Head and Tail speeds of Mean curvature flow with forcing

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Let $g : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function which is periodic with respect to translations by $\mathbb{Z}^d$. We consider the evolution of sets $(\Omega_t)_{t>0}$ in $\mathbb{R}^d$ by the flow

$$V = -\varepsilon\kappa + g(\frac{x}{\varepsilon}) \text{ on } \partial\Omega_t.$$

Here $\kappa = \kappa_{x,t}$ denotes the mean curvature of $\partial\Omega_t$ at given boundary point, positive if convex with respect to Ω_t.

Note that zooming in by the coordinate change $(x,t) \to (x/\varepsilon, t/\varepsilon)$ we have

$$V = -\kappa + g(x) \text{ on } \partial\tilde{\Omega}_t.$$

We are interested in the behavior of Ω_t or $\tilde{\Omega}_t$ as $\varepsilon \to 0$.

Literature in periodic media

◦ Lions-Souganidis (2005): Homogenization result for positive g, when $|Dg|/g^2$ is sufficiently small.

◦ Cesaroni-Novaga (2013): In laminar setting $g(x) = g(x', x_n)$ for periodic graph solutions $\{x_n = u(x', t)\}$, existence of generalized traveling wave solution with the maximal speed.

◦ Caffarelli-Monneau (2014): Homogenization for positive and Lipschitz g in two space dimensions. A counterexample in three dimensions, based on the existence of an unbounded stationary solution.

◦ Cardaliaguet-Lions-Souganidis (2009): examples of pinning and failure of homogenization with sign changing g.

We employ the level set PDE

$$u_t = F(D^2 u, Du, x) = |Du|[\mathrm{tr}\left\{D^2 u^\varepsilon \left(I - \widehat{Du^\varepsilon} \otimes \widehat{Du^\varepsilon}\right)\right\} + g(x)], \quad (1)$$

which is a singular and nonlinear parabolic equation.

Here the initial data is given as a uniformly continuous function $u_0(x)$, and we will study the zero level set of $u(x,t)$.

We are interested in the description of maximal and minimal asymptotic speed ("head" and "tail" speed) for u_ε in general setting, where we may not have plane-like solutions and homogenization fails.

Theorem

There exist two upper- and lower-semicontinuous functions $\bar{s}, \underline{s} : \mathcal{S}^d \to \mathbb{R}$, $\underline{s} \leq \bar{s}$ *with the following properties:*

- $\bar{s}$ *and* $\underline{s}$ *are continuous.*
- *Let* u^ε *solve* (1) *with initial data* $u_0(x) = (x - x_0) \cdot \nu$, *then*

$$\limsup_{\varepsilon \to 0} u^\varepsilon(x,t) = \bar{s}(\nu)t - (x - x_0) \cdot \nu$$

and

$$\liminf_{\varepsilon \to 0} u^\varepsilon(x,t) = \underline{s}(\nu)t - (x - x_0) \cdot \nu.$$

In unit scale ($\varepsilon = 1$) this means

$$\overline{s}(\nu) = \lim_{t\to\infty} \frac{\sup\left\{x\cdot\nu : u^1(x,t)=0\right\}}{t},$$

$$\underline{s}(\nu) = \lim_{t\to\infty} \frac{\inf\left\{x\cdot\nu : u^1(x,t)=0\right\}}{t}.$$

For any u^1 with “flat” initial data.

Theorem (General Initial data)

If u^ε solve (1) with uniformly continuous initial data u_0, the following holds in the viscosity sense:

$$\limsup^* u^\varepsilon \text{ solves } \quad u_t \leq \bar{s}(-\widehat{Du})|Du|$$

and

$$\liminf_* u^\varepsilon \text{ solves } \quad u_t \geq \underline{s}(-\widehat{Du})|Du|.$$

In particular, if $\bar{s} = \underline{s}$ then u^ε uniformly converges to the unique viscosity solution of $u_t = \bar{s}(-\widehat{Du})|Du|$ with initial data u_0.

Laminar case: travelling waves

Here we assume that $g(x) = g(x', x_n)$ and set $\varepsilon = 1$.

In this case, starting with initial set given as graph $\Gamma_0 = \{x_n = U_0(x')\}$, one can show that the interface Γ_t stays as a graph $\{x_n = U(x,t)\}$ and U solves the PDE

$$\frac{U_t}{\sqrt{(1+|DU|^2)}} = \nabla \cdot (\frac{DU}{\sqrt{1+|DU|^2}}) + g(x') \text{ in } \mathbb{R}^{n-1} \times (0,\infty).$$

It follows that U is locally $C^{1,\alpha}$ in space, uniformly in time, if Γ_t moves with finite speed, which is the case for us if U_0 is linear.

Based on this regularity we obtain generalized traveling wave solutions.

Traveling waves in Laminar media

Consider the direction $\nu \in \mathcal{S}^d$ with $\nu \cdot e_n \neq 0$.

Theorem

Suppose $s_1 := \bar{s}(\nu) > s_2 := \underline{s}(\nu)$. Then there are disjoint, open, non-empty sets E_1, E_2 in $\mathbb{R}^{n-1}$ and functions $U_1 : E_1 \to (-\infty, 0]$, $U_2 : E_2 \to [0, \infty)$ such that the following is true:

(a) *E_i and $U_i(x') + x' \cdot \nu$ are $\mathbb{Z}^{n-1}$-periodic.*

(b) *The sets $\partial E_i \times (-\infty, \infty)$, $i = 1, 2$, are stationary solutions.*

(c) *$U_1 \to -\infty$ as $x \to \partial E_1$ and $U_2 \to +\infty$ as $x \to \partial E_2$.*

(d) *The surfaces $\Gamma_i := \{x_n = U_i(x') + s_i t\}, i = 1, 2,$ moves with $V = -\kappa + g$ away from the "obstacle" $\{x_n = s_i t\}$.*

Traveling waves in Laminar media

While our approach allows to describe traveling waves both at maximal and minimal speed, we only recover partial traveling waves away from their highest and lowest positions, as described in (c). For instance in the scenario where there exists multiple localized traveling waves at the same asymptotic speed at s_1, our method appears to capture the most external profile of these waves.

When $\nu = e_n$, the maximal traveling wave with maximal speed was constructed by Cesaroni and Novaga using variational method.

Homogenization in Laminar media

Corollary

If homogenization fails for graph solutions in laminar media, then there exists a set E of prescribed mean curvature in $\mathbb{T}^{n-1}$ with $C^{1,\alpha}$ boundary, satisfying $-\kappa = g$ on $\partial E \cap \mathbb{T}^{n-1}$.

Hence if we can guarantee that such E does not exist in $\mathbb{T}^{n-1}$ for a given function g, then we would have proven homogenization in n-dimensional laminar setting.

It seems difficult however to understand properties of sets of prescribed mean curvature in general, when they are not local minimizers of an energy.

Using the local traveling waves constructed above, one can conclude the following:

Theorem

For any $0 < c_0 < 1$, there exists $C = C(n, c_0)$ such that If $\mathrm{osc} g < C$ then $\bar{s}(\nu) = \bar{s}(\nu)$ for all directions with $\nu \cdot e_n > c_0$.

Above result when $\nu = e_n$ was known by Cesaroni and Novaga.

We can remove the c_0 dependence from C if we know that any sets of prescribed and strictly positive, mean curvature in $\mathbb{T}^n$ has a lower bound on its volume. This is the case for instance if $\mathrm{osc} g \leq c_n \max g$, due to Ciraolo and Maggi (2017).

Homogenization in general media: criteria

When $\operatorname{osc} g$ is large, one can generate examples of fronts with different head and tail speeds, adopting the arguments of Caffarelli and Monneau. These examples are however restricted to laminar media.

In general media much less is known. Here the best known criteria for homogenization is the smallness of $|Dg|/g^2$, in which case one obtains a global, Lipschitz continuous, traveling wave solution. It is not clear whether one could improve this restriction.

Cell problem approach

Typically to show that homogenization holds for (1), one starts with an Ansatz

$$u^{\varepsilon}(x,t) = u^0(x,t) + \varepsilon v(\frac{x}{\varepsilon}) + o(\varepsilon),$$

where v solves a cell problem given by the limit profile u^0, which in our setting is a linear profile $x \cdot \nu - st$ for a unit vector ν. The idea is then to look for $s = s(\nu)$ for which there exists a $\mathbb{Z}^n$-periodic solution v of the cell problem

$$F(D^2 v, \nu + Dv, y) = s \text{ in } [0,1]^n.$$

The existence and bounds of solution v for the above problem (or an approximate version of it) hence confirms the Ansatz as well as the homogenizatio of u^{ε}.

Furthermore the regularity of the cell solution v translates into the corresponding regularity properties of the limit speed s.

• When $\frac{|Dg|}{g^2}$ is small, v is Lipschitz continuous, and $s(\nu)$ is Lipschitz continuous (Lions-Souganidis)

• In two dimensions, v and $s(\nu)$ are continuous (Caffarelli-Monneau)

• in two dimensional laminar case, v and $s(\nu)$ are Lipschitz continuous. (Cesaroni-Novaga)

Cell problem: difficulties

When homogenization is not expected, there may be no global limit profile for u^ϵ, let alone an asymptotic planar profile. Instead we introduce "obstacle cell problems", which amounts to looking for the maximal subsolution and minimal supersolution of a "cell problem", yielding descriptions for $\limsup$ and $\liminf$ of u^ε.

These solutions are not periodic and feature very low regularity in general...

Obstacle approach and our contribution

The obstacle approach was first introduced by Caffarelli, Souganidis and Wang for random homogenization of elliptic PDEs, and later adopted by Kim and Požár for free boundary problems. They share the common feature with our problem that there are no standard cell problems one can expect to solve, either due to the non-periodic environment or non-periodic evolution of the free boundaries.

On the other hand, in above results homogenization is expected to hold, and the obstacle solutions are asymptotically regular in these settings.

Our contribution in this paper is thus introducing a “cell problem” type approach for a problem where homogenization is not expected to occur in general, or more precisely when large-scale regularity is missing.

Head and Tail speed: Definition

Set $\varepsilon = 1$. For given $s \in \mathbb{R}$ and $\nu \mathcal{S}^n$, we define $\bar{u}_{s,\nu}$ by the maximal subsolution of (1) which lies below the obstacle $O_{s,\nu}(x,t) := st - \nu \cdot x$. Then we define the head speed by

$$\bar{s}(\nu) := \inf\{s : \bar{u}_{s,\nu} < O_{s,\nu} \text{ after some time } t > T\}.$$

Similarly $\underline{s}(\nu)$ can be defined using $\underline{u}_{s,\nu}$: the minimal supersolution which lies above the obstacle $O_{s,\nu}$.

When defined in finite domain, we impose boundary conditions to coincide with obstacle.

Localization of obstacle solutions

To characterize the asymptotic normal velocity in terms of $\bar{s}$ and $\underline{s}$ for initial data that are not asymptotically affine, it is necessary to localize the obstacle solutions to conclude that their head and tail speed is obtained independent of its constraint on far-away lateral boundary.

In other words, we should show that the obstacle solution does not change its profile too much locally, if we alter its boundary conditions far away from the given neighborhood.

We achieve this by showing that comparison principle holds between two obstacle solutions even if their boundary data are not ordered, if the domain of comparison is far away from the boundary of the original domain.

Localization

Let u and v satisfy

$$u(x,t) \leq u(x-\xi, t-\tau_1) \text{ and } v(x,t) \leq v(x+\xi, t+\tau_2),$$

where $|\xi \cdot \nu| \leq \tau_1, \tau_2$.

Proposition (Local comparison)

Let u, v be a subsolution and supersolution with above property. Suppose $\tau_1, \tau_2 \leq e^{-LT}$, where L is the Lipschitz constant for g. Then the following holds: If $u(\cdot, 0) < v(\cdot, 0)$ in $|(x-x_0)| \leq R(0)$, then

$$u(\cdot, t) \leq v(\cdot, t) \text{ in } \{|x-x_0| \leq R(t)\} \text{ for } 0 \leq t \leq T,$$

Here $R(T) := C(|\xi|)e^{2L(T-t)}$.

Birkhoff properties

The following property is central in proving the "local comparison".

The maximal and minimal obstacle solutions feature space-time monotonicity (Birkhoff) properties. For instance $\bar{u}_{s,\nu}$ satisfy

$$\bar{u}_{s,\nu}(x+\xi, t+\tau) \leq \bar{u}_{s,\nu}(x,t),$$

if $\xi \in \mathbb{Z}^d$ and if $s\tau - \xi \cdot \nu \leq 0$. In particular if ν is irrational then for any small $\tau > 0$ one can find $\xi \in \mathbb{Z}^d$ such that $s\tau \leq \xi \cdot \nu \leq C\tau$.

Proof of local comparison

To prove the local comparison, consider $\tilde{v}(x,t;h) := \inf_{|x-y|\leq h(t)} v(y,t)$, where $h(t) > 0$ decreases fast enough over time so that $\tilde{v}$ is a supersolution to the flow. Since $u < v$ at $t = 0$, $u < \tilde{v}$ for

Suppose that $\tilde{v}(\cdot; h)$ crosses u from above at (x_0, t_0) in $\{|x| \leq R(t)\} \times \{0 \leq t \leq t_0\}$ for the first time. Note that for $t \leq t_0$ and for $|x| \leq R(t-\tau_1)$,

$$u(x,t) \leq u(x-\xi, t-\tau_1) \leq \tilde{v}(x-\xi, t-\tau_1) \leq \tilde{v}(x, t+\Delta t).$$

In particular we have, for $\Delta t = \tau_2 - \tau_1$,

$$u(\cdot, t-\Delta t) \leq \tilde{v}(\cdot, t) \text{ in } \Sigma(t) := |x - x_0| \leq R(t-t_2) - R(t).$$

Due to the finite propagation property, if Δt is small enough compared to $h^2(t)$, then it follows that

$$u(\cdot, t) \leq \tilde{v}(\cdot, t; h/2) \text{ in } \Sigma(t).$$

Proof of local comparison

Let us define the space-time domain $\Sigma := \cup_{t_0-1\leq t\leq t_0}\Sigma(t)\times\{t\}$, then

$$u \leq \tilde{v}(\cdot; h/2) \quad \text{in } \Sigma.$$

In addition, by our assumption on comparison domain,

$$u \leq \tilde{v}(\cdot; h) \quad \text{in } \Sigma \cap \{t = t_0 - 1\} \subset \{|x| \leq R(t_0 - 1)\}.$$

If $\Sigma(t)$ is wide enough, a local perturbation of $\tilde{v}$ yields that $u \leq \tilde{v}(\cdot; h)$ at (x_0, t_0), which is a contradiction.

A local perturbation by Inf-convolution

Lemma

If u is a supersolution, then so is

$$\tilde{u}(x,t) := \inf_{|x-y| \leq h(t)\varphi(x)} u(y,t),$$

if

$$\varphi(x) = 1 - c|(x-x_0)|^2 \text{ where } c << 1.$$

This type of local perturbation is originally introduced in Athanasoupolos-Caffrarelli-Salsa (1996) for the Stefan problem.

Detachment from the obstacle

Local comparison yields the following result:

Proposition

Let ν be a irrational direction and let $u^\varepsilon := \bar{u}_{s,\nu}$ be given with the obstacle speed $s > \bar{s}(\nu) + \delta$ and defined in $|x| \leq Ce^{2LT}, t \in [0, T]$. Then we have

$$u^\varepsilon(x,t) \leq O_{s,\nu}(x,t) - \frac{\delta}{2}(t - t_0) \quad \text{in } |x| \leq Ce^{2LT}/2$$

where $t_0 = t(\nu)$.

Hence $\bar{s}$ describe the maximal homogenized speed for all solutions u^ε, regardless of their boundary data assigned far away from the reference point. Thus $\bar{s}$ can be uniquely characterized as the "head speed" for solutions with general initial data.

Continuity properties

Another consequence of local comparison is the continuity of $\bar{s}$ and $\underline{s}$ as ν varies. Here one needs to compare fronts with different direction of propagations, for which standard comparison principle fails.

Open Questions

- In the graph (laminar) setting, it is shown that small oscillation of g guarantees homogenization. Would the same hold for general periodic media?
- Can we characterize further the head and tail speeds in terms of g? (In laminar setting, when $\nu = e_n$, head speed $\bar{s}$ has a variational formula by Cesaroni-Novaga).
- Generalized traveling waves in general media?

Thank you for your attention!