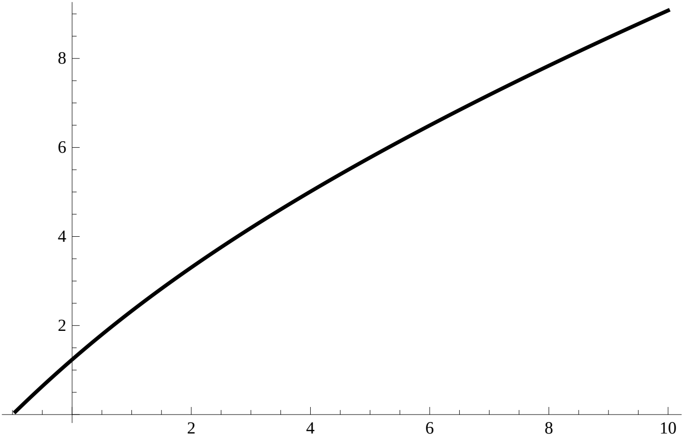

Figure: Case $d = 2$, $p = 3$, $B = 1$: plot of $\alpha \mapsto (2\pi)^{\frac{2}{p}-1}\,\mu_{\mathbf{B}}(\alpha)$

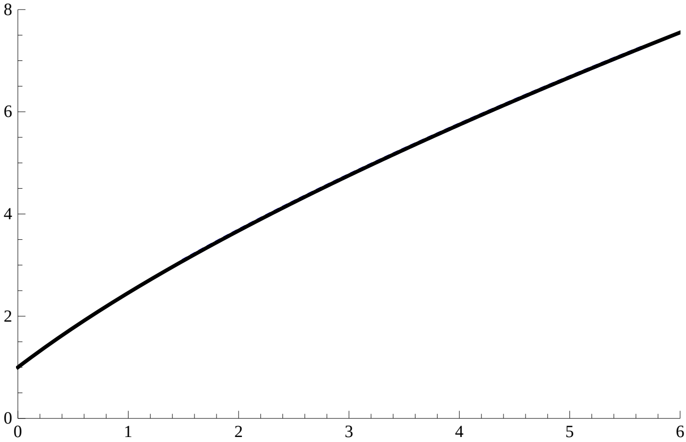

Figure: Case $d = 2$, $p = 1.4$, $B = 1$: plot of $\beta \mapsto \nu_{\mathbf{B}}(\beta)$. The horizontal axis is measured in units of $(2\,\pi)^{1-\frac{2}{p}}\,\beta$

Preliminaries: constants in the inequalities

Assume that $p > 2$ and let C_p denote the best constant in

$$\|\nabla u\|_2^2 + \|u\|_2^2 \geq \mathsf{C}_p \, \|u\|_p^2 \quad \forall \, u \in \mathrm{H}^1(\mathbb{R}^d)$$

Consider also the inequality

$$\|\nabla u\|_2^{d\,(1-\frac{2}{p})} \, \|u\|_2^{2-d\,(1-\frac{2}{p})} \geq \mathsf{S}_p \, \|u\|_p^2 \quad \forall \, u \in \mathrm{H}^1(\mathbb{R}^d)$$

By scaling and optimization over the scaling parameter, we find that

$$\mathsf{S}_p = \tfrac{1}{2\,p} \, \big(2\,p - d\,(p-2)\big)^{1-d\,\frac{p-2}{2\,p}} \, \big(d\,(p-2)\big)^{\frac{d\,(p-2)}{2\,p}} \, \mathsf{C}_p$$

General lower estimates

Proposition

Let $d = 2$ or 3. For any $p \in (2, +\infty)$, any $\alpha > -\Lambda = -\Lambda[\mathbf{B}] < 0$

$$\mu_{\mathbf{B}}(\alpha) \geq \mu_{\rm interp}(\alpha) := \begin{cases} \mathsf{S}_p\,(\alpha+\Lambda)\,\Lambda^{-d\,\frac{p-2}{2\,p}} & \text{if } \alpha \in \left[-\Lambda, \frac{\Lambda\,(2\,p-d\,(p-2))}{d\,(p-2)}\right] \\ \mathsf{C}_p\,\alpha^{1-d\,\frac{p-2}{2\,p}} & \text{if } \alpha \geq \frac{\Lambda\,(2\,p-d\,(p-2))}{d\,(p-2)} \end{cases}$$

Let $t \in [0,1]$. From the diamagnetic inequality $\|\nabla|\psi|\|_2 \leq \|\nabla_{\mathbf{A}}\psi\|_2$ and from the inequality with $\lambda = \frac{\alpha+\Lambda\,t}{1-t}$, we deduce that

$$\begin{aligned} \|\nabla_{\mathbf{A}}\psi\|_2^2 + \alpha\,\|\psi\|_2^2 \geq t\,\left(\|\nabla_{\mathbf{A}}\psi\|_2^2 - \Lambda\,\|\psi\|_2^2\right) \\ + (1-t)\,\left(\|\nabla|\psi|\|_2 + \frac{\alpha+\Lambda\,t}{1-t}\,\|\psi\|_2^2\right) \\ \geq \mathsf{C}_p\,(1-t)^{\frac{d\,(p-2)}{2\,p}}\,(\alpha+t\,\Lambda)^{1-d\,\frac{p-2}{2\,p}}\,\|\psi\|_p^2 \end{aligned}$$

and optimize on $t \in [\max\{0, -\alpha/\Lambda\}, 1]$

Lower estimates for $d = 2$ and a constant magnetic field

Assume that $\mathbf{B} = (0, B)$ is constant, $d = 2$ and choose

$$\mathbf{A}_1 = \tfrac{B}{2} x_2 \,, \quad \mathbf{A}_2 = -\tfrac{B}{2} x_1 \quad \forall\, x = (x_1, x_2) \in \mathbb{R}^2$$

Proposition

(Loss, Thaller, 1997) Consider a constant magnetic field with field strength B in two dimensions. For every $c \in [0, 1]$, we have

$$\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} \psi|^2 \, dx \ge \left(1 - c^2\right) \int_{\mathbb{R}^2} |\nabla \psi|^2 \, dx + c\, B \int_{\mathbb{R}^2} \psi^2 \, dx$$

and equality holds with $\psi = u\, e^{iS}$ and $u > 0$ if and only if

$$\left(-\partial_2 u^2,\, \partial_1 u^2\right) = \frac{2\, u^2}{c} \left(\mathbf{A} + \nabla S\right)$$

Idea of the proof

$$\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}}\psi|^2\,dx = \int_{\mathbb{R}^2} |\nabla u|^2\,dx + \int_{\mathbb{R}^2} |\mathbf{A}+\nabla S|^2\,u^2\,dx$$
$$= \left(1-c^2\right)\int_{\mathbb{R}^2} |\nabla u|^2\,dx + \underbrace{\int_{\mathbb{R}^2}\left(c^2\,|\nabla u|^2 + |\mathbf{A}+\nabla S|^2\,u^2\right)dx}_{\geq \int_{\mathbb{R}^2} 2\,c\,|\nabla u|\,|\mathbf{A}+\nabla S|\,u\,dx}$$

with equality only if $c\,|\nabla u| = |\mathbf{A}+\nabla S|\,u$

$$2\,|\nabla u|\,|\mathbf{A}+\nabla S|\,u = |\nabla u^2|\,|\mathbf{A}+\nabla S| \geq \left(\nabla u^2\right)^\perp\cdot(\mathbf{A}+\nabla S)$$

where $\left(\nabla u^2\right)^\perp := \left(-\partial_2 u^2,\,\partial_1 u^2\right)$
Equality case: $\left(-\partial_2 u^2,\,\partial_1 u^2\right) = \gamma\,(\mathbf{A}+\nabla S)$ for $\gamma = 2\,u^2/c$
Integration by parts yields

$$\int_{\mathbb{R}^2}\left(c^2\,|\nabla u|^2 + |\mathbf{A}+\nabla S|^2\,u^2\right)dx \geq B\,c\int_{\mathbb{R}^2} u^2\,dx$$

Lower estimate ($d = 2$, constant magnetic field): a result

Proposition

Consider a constant magnetic field with field strength B in two dimensions. Given any $p \in (2, +\infty)$, and any $\alpha > -B$, we have

$$\mu_{\mathbf{B}}(\alpha) \geq \mathsf{C}_p \left(1 - c^2\right)^{1-\frac{2}{p}} (\alpha + c\,B)^{\frac{2}{p}} =: \mu_{\rm LT}(\alpha)$$

with

$$c = c(p, \eta) = \frac{\sqrt{\eta^2 + p - 1} - \eta}{p - 1} = \frac{1}{\eta + \sqrt{\eta^2 + p - 1}} \in (0, 1)$$

and $\eta = \alpha\,(p-2)/(2\,B)$

Upper estimate (1): $d = 2$, constant magnetic field

For every integer $k \in \mathbb{N}$ we introduce the special symmetry class

$$\psi(x) = \left(\tfrac{x_2 + i\,x_1}{|x|}\right)^k \, v(|x|) \quad \forall\, x = (x_1, x_2) \in \mathbb{R}^2 \qquad (\mathcal{C}_k)$$

(E. - Lions, 1989): if $\psi \in \mathcal{C}_k$, then

$$\frac{1}{2\,\pi} \int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} \psi|^2 \, dx = \int_0^{+\infty} |v'|^2 \, r \, dr + \int_0^{+\infty} \left(\tfrac{k}{r} - \tfrac{B\,r}{2}\right)^2 \, |v|^2 \, r \, dr$$

and optimality is achieved in $\mathcal{C}_k$

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For $k = 0$, test function $v_\sigma(r) = e^{-\,r^2/(2\,\sigma)}$: an optimization on $\sigma > 0$ provides an explicit expression of $\mu_{\rm Gauss}(\alpha)$ such that

Proposition

If $p > 2$, then

$$\mu_{\mathbf{B}}(\alpha) \le \mu_{\rm Gauss}(\alpha) \quad \forall\, \alpha > -\Lambda[\mathbf{B}]$$

This estimate is not optimal because v_σ does not solve the Euler-Lagrange equations

Upper estimate (2): $d = 2$, constant magnetic field

A numerical upper estimate. The Euler-Lagrange equation in $\mathcal{C}_0$ is

$$-v'' - \frac{v'}{r} + \left(\tfrac{B^2}{4}\, r^2 + \alpha\right) v = \mu_{\rm EL}(\alpha) \left(\int_0^{+\infty} |v|^p \, r \, dr\right)^{\frac{2}{p}-1} |v|^{p-2}\, v$$

We can restrict the problem to positive solutions such that

$$\mu_{\rm EL}(\alpha) = \left(\int_0^{+\infty} |v|^p \, r \, dr\right)^{1-\frac{2}{p}}$$

and then we have to solve the reduced problem

$$-v'' - \frac{v'}{r} + \left(\tfrac{B^2}{4}\, r^2 + \alpha\right) v = |v|^{p-2}\, v$$

Then, we get a numerical upper estimate from this test function:

$$\mu_{\mathbf{B}}(\alpha) \le \mu_{\rm EL}(\alpha)$$

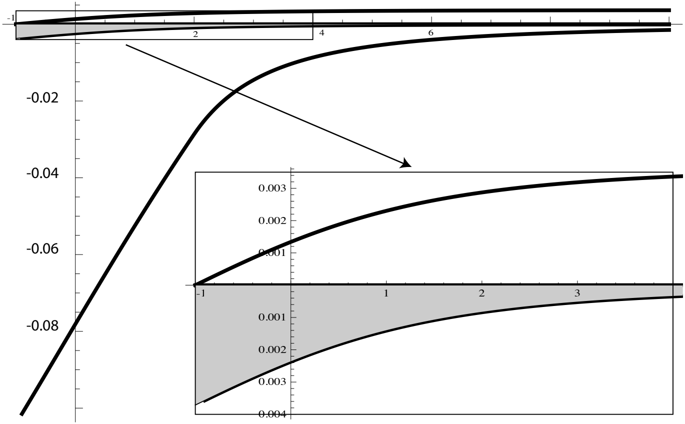

Figure: Case $d = 2$, $p = 3$, $B = 1$: comparison of the upper estimates $\alpha \mapsto \mu_{\rm Gauss}(\alpha)$ and $\alpha \mapsto \mu_{\rm EL}(\alpha)$ with the lower estimates $\alpha \mapsto \mu_{\rm interp}(\alpha)$ and $\alpha \mapsto \mu_{\rm LT}(\alpha)$

Plots represent the curves $\log_{10}(\mu_{\rm Gauss}/\mu_{\rm EL})$, $\log_{10}(\mu_{\rm LT}/\mu_{\rm EL})$ and $\log_{10}(\mu_{\rm interp}/\mu_{\rm EL})$ so that $\alpha \mapsto \mu_{\rm EL}(\alpha)$ corresponds to a straight line at level 0. The exact value associated with $\mu_{\mathbf{B}}$ lies in the grey area

Numerical stability of radial optimal functions

Let us denote by ψ_0 an optimal function in $(\mathcal{C}_0)$ such that

$$-\psi_0'' - \frac{\psi_0'}{r} + \left(\tfrac{B^2}{4}\,r^2 + \alpha\right)\psi_0 = |\psi_0|^{p-2}\,\psi_0$$

and consider the test function

$$\psi_\varepsilon = \psi_0 + \varepsilon\, e^{i\,\theta}\, v$$

where $v = v(r)$ and $e^{i\,\theta} = (x_1 + i\,x_2)/r$
As $\varepsilon \to 0_+$, the leading order term is

$$2\,\pi\left[\int_{\mathbb{R}^2} |v'|^2\,dx + \int_{\mathbb{R}^2}\left(\left(\tfrac{1}{r} - \tfrac{B\,r}{2}\right)^2 + \alpha\right)|v|^2\,dx - \tfrac{p}{2}\int_0^{+\infty}|\psi_0|^{p-2}\,v^2\,r\,dr\right]\varepsilon^2$$

and we have to solve the eigenvalue problem

$$-v'' - \frac{v'}{r} + \left(\left(\tfrac{1}{r} - \tfrac{B\,r}{2}\right)^2 + \alpha\right)v - \tfrac{p}{2}\,|\psi_0|^{p-2}\,v = \mu\,v$$

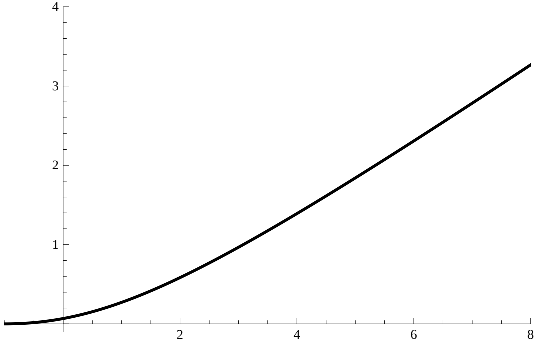

Figure: Case $p = 3$ and $B = 1$: plot of the eigenvalue μ as a function of α A careful investigation shows that μ is always positive, including in the limiting case as $\alpha \to (-B)_+$, thus proving the numerical stability of the optimal function in $\mathcal{C}_0$ with respect to perturbations in $\mathcal{C}_1$

An open question of symmetry

– (Bonheure, Nys, Van Schaftingen, 2016) for a fixed $\alpha > 0$ and for $\mathbf{B}$ small enough, the optimal functions are radially symmetric functions, *i.e.*, belong to $\mathcal{C}_0$

This regime is equivalent to the regime as $\alpha \to +\infty$ for a given $\mathbf{B}$, at least if the magnetic field is constant

– Numerically our upper and lower bounds are (in dimension $d = 2$, for a constant magnetic field) numerically extremely close

– The optimal function in $\mathcal{C}_0$ with respect to perturbations in $\mathcal{C}_1$

CONJECTURE: *Prove that the optimality case is achieved among radial function if $d = 2$ and $\mathbf{B}$ is a constant magnetic field, large or small.*

Aharonov-Bohm magnetic fields

On the two-dimensional Euclidean space $\mathbb{R}^2$, let us introduce the Aharonov-Bohm magnetic potential

$$\mathbf{a}(\mathbf{x}) = a\left(\frac{-x_2}{|\mathbf{x}|^2}, \frac{x_1}{|\mathbf{x}|^2}\right), \quad a \in \mathbb{R}; \ \ \mathbf{b} = \operatorname{curl} \mathbf{a}.$$

Magnetic Schrödinger energy (polar coordinates), $\Psi = w\, e^{i\phi}$, $w = |\Psi|$:

$$\int_{\mathbb{R}^2} |(i\nabla + \mathbf{a})\,\Psi|^2\, d\mathbf{x} = \int_0^{+\infty}\!\!\int_{-\pi}^{\pi} \Big(|\nabla w|^2 + |w|^2\big(|\partial_r \phi|^2 + \frac{1}{r^2}\,|\,\partial_\theta \phi - i\,a\,\phi|^2\big)\Big)\, r\, \mathrm{d}\theta\, \mathrm{d}r$$

We are in particular interested in proving new Hardy type inequalities in dimension 2, like:

$$\int_{\mathbb{R}^2} |(i\nabla + \mathbf{a})\,\Psi|^2\, d\mathbf{x} \ge \tau \int_{\mathbb{R}^2} \frac{V(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2}\, |\Psi|^2\, \mathrm{d}\mathbf{x} \quad \forall\, V \in \mathrm{L}^q(\mathbb{S}^1), \quad q \in (1, +\infty)$$

$$\Longrightarrow \tau = \tau\left(a, \|V\|_{\mathrm{L}^q(\mathbb{S}^1)}\right) \ ?$$

Hardy inequalities

(Hoffmann-Ostenhof, Laptev, 2015) proved Hardy's inequality

$$\int_{\mathbb{R}^d} |\nabla \Psi|^2 \, \mathrm{d}\mathbf{x} \geq \tau \int_{\mathbb{R}^d} \frac{V(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2} \, |\Psi|^2 \, \mathrm{d}\mathbf{x}$$

where the constant τ depends on the value of $\|V\|_{\mathrm{L}^q(\mathbb{S}^{d-1})}$ and $d \geq 3$

In dimension $d = 2$, the usual Hardy inequality does not hold true, but:

$$\mathbf{a}(\mathbf{x}) = a \left(\frac{-x_2}{|\mathbf{x}|^2}, \frac{x_1}{|\mathbf{x}|^2} \right), \quad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \,, \quad a \in \mathbb{R}$$

and recall the inequality (Laptev, Weidl, 1999)

$$\int_{\mathbb{R}^2} |(i\,\nabla + \mathbf{a})\,\Psi|^2 \, \mathrm{d}\mathbf{x} \geq \min_{k \in \mathbb{Z}} (a - k)^2 \int_{\mathbb{R}^2} \frac{|\Psi|^2}{|\mathbf{x}|^2} \, \mathrm{d}\mathbf{x}$$

Preliminaries: a simple interpolation on the circle

On $(-\pi,\pi] \approx \mathbb{S}^1 \ni s$, let us consider the uniform probability measure $\mathrm{d}\sigma = \mathrm{d}s/(2\pi)$
The inequality

$$\|\psi'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \ge \mu_{0,p}(\alpha)\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)} \tag{5}$$

holds for some concave $(0,+\infty) \in \alpha \mapsto \mu_{0,p}(\alpha)$

Lemma

− *If* $p > 2$ *and* $0 < \alpha \le 1/(p-2)$, *then* $\mu_{0,p}(\alpha) = \alpha$

− *If* $p = -2$ *and* $\alpha = 1/(p-2) = -1/4$, *then* $\mu_{0,p}(-1/4) = -1/4$

− *In both cases, equality achieved only by constant functions*

Case $p = -2$ (Exner, Harrell, Loss, 1998)):

$$\|\psi'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \frac{1}{4}\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)} \ge \frac{1}{4}\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$$

Case $p > 2$: by the Bakry-Emery method for instance.

Carré du champ method

Let $\mathcal{F}[u] := \|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \frac{1}{p-2}\left(\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)} - \|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}\right)$ and consider a positive solution of the parabolic equation

$$\frac{\partial u}{\partial t} = u'' + (p-1)\,\frac{|u'|^2}{u}$$

If $p = -2$ (new application of the *carré du champ* method)

$$\frac{d}{dt}\mathcal{F}[u(t,\cdot)] = -\underbrace{\int_{-\pi}^{\pi}\left(|u''|^2 - |u'|^2\right)\mathrm{d}\sigma}_{\le 0 \text{ (Poincaré)}} - \int_{-\pi}^{\pi}\frac{|u'|^4}{u^2}\,\mathrm{d}\sigma$$

If $p > 1$, $p \neq 2$, the method is well known (Bakry, Emery, 85)

Magnetic rings: magnetic interpolation in 1d, with Aharonov-Bohm potential

We want to characterize the *optimal constant* in the inequality

$$\|\psi' - i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \ge \mu_{a,p}(\alpha)\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}$$

written for any $\psi \in X_a$

$$\mu_{a,p}(\alpha) := \inf_{\psi \in X_a \setminus \{0\}} \frac{\int_{-\pi}^{\pi} \left(|\psi' - i\,a\,\psi|^2 + \alpha\,|\psi|^2\right) \mathrm{d}\sigma}{\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}$$

$p = -2$ (Exner, Harrell, Loss, 1998)
$p = +\infty$ (Galunov, Olienik, 1995) (Ilyin, Laptev, Loss, Zelik, 2016)

Using a Fourier series $\psi(s) = \sum_{k \in \mathbb{Z}} \psi_k\, e^{iks}$, we obtain

$$\|\psi' - i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} = \sum_{k \in \mathbb{Z}} (k-a)^2\,|\psi_k|^2 \ge \min_{k \in \mathbb{Z}} (k-a)^2\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$$

In the particular (important) case $a \in (0, 1/2)$,

$\psi \mapsto \|\psi' - i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$ is coercive for any $\alpha > -a^2$

Magnetic interpolation on the circle: Magnetic flux, a reduction

Assume that $a : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic function such that its restriction to $(-\pi, \pi] \approx \mathbb{S}^1$ is in $\mathrm{L}^1(\mathbb{S}^1)$ and define the space

$$X_a := \left\{ \psi \in C_{\mathrm{per}}(\mathbb{R}) \, : \, \psi' - i\, a\, \psi \in \mathrm{L}^2(\mathbb{S}^1) \right\}$$

– According to (Ilyin, Laptev, Loss, Zelik, 2016)

$$\psi(s) \mapsto e^{i \int_{-\pi}^{s} (a(s) - \bar{a}) \, \mathrm{d}\sigma} \, \psi(s)$$

where $\bar{a} := \int_{-\pi}^{\pi} a(s) \, \mathrm{d}\sigma$ is the *magnetic flux*, reduces the problem to

a is a constant function

– For any $k \in \mathbb{Z}$, ψ by $s \mapsto e^{iks} \, \psi(s)$ shows that $\mu_{a,p}(\alpha) = \mu_{k+a,p}(\alpha)$

$$a \in [0, 1]$$

– $\mu_{a,p}(\alpha) = \mu_{1-a,p}(\alpha)$

because $|\psi' - i\, a\, \psi|^2 = |\chi' + i\,(1-a)\,\chi|^2 = \left|\overline{\psi}' - i\, a\, \overline{\psi}\right|^2$ if
$\chi(s) = e^{-is}\, \overline{\psi(s)}$ and so, enough to consider $a \in [0, 1/2]$.

An interpolation result for the magnetic ring

Theorem

For any $p > 2$, $a \in \mathbb{R}$, and $\alpha > -a^2$, $\mu_{a,p}(\alpha)$ is achieved and

- (i) *if $a \in (0, 1/2)$ and $a^2\,(p+2) + \alpha\,(p-2) \le 1$, then $\mu_{a,p}(\alpha) = a^2 + \alpha$ and equality in (5) is achieved only by the constant functions*
- (ii) *if $a \in (0, 1/2)$ and $a^2\,(p+2) + \alpha\,(p-2) > 1$, then $\mu_{a,p}(\alpha) < a^2 + \alpha$ and equality in (5) is not achieved by the constant functions*

The function $a \mapsto \mu_{a,p}(\alpha)$ is monotone increasing and concave.

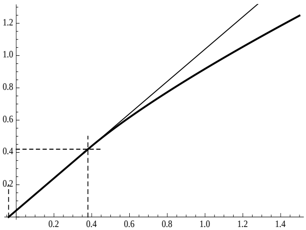

Figure: $\alpha \mapsto \mu_{a,p}(\alpha)$ with $p = 4$ and $a = 0.2$

Reformulation of the interpolation problem I

Any minimizer $\psi \in X_a$ of $\mu_{a,p}(\alpha)$ satisfies the Euler-Lagrange equation

$$(H_a + \alpha)\,\psi = |\psi|^{p-2}\,\psi$$

up to a multiplication by a constant and $v(s) = \psi(s)\,e^{ias}$ satisfies the condition

$$v(s + 2\pi) = e^{2i\pi a}\,v(s) \quad \forall\, s \in \mathbb{R} \tag{6}$$

Hence

$$\mu_{a,p}(\alpha) = \min_{v \in Y_a \setminus \{0\}} \mathsf{Q}_{p,\alpha}[v]$$

where $Y_a := \{v \in C(\mathbb{R}) \,:\, v' \in \mathrm{L}^2(\mathbb{S}^1)\,,\ (6) \text{ holds}\}$ and

$$\mathsf{Q}_{p,\alpha}[v] := \frac{\|v'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|v\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}{\|v\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}$$

Reformulation of the interpolation problem II

With $v = u\, e^{i\phi}$ the boundary condition becomes

$$u(\pi) = u(-\pi)\,, \quad \phi(\pi) = 2\pi\,(a+k) + \phi(-\pi) \tag{7}$$

for some $k \in \mathbb{Z}$, and $\|v'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} = \|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \|u\,\phi'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$
Hence

$$\mu_{a,p}(\alpha) = \min_{(u,\phi)\in Z_a\setminus\{0\}} \frac{\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \|u\,\phi'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}$$

where $Z_a := \{(u,\phi) \in C(\mathbb{R})^2\,:\, u',\, u\,\phi' \in \mathrm{L}^2(\mathbb{S}^1)\,,\ \text{(7) holds}\}$

Reformulation of the interpolation problem III

We use the Euler-Lagrange equations

$$-u'' + |\phi'|^2\,u + \alpha\,u = |u|^{p-2}\,u \quad \text{and} \quad (\phi'\,u^2)' = 0$$

Integrating the second equation, and *assuming that u never vanishes*, we find a constant L such that $\phi' = L/u^2$. Taking (7) into account, we deduce from

$$L\int_{-\pi}^{\pi}\frac{\mathrm{d}s}{u^2} = \int_{-\pi}^{\pi}\phi'\,\mathrm{d}s = 2\pi\,(a+k)$$

that

$$\|u\,\phi'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} = L^2\int_{-\pi}^{\pi}\frac{\mathrm{d}\sigma}{u^2} = \frac{(a+k)^2}{\|u^{-1}\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}$$

Hence

$$\phi(s) - \phi(0) = \frac{a+k}{\|u^{-1}\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}\int_{-\pi}^{s}\frac{\mathrm{d}s}{u^2}$$

Let us define

$$\mathcal{Q}_{a,p,\alpha}[u] := \frac{\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + a^2 \, \|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha \, \|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}$$

Lemma

For any $a \in (0, 1/2)$, $p > 2$, $\alpha > -a^2$,

$$\mu_{a,p}(\alpha) = \min_{u \in \mathrm{H}^1(\mathbb{S}^1)\setminus\{0\}} \mathcal{Q}_{a,p,\alpha}[u]$$

is achieved by a function $u > 0$

Proofs

− The existence proof is done on the original formulation of the problem using the diamagnetic inequality
− $\psi(s)\, e^{ias} = v_1(s) + i\, v_2(s)$, solves

$$-\, v_j'' + \alpha\, v_j = (v_1^2 + v_2^2)^{\frac{p}{2}-1}\, v_j\,, \quad j = 1\,,\ 2$$

and the Wronskian $w = (v_1\, v_2' -\ v_1'\, v_2)$ is constant so that $\psi(s) = 0$ is incompatible with the twisted boundary condition
− if $a^2\,(p+2) + \alpha\,(p-2) \le 1$, then $\mu_{a,p}(\alpha) = a^2 + \alpha$ because

$$\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + a^2\,\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)} = (1-4\,a^2)\,\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + 4\,a^2\left(\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \tfrac{1}{4}\,\|u^{-1}\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\right)$$

if $a^2\,(p+2) + \alpha\,(p-2) > 1$, the test function $u_\varepsilon := 1 + \varepsilon\, w_1$

$$\mathcal{Q}_{a,p,\alpha}[u_\varepsilon] = a^2 + \alpha + \big(1 - a^2\,(p+2) -\ \alpha\,(p-2)\big)\,\varepsilon^2 + o(\varepsilon^2)$$

proves the linear instability of the constants and $\mu_{a,p}(\alpha) < a^2 + \alpha$

$$\mathcal{Q}_{a,p,\alpha}[u] := \frac{\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + a^2\,\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}\,,$$

$$\mu_{a,p}(\alpha) = \min_{u\in\mathrm{H}^1(\mathbb{S}^1)\setminus\{0\}} \mathcal{Q}_{a,p,\alpha}[u]$$

$$\mathsf{Q}_{p,\alpha}[u] = \mathcal{Q}_{a=0,p,\alpha}[u]\,,\quad \nu_p(\alpha) := \inf_{v\in\mathrm{H}^1_0(\mathbb{S}^1)\setminus\{0\}} \mathsf{Q}_{p,\alpha}[v]$$

Proposition

$\forall\, p > 2$, $\alpha > -a^2$, we have $\mu_{a,p}(\alpha) < \mu_{1/2,p}(\alpha) \le \nu_p(\alpha) = \mu_{1/2,p}(\alpha)$

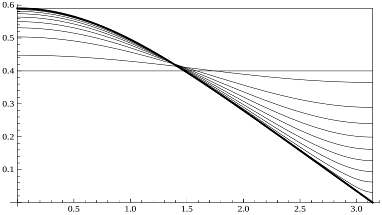

Figure: $p = 4$, $\alpha = 0$, $a = 0.40, 0.41, \ldots 0.49$; $u'' + u^{p-1} = 0$

Consequence: a Keller-Lieb-Thirring inequality

Magnetic Schrödinger operator $H_a - V = -\left(\frac{d}{ds} - i\,a\right)^2 - V$

$-$ The function $\alpha \mapsto \mu_{a,p}(\alpha)$ is monotone increasing, concave, and therefore has an inverse, denoted by $\alpha_{a,p} : \mathbb{R}^+ \to \mathbb{R}^+$, which is monotone increasing, and convex

Corollary

Let $p > 2$, $a \in (0, 1/2)$, $q = p/(p-2)$ and assume that V is a non-negative function in $\mathrm{L}^q(\mathbb{S}^1)$. Then

$$\lambda_1(H_a - V) \geq -\,\alpha_{a,p}\left(\|V\|_{\mathrm{L}^q(\mathbb{S}^1)}\right)$$

and $\alpha_{a,p}(\mu) = \mu - a^2$ iff $4\,a^2 + \mu\,(p-2) \leq 1$ (optimal V is constant)

Equality is achieved

A new Hardy inequality

Proposition

Let $p > 2$, $a \in (0, 1/2)$, $q = p/(p-2)$ and assume that V is a non-negative function in $\mathrm{L}^q(\mathbb{S}^1)$. Then the following inequality holds

$$\int_{\mathbb{R}^2} |(i\,\nabla + \mathbf{a})\,\Psi|^2\,d\mathbf{x} \geq \tau \int_{\mathbb{R}^2} \frac{V(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2}\,|\Psi|^2\,\mathrm{d}\mathbf{x} \quad \forall\, V \in \mathrm{L}^q(\mathbb{S}^1)\,, \quad q \in (1, +\infty)\,,$$

with $\tau > 0$ given by

$$\alpha_{a,p}\left(\tau\,\|V\|_{\mathrm{L}^q(\mathbb{S}^1)}\right) = 0\,.$$

Moreover, $\tau = a^2/\|V\|_{\mathrm{L}^q(\mathbb{S}^1)}$ if $4\,a^2 + \|V\|_{\mathrm{L}^q(\mathbb{S}^1)}\,(p-2) \leq 1$

Corollary

For any $a \in (0, 1/2)$, by taking V constant, small enough in order that $4\,a^2 + \|V\|_{\mathrm{L}^q(\mathbb{S}^1)}\,(p-2) \leq 1$, we recover the inequality

$$\int_{\mathbb{R}^2} |(i\,\nabla + \mathbf{a})\,\Psi|^2\,\mathrm{d}\mathbf{x} \geq a^2 \int_{\mathbb{R}^2} \frac{|\Psi|^2}{|\mathbf{x}|^2}\,\mathrm{d}\mathbf{x}$$

Proof

Let $\tau \geq 0$, $\mathbf{x} = (r, \theta) \in \mathbb{R}^2$ be polar coordinates in $\mathbb{R}^2$

$$\int_{\mathbb{R}^2} \left(|(i\,\nabla + \mathbf{a})\,\Psi|^2 - \tau\, V\,(x/|x|)\,\frac{|\Psi|^2}{|x|^2} \right) \mathrm{d}\mathbf{x}$$

$$= \int_0^\infty \int_{\mathbb{S}^1} \Big(\underbrace{r\,|\partial_r \Psi|^2}_{\geq 0} + \frac{1}{r}\,\left(|\partial_\theta \Psi - i\,a\,\Psi|^2 - \tau\, V\,|\Psi|^2 \right) \Big) \mathrm{d}\theta\,\mathrm{d}r$$

$$\geq \lambda_1\,(H_a - \tau\, V) \int_0^\infty \int_{\mathbb{S}^1} \frac{1}{r}\,|\Psi|^2\,\mathrm{d}\theta\,\mathrm{d}r$$

$$\geq -\,\alpha_{a,p}(\tau\,\|V\|_{\mathrm{L}^q(\mathbb{S}^1)}) \int_0^\infty \int_{\mathbb{S}^1} \frac{1}{r}\,|\Psi|^2\,\mathrm{d}\theta\, dr$$

– If $\tau = 0$, then $\alpha_{a,p}(\tau\,\|V\|_{\mathrm{L}^q(\mathbb{S}^1)}) = \alpha_{a,p}(0) = -\,a^2$
– $\alpha_{a,p}(\tau\,\|V\|_{\mathrm{L}^q(\mathbb{S}^1)}) > 0$ for τ large

$\Longrightarrow\ \exists\,!\ \tau > 0$ such that $\alpha_{a,p}(\tau\,\|V\|_{\mathrm{L}^q(\mathbb{S}^1)}) = 0$

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Thank you for your attention!