Extremum Problems for Laplacian Eigenvalue

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Abstract

Eigenvalue Problems for Laplacians are among most studied ones in classical analysis, partial differential equations, calculus of variations and mathematical physics. In this lecture I shall discuss some recent progress on a couple extremum problems involving Dirichlet eigenvalues of the Laplacian. These problems have origins in shape optimization, pattern formation,..., and even the data science. We will show how they are related to harmonic maps into singular spaces and the some recent works on free boundary value problems involving vector valued functions.

Extremum Problems

I. Let Ω be a bounded, smooth domain in $\mathbb{R}^n$ and let $m \geq 1$ be a positive integer. Find a partition of $\Omega : \Omega = \cup_{j=1}^m \Omega_j$ such that it minimizes the functional $F(\lambda_{k_1}(\Omega_1), \ldots, \lambda_{k_m}(\Omega_m))$, where $\lambda_{k_j}(\Omega_j)$ is the k_j-th Dirichlet eigenvalue of Ω_j with the vanishing Dirichlet condition. Here, F is nondecreasing in each of the variables.

II. Find an $\Omega \subseteq \mathbb{R}^n$ with a given volume $vol(\Omega) = V_0 >$ such that it minimizes the functional F as above.

A special case is the following optimal partition problem:

Let Ω be a bounded, smooth domain in $\mathbb{R}^n$ and let $m \geq 1$ be a positive integer. Find a partition of Ω into m with mutually disjoint subsets $\Omega_j,\ j = 1, 2, \ldots, m$, such that the value $\displaystyle\sum_{j=1}^{m} \lambda_1(\Omega_j)$ is minimized among all possible partitions of Ω.
Here $\lambda_1(A)$ denotes the first Dirichlet eigenvalue of the Laplacian Δ on A with the zero Dirichlet boundary condition on ∂A.

Other special cases are the following problems:

1. $min\{(\lambda_k(\Omega) : \Omega \subseteq \mathbb{R}^n, |\Omega| = 1\}$;
2. $min\{\sum_{j=1}^{k} \lambda_j(\Omega) : \Omega \subseteq \mathbb{R}^n$ with $|\Omega| = 1\}$.

Typical questions are:

1) The existence and the regularity of minimizing domains Ω_k;

2) Uniform estimates on diameter and boundary surface area;

3) Asymptotics as k tends to infinite.

There are many works on the existence theory : D. Bucur, G.Buttazzo, Dal Maso, A. Henrot, I.Figueriredo, D.Mazzoleni, A.Pratelli, B. Velichkov, S.Terracini et all.......

Nodal Partition: Let $u \neq 0$ be an eigenfunction of $-\Delta$ on Ω with Dirichlet boundary condition. Let $\mu(u) = \#$ of nodal domains of u. We say a $k-$nodal partition of Ω if there is an eigenfunction u (of $-\Delta$ on $\Omega \cdots$) such that $\mu(u) = k$, and $\Omega = \amalg_{j=1}^{k} \Omega_j$, Ω_j's are nodal domains of u.

Question:

How large $\mu(u_k)$ can be?

Courant's Theorem$\Rightarrow$ $\mu(u_k) \leq k$.
Here, u_k is the k-th eigenfunction of $-\Delta$ on Ω with the Dirichlet boundary condition.

A. Pleijel's Theorem

Let $\mu_k = \max\{\mu(u) : u \in E(\lambda_k) \setminus \{0\}\}$. Then $\overline{\lim}_{k\to\infty}\mu_k/k = \eta_0(N) < 1$.

Proof.

Suppose $u \in E(\lambda_k) \setminus \{0\}$ has μ_k nodal domains, $\Omega = \amalg_{j=1}^{\mu_k}\Omega_j$, then one of the Ω_j's has volume $\leq |\Omega|/\mu_k$. (Note $\mu_k \leq k$) Faber-Krahn inequality$\Rightarrow$ $\lambda_k(\Omega) = \lambda_1(\Omega_j) \geq \lambda_1(B)$, where $B \subset \mathbb{R}^N$ with $|B| = |\Omega_j| \leq |\Omega|/\mu_k$. Take for example $N = 2$, then $\lambda_1(B) \geq \frac{\mu_k}{|\Omega|}\pi j_0^2 = \pi(2\cdot 4)^2\frac{\mu_k}{|\Omega|}$. On the other hand, Weyl Asymptotic Formula$\Rightarrow$ $\lambda_k(\Omega) \simeq \frac{4\pi k}{|\Omega|}$.

$$\therefore (2\cdot 4)^2\mu_k \leq 4k$$

$$\eta_0(2) \leq \frac{4}{(2\cdot 4)^2} < 1.$$

□

Spectral Equal Partition

A partition of $\Omega = \amalg_{j=1}^{m} \Omega_j$ is called a spectral equal partition if $\lambda_1(\Omega_j)$'s are all equal.

A nodal partition is a spectral equal partition. The converse is not true.

ℓ^p-Mininal Partitions

$\Omega = \amalg_{j=1}^{m} \Omega_j$ is a ℓ^p-Minimal partition if for any partition of Ω into $m-$subsets, $\Omega = \amalg_{j=1}^{m} \Omega'_j$, then

$$\left(\sum_{j=1}^{m} \lambda_1(\Omega_j)^p \right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^{m} \lambda_1(\Omega'_j)^p \right)^{\frac{1}{p}} .$$

ℓ^∞−minimal partition

- First, for a given partition, one finds $max\lambda_1(\Omega_j), 1 \leq j \leq m$, then one minimizes this maximum value among partitions $\Omega = \cup_{j=1}^m \Omega_j$.
- Any ℓ^∞−minimal partition is a spectral equal partition.
- Existence, regularity & regularity of free interfaces of minimal partitions have been studied mostly in 2D only.

If u_k is the k^{th}-eigenfunction with the Dirichlet boundary condition, and if u_k has k-nodal domains, then this nodal partition is a ℓ^∞-minimal partition.

Conjectures (C-L)

For $2-D$, $|\Omega|=1$.

(a) $\lim_{m\to\infty} \ell_m^1(\Omega) = c_0$ exists.

(b) $\lim_{m\to\infty} \ell_m^1(\Omega) = \lambda_1(Hexa)$. Here $\ell_m^1(\Omega) = \sum_{j=1}^m \lambda_1(\Omega_j)/m^2$.

We shall prove that the part (a) is true; and it valids also for high dimensions also.

Problem P*

Let

$$\sum = \{y \in \mathbb{R}^m,\ F(y) = 0\} \quad \text{where } F(y) = \sum_{k \neq \ell} y_k^2 y_\ell^2 .$$

Find $v \in H_0^1(\Omega, \Sigma)$ such that

$$\int_\Omega v_k^2(x)dx = 1, \quad k = 1, \ldots, m,$$

and that v minimizes $\int_\Omega |\nabla v|^2 \, dx$ among all such maps in $H_0^1(\Omega, \Sigma)$.

Remark 1

⋆ Problem P^* has an absolute minimizer $u \in H_0^1(\Omega, \Sigma)$.
Without loss of generality one may assume $u = (u_1, \ldots, u_m)$ with each $u_j(x) \geq 0$ on Ω.
Let $\Omega_j = \{x \in \Omega : u_j(x) > 0\}$.
Then Ω_j's, $j = 1, \ldots, m$, give a partition of Ω. Moreover,

$$\sum_{j=1}^{m} \lambda_1(\Omega_j) \leq \sum_{j=1}^{m} \int_\Omega |\nabla u_j|^2(x)dx = \int_\Omega |\nabla u|^2(x)dx.$$

Remark 2

$\star$ Let $\Omega_j,\ j = 1, \ldots, m$ be a partition of Ω (as above).
Let

$$\begin{cases} \Delta v_j + \lambda_1(\Omega_j) v_j = 0 & \text{in } \Omega_j \\ v_j|_{\partial\Omega_j} = 0. \end{cases}$$

Define

$$u_j(x) = \begin{cases} v_j(x) & \text{if } x \in \Omega_j \\ 0 & \text{otherwise} \end{cases}$$

and $u(x) = (u_1(x), \ldots, u_m(x))$. Then $u \in H_0^1(\Omega, \Sigma)$. Moreover

$$\int_\Omega |\nabla u|^2(x) dx = \sum_{j=1}^{m} \lambda_1(\Omega_j).$$

Corollary

Problem P $\iff$ *Problem* P^*.
In particular, Problem P *has a solution.*

We connected the eigenvalue partition problem to a harmonic map problem into a singular space. One can even formulate the associated gradient flows. This observation turns out to be relevant and useful in recent applications to Data Analysis.

Monotonicity and Branching Orders

Let $u : \Omega \to \Sigma$ be an energy minimizing (stationary) map. Let

$$D_a(r) = \int_{B_r(a)} |\nabla u|^2(x)dx, \quad H_a(r) = \int_{\partial B_r(a)} d_\Sigma^2(u, p).$$

Lemma

$N_a(r) = \dfrac{rD_a(r)}{H_a(r)}$ *is monotone increasing.*

Lemma

Let $N(a) = \lim_{r\to 0^+} N_a(r)$.
Then $N(a)$ *is either equal to* 1 *(with* $p = u(a)$*)*
or it is strictly larger than 1 $(N(a) \geq 1 + \delta_n)$.

Basic Facts Established in [CL]

Theorem

If u is an energy minimizing map in $H^1(\Omega, \Sigma)$, then u is locally uniformly Lipschitz continuous.

Remark

It implies subdomains in an optimal partition are Lipschitz. Let

$$\Gamma = \{x \in \Omega : u(x) = 0\}$$
$$\Gamma^* = \{x \in \Gamma : N(x) = 1\}.$$

Theorem

(a) $\Gamma \setminus \Gamma^*$ *is a relatively closed subset of Hausdorff dimension* $\leq n - 2$.

(b) Γ^* *is locally* $C^{1,\alpha}$ *away from* $\Gamma \setminus \Gamma^*$

Sketch of Proof

(a) follows from Federer's dimension reduction principle.
There are four steps for proving (b).

- *Step* 1: Almost two half ball property
- *Step* 2: clean-up lemma
  (scalar case $\implies \Gamma^* \in C^{2,\alpha}$)
- *Step* 3: Reifenberg vanishing property of Γ^*
  and Jerison-Kenig Harnack chain and NTA properties
- *Step* 4: boundary Harnack $\implies \Gamma^* \in C^{1,\alpha}$

Further Results:
For every $S \in \mathscr{F}$, we let

$$S_j = \{a \in S : \text{Inv}\,(T) \leq j \text{ for all tangent cones } T \text{ of } S \text{ at } a\}.$$

Theorem

$S_0 \subset S_1 \subset \cdots \subset S_{n-2} = S$ and $\dim_H S_j \leq j$.

O.Alper proved that S_{n-2} has locally bounded H^{n-2} measure.
S.Snelson showed the similar statements for its gradient flow.
Global structure of the partition and S remains to be difficult to understand.

Let Ω be a bounded domain in $\mathbb{R}^N$.

$$\begin{cases} -\Delta u = \lambda u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

(Normalized) Eigenfunctions: $u_k(x)$ with $\int_\Omega u_k^2(x)dx = 1$,

$$\begin{cases} -\Delta u_k = \lambda_k u_k & \text{in} \quad \Omega, \\ u_k = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

- Eigenvalues λ_k: $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$
- $\{u_k\}$ form an orthonormal basis of $L^2(\Omega)$.
- Weyl Asymptotic Formula:

$$\lambda_k \simeq C_N \left(\frac{k}{|\Omega|}\right)^{2/N} \quad \text{as} \quad k \to +\infty.$$

- Polya Conjecture: $\lambda_k \ge C_N \left(\frac{k}{|\Omega|}\right)^{2/N}$.

Rayleigh-Faber-Krahn Inequality

$$\lambda_1(B) \leq \inf\{\lambda_1(\Omega) : \Omega \subset \mathbb{R}^N \quad \text{with} \quad |\Omega| = 1\}.$$

Here B is a ball in $\mathbb{R}^N$ with $|B| = 1$.

Extremum Problems:

$$(\star) \qquad \inf\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^N \quad \text{with} \quad |\Omega| = 1\} = \Lambda(k, N).$$

Basic Questions

(1) What type domains Ω are admissible in $(\star)$?

(2) How to handle infinite stretchings & splitings?

Theorem

There is a convex domain that solves

$$\inf\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^N \quad \textit{is convex and} \quad |\Omega| = 1\}.$$

Theorem

There is a bounded Quasi-Open set of finite primeter $\Omega_\star$ that solves

$$\inf\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^N \quad \textit{with} \quad |\Omega| = 1\}.$$

Weak version of Spliting Inequality

Assumption

There is a multiply-connected domain $\Omega_k^\star$ that solves

$$(\star) \quad \inf\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^N \quad \text{Q.O. with} \quad |\Omega| = 1\} = \Lambda(k, N).$$

Theorem

$$\Lambda(k, N)^{N/2} = \sum_{j=1}^{m} \Lambda(k_j, N)^{N/2}.$$

Here $\Omega_k^\star = \amalg_{j=1}^m \Omega_{k_j}$, $k_1 + \cdots + k_m = k$, $\sum_{j=1}^m |\Omega_{k_j}| = 1$.

Strong version of Spliting Inequality

Theorem

There is a minimizing sequence $\{\Omega_n\}$ *for* $(\star)$ *such that for all* n *large,*

$$\Omega_n = \amalg_{j=1}^{m} \Omega_{n,k_j}, \quad \sum_{j=1}^{m} k_j = k.$$

Moreover, properly scaled $\{\Omega_{n,k_j}\}$ *would form a minimizing sequence for* $(\star)$ *when* $k = k_j$*. In particular,*

$$\Lambda(k,N)^{N/2} \geq \sum_{j=1}^{N} \Lambda(k_j,N)^{N/2}.$$

Main Theorem

The minimization problem

$$(\star) \quad \inf\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^N \quad \text{Q.O.}, \quad |\Omega| = 1\}$$

has a solution $\Omega_k^\star$ which in general would be $m-$connected with $m < k$.

Theorem [Kriventsov-L]

The minimization problem has a solution $\Omega_k^\star$ such that:
1) It is bounded and of finite primeter with partially smooth boundary.
This applies to, in particular, the following problem:

$$(\star) \quad \inf\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^N \quad \text{Q.O.}, \quad |\Omega| = 1\}$$

2) It, in addition, has a smooth boundary in both $2D$ and $3D$ when the functional is non-degenerate. For example, the problem:
$min\{\sum_{j=1}^{k} \lambda_j(\Omega) : \Omega \subseteq \mathbb{R}^n$ with $|\Omega| = 1\}$,
is non-degenerate.

Theorem

Let Ω be a bounded domain in $\mathbb{R}^2$ with $|\Omega| = 1$. Then $\lim_{m\to\infty} \ell_m^1(\Omega) = c_0$. *Here* $\ell_m^1(\Omega) = \sum_{j=1}^m \lambda_1(\Omega_j)/m^2$, *and* $\Omega = \sqcup_{j=1}^m \Omega_j$ *is a* ℓ^1*-minimal m-partition.*

Remark: The similar statement is also valid for higher dimensions $d \geq 3$ with

$$\ell_m^1(\Omega) = \sum_{j=1}^m \lambda_1(\Omega_j)/m^{1+2/d}.$$

Remark: It is very easy to show

$$0 < A \leq \ell_m^1(\Omega) \leq B \leq \infty,$$

for positive constant A, B, and for all m.

Sketch the proof. (2D Case)

Step 1 Define $a(Q) = \lim_{m\to\infty} \sum_{j=1}^{m^2} \lambda_1(\Omega_j)/m^+$.

Let $m_0 \in \mathbb{N}$ be such that

$$a(Q) \sim \ell^1_{m_0^2}(Q).$$

Here, Q is the unit cube. For all $m \in \mathbb{N}$, $m \gg m_0^2$, write $m = k^2 m_0^2 + O(k)$.

Divide Q into k^2 equal small cubes, and put in each small cubes a scaled optimized partition (into m_0^2 subdomains). This, by a simple scaling, one has

$$\ell^1_{k^2 m_0^2}(Q) \leq \ell'_{m_0^2}(Q).$$

The latter implies $\overline{\lim_m} \ell^1_m(Q) \leq \underline{\lim_m} \ell^1_m(Q)$

Step 2 For any bounded domain $\Omega \subset \mathbb{R}^2$, $\overline{\lim_{m\to\infty}} \ell_m^1(\Omega) \leq \lim_{m\to\infty} \ell_m^1(Q)$. Here, $|\Omega| = |Q| = 1$.
We note that, for any $\epsilon > 0$, there is $k \in \mathbb{N}, \sqcup_{j=1}^k Q_j \leq \Omega \leq \sqcup_{j=1}^k Q_j \cup \sqcup_{i=1}^\ell Q_i'$, where $\{Q_j\}$'s and $\{Q_i\}$'s are small diactic cubes of the same size and $\ell = o(k), \sum_{j=1}^\ell |Q_i'| < \epsilon$. $\ell_m^1(\Omega) \leq \ell_m^1\left(\sqcup_{j=1}^k Q_j\right) \leq (1+2\epsilon)\, a(Q)$, for $m \gg 1$ and $0 < \epsilon \ll 1$.

Step 3 $\underline{\lim_m} \ell_m^1(\Omega) \geq \lim_m \ell_m^1(Q)$

We note first that $\ell_m^1(\Omega) \geq \ell_m^1\big(\sqcup_{j=1}^k Q_j \bigcup \sqcup_{i=1}^\ell Q_i'\big)$.

Next, one shows, for $m \gg (k+\ell)\frac{1}{|Q_j|} \gg 1$, that $\ell_m^1\big(\sqcup_{j=1}^k Q_j \bigcup \sqcup_{i=1}^\ell Q_i'\big) \geq (1-\epsilon)a(Q)$ for any $\epsilon > 0$. This last step is by a direct computation using cut-off functions, ξ supported away from suitablly small neighborhoods of $\big(\sqcup_{j=1}^k \partial Q_j\big) \bigcup \big(\sqcup_{i=1}^\ell \partial Q_i'\big)$ and the minimizing maps $U = (u', \ldots, u^m)$. More precisely, one can obtain the conclusion by calculating $\int |\nabla(\xi U)|^2\, dx$ and $\int |\xi U|^2\, dx$ etc.

Conjecture For bounded domain $\Omega \subseteq \mathbb{R}^2$ with $|\Omega| = 1$, one has $\lim_{m \to \infty} \ell'_m(\Omega) = \lambda_1(H)$. H is a regular hexagon with area of $H = 1$.